2nd Feb. 21

Dualities in local algebra

Topics

- Cellular approximations
- Local cohomology & derived completions
- Gorenstein rings
- Local duality

Additional reference:

Homotopy invariant commutative algebra over fields, by J.P.C. Greenlees.

Top. Spaces ⇝ Algebraic Homotopy Theory ⇝ Alg. Geom. Num. Th.

Basic problem: — x —

$\mathcal{C}$ any category.

$M \in ob(\mathcal{C})$

- What objects of $\mathcal{C}$ can be built out of M?
- Given $A \in ob(\mathcal{C})$, can we find "best" approximation of A by M?

"M-Cellular approximation".

(I) R ring ($1 \in R$) fixed.

M an R-Complex:

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots$$

ΣM $\qquad (\Sigma M)_i = M_{i-1}$

↑

"Suspension of M" $\qquad \partial^{\Sigma M} = -\partial^M$

- These are the objects of $\mathcal{C}(R)$

Maps: $M: \cdots \to M_{i+1} \xrightarrow{\partial} M_i \to M_{i-1} \to \cdots$

$f \downarrow \qquad\qquad \downarrow f_{i+1}, C_i \qquad \downarrow d_i C_i \qquad \downarrow f_{i-1}$

$N: \longrightarrow N_{i+1} \xrightarrow{\partial} N_i \to N_{i-1} \to \cdots$

$H_*(M) := \{H_i(M)\}_{i \in \mathbb{Z}} \qquad H_i(M) := \ker(\partial_i) / \operatorname{Im}(\partial_{i+1})$

Given morphism $f: M \to N$ one gets

$$H_*(f): H_*(M) \xrightarrow{\sim} H_*(N)$$

f is a quasi-iso. if $H_*(f)$ is an iso.

In localized: $\mathcal{C}(R)/_{\sim} =: \mathcal{D}(R)$ "Derived category"

Homotopy equivalence of maps:

$$f, g: M \to N \qquad f \equiv g$$

Mean homotopic.

- i.e. $\exists\ h: M \to \Sigma^{-1} N$

$$h_i: M_i \to N_{i+1}$$

s.t. $\partial^N h + h \partial^M = f - g$.

$$\mathrm{Hom}_K(M,N) := \mathrm{Hom}_{\mathcal{C}}(M,N)/\simeq$$

Then $f: M \to N$ is a homotopy equivalence, if $\exists\ g: N \to M$ s.t.

$$g \circ f \simeq id^M \quad \text{and} \quad f \circ g \simeq id^N$$

Homotopy equiv. $\Rightarrow$ quasi-isom.

Semi-projective complexes: $P^\cdot \in \mathcal{C}(R)$

is semi-projective if it has a filtration

$$\{0\} = P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) \subseteq \cdots \subseteq P$$

s.t. ① $P(n)$ subcomplexes of P

② $\bigcup P(n) = P$

③ $\dfrac{P(n+1)}{P(n)}$ is graded projective R-module with

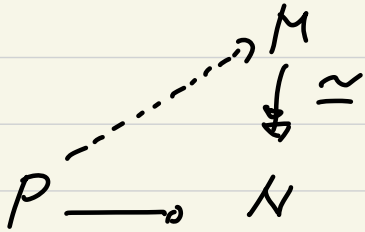
$\partial = 0$

i.e. $\frac{P(n+1)}{P(n)} = \oplus \Sigma^i Q_i$ where Q_i projective

$\partial P(n+1) \subseteq P(n)$

i.e. $P(n+1)$ is obtained from $P(n)$ by "gluing" projective modules.

Facts: (1) P semi-proj. has lifting property:



i.e. $M \xrightarrow{\simeq} N$ in duals

$$\underset{K}{Hom}(P, M) \xrightarrow{\simeq} \underset{K}{Hom}(P, N)$$

$\curvearrowright$ abelian groups

(2) P, Q semi-projective and

$$P \xrightarrow{\sim} Q \Rightarrow P \xrightarrow{\equiv} Q$$

f quasi-iso $\Rightarrow$ f a htpy equiv.

(3) Any $M \in C(R)$ has

$$P \xrightarrow{\sim} M \quad \leftarrow \text{ can make this functorial.}$$

$\uparrow$ semi-projective.

$$(\text{semi-Proj})/_{\equiv} \xrightarrow{\sim} C(R)/_{\sim} =: DR$$

$\uparrow$ Equiv.

Model for the derived category of R.

$\mathrm{Hom}_D(M, N) :=$ Morphisms in $D(R)$

$$= \operatorname{Hom}_K(P_M, N)$$

$P_M \to M$ is a semi-proj. resln.

$$\overset{*}{\operatorname{Hom}}_D(M, N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_D(M, \Sigma^i N)$$

$D(R)$ is a triangulated category.

$f: M \to N$ morphism

$$M \to N \to \operatorname{Cone}(f) \to \Sigma M \to \Sigma N$$

$\uparrow$ Mapping Cone of f

These diagrams are the Δ^k in $D(R)$:

$$M \to N \to \operatorname{Cone}(f) \to \Sigma^k M$$

For any $W \in D(R)$ this induces:

$$\cdots \to \underset{D}{\mathrm{Hom}}(W, M) \to \underset{D}{\mathrm{Hom}}(W, N) \to \underset{D}{\mathrm{Hom}}(W, \mathrm{Cone}(f)) \to \underset{D}{\mathrm{Hom}}(W, \Sigma M) \to \cdots$$

l.e.s. of abelian groups.

(II) Fix $M \in D(R)$ an R-complex.

$\mathrm{Thick}(M) :=$ Thick subcategory gen. by M

$=$ Obj $(D(R))$ that can be built out of M using:

- Finite sums
- Σ
- Mapping cones
- Retracts:

$$Y \underset{f}{\overset{g}{\rightleftarrows}} X \in \mathrm{Thick}(M)$$

$gf = \mathrm{id}^Y$ in $D(R)$

Then $Y \in \mathrm{Thick}(M)$

(i.e. direct summands of objects in Thick(M) are in it)

Loc(M) := Allow also arbitrary sums

$$\mathrm{Thick}(M) \subseteq \mathrm{Loc}(M)$$

"finitely built from M" ↑ "built from M".

Example: $M := R$

Thick(R) are complexes iso. to

$$0 \to P_b \to \cdots \to P_a \to 0$$ where

each P_i is f.g. projective.

("perfect" complexes)

- These are precisely the (compact) objects in $D(R)$:

For any $M \in D(R)$ and $\{X_i\}_{i \in I}$

$$\bigoplus_i \operatorname{Hom}_D(M, X_i) \longrightarrow \operatorname{Hom}_D(M, \bigoplus X_i)$$

M Small if this is a bijection $\forall \{X_i\}$.

Recap: Thick(R) = Small objects in $D(R)$.

Loc(R) = $D(R)$

Equivalent to each complex having a

semi-proj. replacement.

What about a general M?

Fact: (Dror-Farjoun / Neeman): $M \in D(R)$

$$Loc(M) \hookrightarrow D(R)$$

This has a right adjoint:

$$Cell_M(-): D(R) \to Loc(M)$$

"M-cellular approximation"

i.e. $\forall\ X \in D(R)$, $\exists$ morphism

$$Cell_M(X) \to X \quad \text{s.t.}$$

(1) $Cell_M(X) \in Loc(M)$

(2) $\forall\ W \in Loc(M)$, the induced map

$$Hom_D(W, Cell_M(X)) \xrightarrow{\simeq} Hom_D(W, X)$$

Equivalently: $\underset{D}{\mathrm{Hom}}^*(M, \mathrm{Cell}_M(X)) \xrightarrow{\sim} \underset{D}{\mathrm{Hom}}^*(M, X)$

Complete $\mathrm{Cell}_M(X) \to X$ to a Δ-

$$\mathrm{Cell}_M(X) \to X \to L_M(X) \to \Sigma\,\mathrm{Cell}_M(X)$$

Then: $\underset{D}{\mathrm{Hom}}^*(M, L_M(X)) = 0$

M- trivial or M- local.

Ex: $R = \mathbb{Z}$ $M = \mathbb{Z}/p$ p prime

$$L_M(X) = X\left[\tfrac{1}{p}\right]$$

In general, $\mathrm{Cell}_M(X)$ is "complicated".

Except when M is small; i.e. finitely built out of R.

(11) Classical Morita Theory

(H. Bass: Oregon lecture notes)

R ring, M module.

$$E := \underset{\text{I}}{\text{End}_R(M)} = \text{Hom}_R(M, M)$$

viewed as a ring.

$$\text{Mod } R \rightleftarrows \text{Mod } E^{op}$$

$$X \longrightarrow \text{Hom}_R(M, X)$$

right E-module

$$Y \otimes_E M \longleftarrow Y$$

- This is an adjoint pair. It induces an equivalence:

$$\begin{array}{ccc} \operatorname{add} M & \rightleftarrows & \operatorname{add} E^{\mathbb{P}} \\ \uparrow & & \uparrow \\ \text{Direct summands of} & & \text{f.g. projective} \\ \text{finite direct sums of} & & E\text{-modules} \\ M & & \end{array}$$

$$\operatorname{Hom}_R(M, X) \otimes_E M \twoheadrightarrow X$$

$$Y \longrightarrow \operatorname{Hom}(M, Y \otimes_E M)$$

When M is finitely presented, we also get:

$$\operatorname{Add} M \rightleftarrows \operatorname{Proj} E^{\mathbb{P}}$$

$$-\times-$$

$$M = R^n \qquad \operatorname{End}_R(R^n) = M_n(R)$$

$$\operatorname{Mod} R \rightleftarrows \operatorname{Mod} M_n(R)^{\mathbb{P}}$$