DUALITIES IN TOPOLOGY & ALGEBRA

ICTS (1-14$^{\text{th}}$ February, 2021)

LOCALIZATION OF SPACES

LOCALIZATION

All rings are assumed to be integral domains and with 1.

For a multiplicatively closed subset S of a ring R, define

$$S^{-1}R = \{[r/s] \mid r \in R, s \in S\}, \quad r/s \sim r'/s' \text{ if } rs' = r's.$$

- Nomenclature: *R localized away from S*
- $S^{-1}R$ is a ring:

$$[r/s] + [r'/s'] = [(rs' + r's)/ss'], \quad [r/s] \cdot [r'/s'] = [rr'/ss'].$$

- Localization homomorphism:

$$R \to S^{-1}R, \quad r \mapsto [r/1].$$

- $(R - 0)^{-1}R$ is the field of fractions of R.

- Localized at a prime: If $\mathfrak{p}$ be a prime ideal, then $(R-\mathfrak{p})^{-1}R$ is called R localized at $\mathfrak{p}$; denoted by $R_{\mathfrak{p}}$.

- Let $\mathfrak{m} = (x-a)$ and $R = \mathbb{C}[x]$. In $R_{\mathfrak{m}}$, any polynomial $p(x)$ such that $p(a) \neq 0$ is invertible.

- The localized ring $\mathbb{Z}_{(3)}$ is a subring of $\mathbb{Q}$, where the fractions have denominators coprime to 3.

- Let ℓ be a set of primes in $\mathbb{Z}$. If S is the multiplicative set generated by primes *not* in ℓ, then $S^{-1}\mathbb{Z}$ is the ring $\mathbb{Z}$ localized at ℓ, written as $\mathbb{Z}_{\ell}$. It is a subring of $\mathbb{Q}$ with $\mathbb{Z}_{\varnothing} = \mathbb{Q}$ and $\mathbb{Z}_{\text{all primes}} = \mathbb{Z}$.

- Localizing modules: For an R-module M, consider

$$S^{-1}M = M \otimes_R S^{-1}R.$$

ABELIAN GROUPS

The localization of an abelian group A with respect to a set of primes ℓ is the $\mathbb{Z}_\ell$-module $A_\ell = A \otimes_\mathbb{Z} \mathbb{Z}_\ell$

- $\mathbb{Z}_\ell \otimes_\mathbb{Z} \mathbb{Z}_{\ell'} \cong \mathbb{Z}_{\ell \cap \ell'}$

  In particular $\mathbb{Z}_\ell \otimes_\mathbb{Z} \mathbb{Z}_\ell = \mathbb{Z}_\ell$, $\mathbb{Q} \otimes_\mathbb{Z} \mathbb{Z}_\ell = \mathbb{Q}$.

- $\mathbb{Z}_\ell \times_\mathbb{Q} \mathbb{Z}_{\ell'} \cong \mathbb{Z}_{\ell \cup \ell'}$

  The left hand term is the fibre product of the inclusions $\iota : \mathbb{Z}_\ell \hookrightarrow \mathbb{Q}$ and $\iota' : \mathbb{Z}_{\ell'} \hookrightarrow \mathbb{Q}$. If $\ell \cap \ell' = \varnothing$, then

$$\iota + \iota' : \mathbb{Z}_\ell \times \mathbb{Z}_{\ell'} \to \mathbb{Q}$$

is surjective.

Fact: An abelian group is a $\mathbb{Z}_\ell$-module if and only if $A \cong A_\ell$ via the localization homomorphism.

- Flatness: Localization preserves exactness, i.e., if

$$0 \to A \to B \to C \to 0$$

is a s.e.s of abelian groups, then $0 \to A_\ell \to B_\ell \to C_\ell \to 0$ is also exact.

The functor $- \otimes \mathbb{Z}_\ell$ is right exact. If $a/s \in A_\ell$ maps to $0 \in B_\ell$, then $ta = 0$ in B for some $t \in S$. This implies that $a/s \sim 0/t$.

- Two out of three: In a s.e.s of abelian groups if two of the groups are $\mathbb{Z}_\ell$-modules, so is the third.

Follows from the commutative diagram and five lemma

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle -\otimes\mathbb{Z}_\ell} & & \downarrow{\scriptstyle -\otimes\mathbb{Z}_\ell} & & \downarrow{\scriptstyle -\otimes\mathbb{Z}_\ell} & & \\
0 & \longrightarrow & A_\ell & \longrightarrow & B_\ell & \longrightarrow & C_\ell & \longrightarrow & 0
\end{array}
$$

CONNECTIONS TO TOPOLOGY

In a long exact sequence

$$\cdots \to A_n \to B_n \to C_n \to A_{n-1} \to \cdots$$

of abelian groups, if two of three families $\{A_n\}, \{B_n\}, \{C_n\}$ are $\mathbb{Z}_\ell$-modules, then so is the third. This follows by applying $- \otimes \mathbb{Z}_\ell$ and five lemma.

- Fibrations: Let $F \hookrightarrow E \to B$ be a (Serre) fibration of connected spaces with abelian fundamental groups. Then we have a l.e.s

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

Thus, if two of the families $\{\pi_*(F)\}, \{\pi_*(E)\}, \{\pi_*(B)\}$ are $\mathbb{Z}_\ell$-modules, then so is the third.

LOCAL SPACES

A simply connected space X is called ℓ-local if $\pi_*(X)$ are $\mathbb{Z}_\ell$-modules.

$\ell = $ all primes: any space is ℓ-local

$\ell = \varnothing$: if X is ℓ-local, then it is called a rational space since $\pi_j(X)$ is a vector space over $\mathbb{Q}$.

If X is $(r-1)$-connected ($r \geq 2$), then the Hurewicz isomorphism

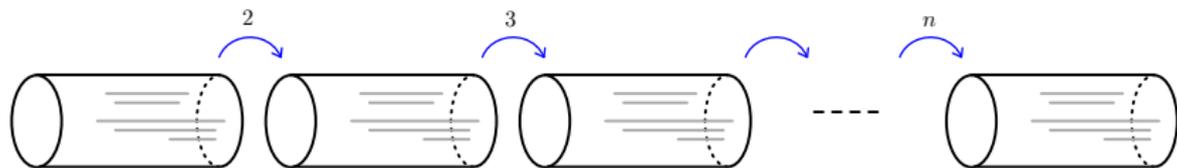$$\pi_r(X) \xrightarrow{\cong} H_r(X; \mathbb{Z})$$

implies that $\pi_r(X)$ is a $\mathbb{Z}_\ell$-module if and only if $H_r(X; \mathbb{Z})$ is. Note that $H_r(X; \mathbb{Z}_\ell) \cong H_r(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$.

Theorem *For a simply connected space X, the following are equivalent:*

(a) $\{\pi_*(X)\}$ are $\mathbb{Z}_\ell$-modules

(b) $\{\widetilde{H}_*(X; \mathbb{Z})\}$ are $\mathbb{Z}_\ell$-modules

(c) $\{\widetilde{H}_*(\Omega X; \mathbb{Z})\}$ are $\mathbb{Z}_\ell$-modules.

Rational circle

For a positive integer k, consider the map $k: S^1 \to S^1, z \mapsto z^k$. Let $C = S^1 \times [0, 1]$ denote the cylinder placed horizontally. Consider the space S_n^1 as indicated below

As $S_n^1 \hookrightarrow S_{n+1}^1$, we define $S_{\mathbb{Q}}^1$ to be the colimit of these spaces.

- It can be shown that $\pi_1(S_{\mathbb{Q}}^1)$ and $H_1(S_{\mathbb{Q}}^1; \mathbb{Z})$ are both $\mathbb{Q}$ while higher homotopy and homology groups are zero.

- As each S_n^1 deforms to the rightmost S^1 and the gluing is usually drawn to look like a telescope, the limiting space is often referred to as the infinite telescope. The leftmost S^1 includes into $S_{\mathbb{Q}}^1$.

Localization revisited

All spaces are assumed to be simply connected, having the homotopy type of a CW complex.

A map $\ell : X \to X_\ell$ of a space X to an ℓ-local space X_ℓ is called a localization of X if it is universal for maps of X into ℓ-local spaces.

Theorem *For a map $\ell : X \to X'$, the following are equivalent:*
(a) ℓ is a localization
(b) ℓ localizes reduced integral homology
(c) ℓ localizes homotopy groups.

- Any two localizations are homotopy equivalent by universality.

- If ℓ is the identity map, then we deduce that X is its own localization if and only if $\{\pi_*(X)\}$ are $\mathbb{Z}_\ell$-modules.

Rational spheres

Enumerate the positive integers relatively prime to ℓ as $\{k_1, k_2, \ldots\}$. For $n \geq 2$ consider

$$S_\ell^n = \left(\vee_{i=0}^\infty S_i^n \right) \cup_\varphi \left(\sqcup_{j=1}^\infty D_j^{n+1} \right)$$

where the disk D_j^{n+1} is attached via

$$\varphi : \partial D_j^{n+1} \to S_{j-1}^n \vee S_j^n, \ \varphi_*[S^n] = [S_{j-1}^n] - k_j[S_j^n].$$

It can be shown that $\widetilde{H}_*(S_\ell^n; \mathbb{Z})$ is concentrated in degree n and equals $\mathbb{Z}_\ell$. As S_ℓ^n is simply connected, the inclusion $S_0^n \hookrightarrow S_\ell^n$ is a localization.

The space $D_\ell^{n+1} = S_\ell^n \times [0,1] / S_\ell^n \times \{0\}$ will be called the ℓ-local $(n+1)$-disk while S_ℓ^n is the ℓ-local n-sphere.

LOCALIZATION: PROPERTIES

- Localizations of spaces exist.
- Localization preserves fibrations and cofibrations.
- For any ℓ-local space X, the set $[S_\ell^n, X]_*$ is a group for $n > 1$ since $\Sigma S_\ell^{n-1} \simeq S_\ell^n$. This group is isomorphic to $\pi_n(X)$ for $n > 1$.
- Given $f : X \to Y$, there exists $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ such that the diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \xrightarrow{\ f_{\mathbb{Q}}\ } & Y_{\mathbb{Q}} \end{array}$$

The rational homotpy type of X is defined to be the (weak) homotopy type of $X_{\mathbb{Q}}$.

- The rationalization of a simply connected Lie group G is the product of odd rational spheres.