Thick and localising subcategories of derived categories.

4th Feb '21

Reference: Finiteness in the derived categories of local rings.

- Dwyer, Greenlees, Iyengar.

(I) The derived category: A recap.

R ring

$\mathcal{C}(R)$:= Category of R-Complexes

Objects: $\cdots \longrightarrow M_{i+1} \xrightarrow{\partial} M_i \longrightarrow M_{i-1} \longrightarrow \cdots$

(not necessarily bounded)

$f: M \longrightarrow N$ Morphism. Induces

$H_*(f): H_*(M) \longrightarrow H_*(N)$

f is a Quasi-iso. if $H_*(f)$ is an iso.

$Q :=$ Quasi-iso's in $\mathcal{C}(R)$

$\mathcal{D}(R) :=$ $Q^{-1}\mathcal{C}(R)$.

?

inverting quasi-iso's.

One can get a handle on this construction using model categories:

- Projective model structure on $\mathcal{C}(R)$:

$We :=$ Weak equivalences = Quasi-isos

$Fi =$ Fibrations: Morphism $f: M \twoheadrightarrow N$ that are surjective.

Cof: Cofibrations: maps $f: M \to N$ s.t. (i) f is injective

(ii) $\mathrm{Ker}(f)$ is semi-projective.

Recall P semi-projective mean a filtration:

$$0 = P(0) \subseteq P(1) \subseteq \dots \subseteq \bigcup_n P(n) = P$$

i.e. $P(n+1)/P(n)$ is a graded proj. R-module with 0 differential.

Example: Any cx.

$$\dots \to P_{i+1} \to P_i \to P_{i-1} \to \dots$$

of projective modules with $P_i = 0 \ \forall\ i \ll 0$

is semi-projective, with $P(n) = 0 \to P_n \to P_{n-1} \to \dots$

$-\times-$

Upshot

$$H_0(\text{semi-proj}) \xrightarrow{\sim} D(R)$$

Thus $\mathrm{Hom}_D(M,N) = \mathrm{Hom}_K(P_M, P_N)$ where

$P_M \xrightarrow{\sim} M$ and $P_N \xrightarrow{\sim} N$

are cofibrant replacements (i.e. projective resolutions).

Key point: $D(R)$ is a Δ *(stable)* category:

So, (i) arbitrary coproducts exist
(ii) one has mapping cones.

A subcategory $0 \neq J \subseteq D(R)$ is *thick* if

it is closed under:

- mapping cones
- retracts (i.e. taking direct summands)
- finite direct sums.

It is *localizing* if it is closed under arbitrary direct sums.

For $M \in D(R)$, write

Thick (M) := smallest thick subcategory containing M.

Loc (M) := smallest localising sub. cat. containing M

Fact: For any $X \in D(R)$, there is a functorial exact Δle.

$$Cell_M(X) \longrightarrow X \longrightarrow L_M(X) \rightarrow \Sigma\, Cell_M(X)$$

s.t. (1) $Cell_M(X) \in Loc(M)$

(2) $\mathrm{Hom}_D(-, L_M(X)) = 0$ on $Loc(M)$

We speak of $Cell_M(X)$ as the M-cellular approximation of X, and $L_M(X)$ as the M-localisation of X.

II Examples

① $R = \mathbb{Z}$ $\quad D(\mathbb{Z})$

(a) $M = \mathbb{Z}/p$ $\quad p$ prime

$\mathrm{Thick}(\mathbb{Z}/p) = \{X \in D(\mathbb{Z}) \mid H_i(X) = 0 \ \forall\, |i| \gg 0$ and $H_i(X)$ is p-power torsion$\}$ + finitely generated

Sketch: $\subseteq$ clear.

$\supseteq$ Say $X \in \mathrm{Mod}\,\mathbb{Z}$ and p-power torsion

$$X := \bigoplus \mathbb{Z}/p^{n_i} \qquad n_i \in \mathbb{N}$$

To check $X \in \mathrm{Thick}(M)$, suffices to verify

$$\mathbb{Z}/p^n \in \mathrm{Thick}(\mathbb{Z}/p) \quad \forall\, n \in \mathbb{N}$$

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0$$

$$1 \longmapsto p^{n-1}$$

This gives an exact Δ^{le}

$$\mathbb{Z}/p \longrightarrow \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n-1} \longrightarrow \Sigma\mathbb{Z}/p$$

$n=2$ $\quad \Sigma^{-1}\mathbb{Z}/p \to \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \to \mathbb{Z}/p \longrightarrow \Sigma\mathbb{Z}/p$

$\therefore \mathbb{Z}/p^2 \in \mathrm{Thick}(M)$

- Then induction.

Thus all p-power torsion modules are in $\mathrm{Thick}(M)$

- A standard truncation completes the proof.

III $^{\text{fully}}$ $\mathrm{Loc}(\mathbb{Z}/p) = \{X \in D(\mathbb{Z}) \mid H_i(X)$ is p-power torsion$\}$

Ex: $\overline{\mathrm{Thick}(\mathbb{Z}/p)} = \mathrm{Thick}(\mathbb{Z}/p^n)$ $\forall$ $n \geq 1$.

Main pt: $\mathbb{Z}/p \in \mathrm{Thick}(\mathbb{Z}/p^n)$.

(6) $M := \bigoplus_{p \text{ prime}} \mathbb{Z}/p$

$$Loc(M) = \{ X \in D(\mathbb{Z}) \mid H_i(X) \text{ [or lim } \forall i\} $$

i.e. $H(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$

Thus $L_M(X) = X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$

$$Cell_M(X) \longrightarrow X \longrightarrow X_{\mathbb{Q}} \longrightarrow$$

— x —

(2) $R = k[x,y]/(x^2, y^2)$ $|x| = 0 = |y|$, k field

$R \twoheadrightarrow \frac{R}{(x,y)} = k$

Ex: $Thick(k) = \{ X \in D(R) \mid H_i(X) = 0 \ \forall\, |i| \gg 0$, $\operatorname{rank} H_i(X) < \infty \}$

Ex: $k \notin Thick(R/2R)$ $\quad \frac{R}{2R} \twoheadrightarrow k$

However $k \in \mathrm{Loc}(R/_{xR})$.

(III) Detecting when $X \in \mathrm{Thick}(M)$ or $\mathrm{Loc}(M)$ is typically a hard problem.

R Commutative Noetherian

- Eg: $R = k[x_1, \dots, x_n]/I$

$\mathrm{Spec}\, R :=$ Prime ideals in R with Zariski topology.

$\mathfrak{p} \in \mathrm{Spec}\, R$

$k(\mathfrak{p}) :=$ Residue field at $\mathfrak{p}$

$= R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$

The **support** of $M \in D(R)$ is the subset

$$\mathrm{supp}_R M := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid k(\mathfrak{p}) \overset{L}{\otimes}_R M \neq 0\}$$

i.e. $\mathrm{Tor}^R_*(k(\mathfrak{p}), M) \neq 0$.

Fact: (1) $\operatorname{Supp}_R M = \emptyset \iff M = 0$.

(2) When $M \in D^b(\operatorname{mod} R)$

$$\| \{ X \in D(R) \mid H_*(X) \text{ f.g. } R\text{-module} \}$$

i.e. $H_i(X) = 0$ for $|i| >> 0$

$H_i(X)$ f.g. R-module.

$$\operatorname{Supp}_R M = \{ p \in \operatorname{Spec} R \mid H_*(M_p) = H_*(M)_p = 0 \}$$

$$= V\left(\operatorname*{ann}_R H_*(M) \right)$$

- In particular, this is a closed subset of $\operatorname{Spec} R$.

Ex: $\operatorname{Supp}_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) = \{(p)\}$

$$\operatorname{Supp}_{\mathbb{Z}} (\mathbb{Q}) = \{(0)\}.$$

(3) If $X \in \mathrm{Loc}(M)$, then

$$\mathrm{Supp}_R X \subseteq \mathrm{Supp}_R M.$$

In particular, $X \in \mathrm{Thick}(M)$, then

$$\mathrm{Supp}_R X \subseteq \mathrm{Supp}_R M.$$

Converse need not hold:

$$\mathrm{Supp}_{\mathbb{Z}} \mathbb{Q} \subseteq \mathrm{Supp}_{\mathbb{Z}} \mathbb{Z} = \mathrm{Spec}\, \mathbb{Z}$$

But $\mathbb{Q} \notin \mathrm{Thick}(\mathbb{Z})$.

However $\mathbb{Q} \in \mathrm{Loc}(\mathbb{Z}) = \mathcal{D}(\mathbb{Z})$

Thm (Hopkins, Neeman) $M, N \in D(R)$ small R-complexes. Then

$$\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R M \Rightarrow N \in \mathrm{Thick}_R(M)$$

Recall: M small means $M \in \mathrm{Thick}(R)$.

① $\mathrm{Supp}_{\mathbb{Z}}(\mathbb{Z}/p^n) = \{(p)\} = \mathrm{Supp}_{\mathbb{Z}}(\mathbb{Z}/p)$

$\Rightarrow \mathrm{Thick}(\mathbb{Z}/p^n) = \mathrm{Thick}(\mathbb{Z}/p)$

② $R = k[x,y]/(x^2, y^2)$ $\mathrm{Spec}\, R = \{(x,y)\}$

$\mathrm{Supp}_R M = \mathrm{Spec}\, R \quad \forall \quad M \neq 0$ in $D(R)$.

But $k \notin \mathrm{Thick}(R)$,

$\notin \mathrm{Thick}(R/_{\mathfrak{m}^2 R})$

So in Hopkins result, smallness is essential.

③ HN theorem leads to a classification of all the thick subcategory of small objects in $D(R)$ (i.e. $\mathrm{Thick}(R)$).

- Connected to nilpotence phenomenon.

(4) Hopkins' proof is "constructive".

- If $N \in \mathrm{Thick}(M)$, M small

one can build N out of M "explicitly".

When M, N are not small, it is hard to detect when

$$N \in \mathrm{Thick}(M).$$

(Except when R is locally complete intersection).

Thm (Neeman): $M, N \in D(R)$.

If $\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R M$, then $N \in \mathrm{Loc}(M)$.

This implies the result of HN:

In any compactly gen. Δ (stable) category

M, N small objects. Then

$N \in \mathrm{Loc}(M) \Rightarrow N \in \mathrm{Thick}(M)$.