Dualities in Topology & Algebra

ICTS (1-14th February, 2021)

Minimal Models

DGA

A differential graded algebra $(\mathscr{A}, d)$ is a graded vector space over a field $\Bbbk$ given by

$$\mathscr{A} = \oplus_{k \geq 0} A^k$$

along with a differential $d : A^k \to A^{k+1}$ such that $d^2 = 0$ and a multiplication

$$m : A^k \otimes A^l \to A^{k+l}$$

such that Leibniz rule holds, i.e.,

$$d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

A dga is called commutative if $xy = (-1)^{|x||y|} yx$.

The multiplication is always assumed to be assocaitive.

Examples

- Differential forms: For a smooth manifold, the de Rham complex $(\Omega^\bullet(M), d)$ is a commutative dga over $\mathbb{R}$ with wedge product of forms as multiplication.

- Singular cochains: For a topological space X, the singular cochains $(C^\bullet(X;R), \delta)$ is a dga over R with cup product as multiplication. However, this is not commutative.

- Singular cohomology: For any space X, its rational singular cohomology $(H^\bullet(X;\mathbb{Q}), 0)$ is a cdga over $\mathbb{Q}$ as the cup product is homotopy commutative.

- Free commutative graded algebra: A graded algebra of the form

$$A = \text{Symmetric}(V^{\text{even}}) \otimes \text{Exterior}(V^{\text{odd}}) (= \Lambda V)$$

where $V = V^{\text{even}} \oplus V^{\text{even}}$ is a graded vector space.

A MODEL FOR SIMPLICES

For each topological space, Sullivan constructed a cdga $A_{PL}(X)$ of forms with $\mathbb{Q}$-coefficients whose cohomology is isomorphic to $H^\bullet(X;\mathbb{Q})$.

Consider the n-simplex

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in R^{n+1} \mid t_0 + \cdots + t_n = 1, 0 \leq t_j \leq 1\}.$$

Consider the restriction to Δ^n all forms in $\mathbb{R}^{n+1}$ of the form

$$\sum \varphi_{i_1 \cdots i_j} dt_{i_1} \wedge \cdots \wedge dt_{i_j},$$

where $\varphi_{i_1 \cdots i_j}$ are polynomials in the t_i's with $\mathbb{Q}$-coefficients. These forms form a cdga over $\mathbb{Q}$, denoted $A_{PL}(\Delta^n)$.

- The cdga $A_{PL}(\Delta^n)$ has cohomology concentrated in degree 0.
- If $\Delta^k \subset \Delta^n$ is a face, then there is map of cdga's $A_{PL}(\Delta^n) \to A_{PL}(\Delta^k)$ obtained by restriction.

A MODEL FOR SIMPLICIAL COMPLEXES

The cdga can be identified with

$$A_{PL}(\Delta^n) \cong \frac{\Lambda(t_0,\dots,t_n,dt_0,\dots,dt_n)}{\left(\sum t_j - 1, \sum dt_j = 0\right)}.$$

Let K be a simplicial complex. Define $A_{PL}(K)$, the cdga of piecewise polynomial forms with $\mathbb{Q}$-coefficients, to be all collections $\omega = \{\omega_\sigma\}_{\sigma\in K}$ that are compatible, i.e., if $\iota : \tau \to \sigma$ is the inclusion of a face, then $\iota^*\omega_\sigma = \omega_\tau$.

Theorem *For a simplicial complex K, the integration map*

$$\rho : A_{PL}(K) \to C^\bullet_{\text{simp}}(K;\mathbb{Q}), \quad \rho(\omega)(\sigma) = \int_\sigma \omega_\sigma$$

induces an isomorphism of algebras, where $C^\bullet_{\text{simp}}(K;\mathbb{Q})$ is the simplicial cochain complex.

Digression: Simplicial sets

A simplicial set is a collection of sets $\mathscr{K} := \{K_n\}_{n\geq 0}$ equipped with maps

$$\partial_i : K_n \longrightarrow K_{n-1},\ s_i : K_n \longrightarrow K_{n+1},\ i = 0,\ldots,n.$$

These maps satisfy a specific set of relations.

- The set $S_n(X)$ of n-simplices in X form a simplicial set with maps

$$\partial_i : S_n(X) \to S_{n-1}(X),\ s_i : S_n(X) \to S_{n+1}(X),\ i = 0,\ldots,n$$

given by

$$\begin{aligned} \partial_i(\sigma)(t_0,\ldots,t_{n-1}) &= \sigma(t_0,\ldots,t_{i-1},0,t_i,\ldots,,t_{n-1}) \\ s_i(\sigma)(t_0,\ldots,t_{n+1}) &= \sigma(t_0,\ldots,t_i+t_{i+1},\ldots,t_{n+1}) \end{aligned}$$

These maps satisfy

$$\begin{aligned} \partial_i\partial_j &= \partial_{j-1}\partial_i,\ i<j \\ s_i s_j &= s_{j+1}s_i,\ i\leq j \end{aligned}$$

A SIMPLICIAL COCHAIN ALGEBRA

Consider a simplicial cdga A_{PL} with $(A_{PL})_n = A_{PL}(\Delta^n)$. The inclusion of the j^{th} face

$$\iota_j : \Delta^{n-1} \hookrightarrow \Delta^n,\ (t_0,\ldots,t_{n-1}) \mapsto (t_0,\ldots,t_{j-1},0,t_j,\ldots,,t_{n-1})$$

induces maps $\partial_j = \iota_j^*$. The projection to the j^{th} face

$$p_j : \Delta^{n+1} \hookrightarrow \Delta^n,\ (t_0,\ldots,t_{n+1}) \mapsto (t_0,\ldots,t_{j-1},t_j+t_{j+1},\ldots,,t_{n+1})$$

induces maps $s_j = p_j^*$.

The cdga $A_{PL}(X)$ is defined to be

$$A_{PL}(X) = \mathrm{Hom}_{\mathrm{Simp}}(S_\bullet(X),(A_{PL})_\bullet).$$

A q-form ω assigns to each singular n-simplex σ an element $\omega_\sigma \in (A_{PL})_n^q$ such that $\partial_i \omega_\sigma = \omega_{\partial_i \sigma}$ and $s_i \omega_\sigma = \omega_{s_i \sigma}$.

Properties

- The cohomology of $A_{PL}(X)$ is isomorphic to $H^\bullet(X;\mathbb{Q})$.
- For a smooth manifold M, choose a triangulation K. Consider the maps

$$A_{PL}(K)\otimes_{\mathbb{Q}}\mathbb{R}\xrightarrow{\simeq}C^\bullet_{\text{simp}}(K;\mathbb{R})\xleftarrow{\simeq}C^\bullet(M;\mathbb{R})\xrightarrow{\simeq}C^\bullet_\infty(M;\mathbb{R})\xleftarrow{\simeq}\Omega^\bullet(M).$$

We expect many rational homotopy invariants of M to be obtainable from $\Omega^\bullet(M)$.

- For path connected spaces X and Y, there is a natural quasi-isomorphism

$$A_{PL}(X)\otimes A_{PL}(Y)\to A_{PL}(X\times Y).$$

Among the models of commutative cochains for topological spaces, Sullivan models play an important role.

SULLIVAN MODELS

A Sullivan cdga is a cdga of the form $(\Lambda V, d)$, i.e., the underlying algebra is free, and if $V = \oplus_{n\geq 1} V^n$, then V should admit a basis $\{x_\alpha\}$ indexed by a well-ordered set such that $d(x_\alpha) \in \Lambda(x_\beta)_{\beta<\alpha}$.

- $\Lambda(x,y)$ with $|x| = 2, |y| = 3$ and $dx = 0, dy = x^2$ along with $x < y$.
  The cohomology of this agrees with $H^\bullet(S^2;\mathbb{Q})$.
- $\Lambda(x,y)$ with $|x| = k, |y| = k-1 \geq 1$ and $dy = x$ along with $x < y$.
  The cohomology is that of a point; such cdga's are called contractible.

A minimal cdga is a Sullivan cdga $(\Lambda V, d)$ such that $d(V) \subseteq \Lambda^{\geq 2} V$.

- $\Lambda(x,y,z,w)$ with $|x| = |y| = 2, |z| = |w| = 3$ and $dx = dy = 0, dz = xy, dw = x^2 - y^2$.

Minimal models

Fact: Any Sullivan cdga is the tensor product of a minimal cdga and contractible cdga's.

A minimal model for a cdga (A,d) is a minimal cdga $(\Lambda V,d)$ along with a quasi-isomorphism $\varphi:(\Lambda V,d)\to(A,d)$.
A minimal model for a space X is a minimal model for $A_{PL}(X)$.

Theorem *Let (A,d) be a connected cdga, i.e., $H^0(A,d)=\Bbbk$.*

(i) There is a minimal model $(\Lambda V,d)$ for (A,d).

(ii) If (A,d) is simply connected, i.e., $H^1(A,d)=0$, and $H^\bullet(A,d)$ is of finite type, then V is also of finite type.

(iii) In the simply connected case, any two minimal models are isomorphic.

In view of (iii), one speaks of the minimal model of a simply connected cdga.

EXAMPLES

- Spheres: Choose a volume form $\omega \in \Omega^3(S^3)$ and consider

$$\varphi : (\Lambda(x), 0) \to (\Omega^\bullet(S^3), d), \;\; x \mapsto \omega.$$

This is quasi-isomorphism and $(\Lambda(x), 0)$ is the $\mathbb{R}$-minimal model for S^3. For S^2 consider

$$\varphi : (\Lambda(x, y), dx = 0, dy = x^2) \to (\Omega^\bullet(S^2), d), \;\; x \mapsto \omega$$

where ω is a volume form and $|x| = 2, |y| = 3$.

- Complex projective spaces: As $H^\bullet(\mathbb{CP}^n; \mathbb{Q}) \cong \mathbb{Q}[a]/(a^{n+1})$ with $|a| = 2$, we may choose $x, y \in A_{PL}(\mathbb{CP}^n)$ of degree 2 and $2n+1$ such that

$$a = [x], \;\; dx = 0, \;\; dy = x^{n+1}.$$

Then $x \mapsto a, y \mapsto 0$ defines a quasi-isomorphism and $(\Lambda(x, y), d)$ is the minimal model.