On the mod p cohomology for GL_2 (II)

(joint with Haoran Wang)

Yongquan Hu

Morningside Center of Mathematics

I.C.T.S. - T.I.F.R.
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1. Main results (II)

2. The proofs

Notation. Keep (mostly) the notation in the talk of Haoran Wang.

- $L = F_v$ for $v|p$, unramified extension over $\mathbb{Q}_p$ of degree f ;
- $\varpi_L \in \mathcal{O}_L \subset L$, $\mathbb{F}_q \cong \mathcal{O}_L/\varpi_L$;
- $G = \mathrm{GL}_2(L)$, Z =center, $P = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $\overline{P} = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$;
- $K = \mathrm{GL}_2(\mathcal{O}_L)$, I =Iwahori ;
- $K_1 = \mathrm{Ker}(K \to \mathrm{GL}_2(\mathbb{F}_q))$, I_1 =pro-p-Iwahori, $Z_1 = Z \cap K_1$;
- $\bar{\rho} : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(\mathbb{F})$ cont., where $\mathbb{F}/\mathbb{F}_p$ finite (will be reducible non-split and strongly generic in main results) ;
- $\pi(\bar{\rho})$ = smooth admissible representation of G corresponding to some globalization of $\bar{\rho}$ in mod p cohomology (i.e. $\pi_v^D(\bar{r})$ in Haoran's talk).

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- Gee, Emerton-Gee-Savitt : $\mathrm{soc}_K \pi(\bar{\rho}) = \oplus_{\sigma \in W(\bar{\rho})} \sigma$
- EGS : $\pi(\bar{\rho})^{I_1}$
- HW, LMS, Le : $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$
- HW, BHHMS : $\pi(\bar{\rho})[\mathfrak{m}_{K_1}^2]$, $\pi(\bar{\rho})[\mathfrak{m}_{I_1}^3]$, and Gelfand-Kirillov dimension
- Breuil-Diamond, H., Dotto-Le : (partial) local-global compatibility.

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Remark : In general, not known "f.g. $\Rightarrow$ finite length".

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$$\pi_0 = \mathrm{Ind}_P^G \chi_1\omega^{-1} \otimes \chi_2, \quad \pi_f = \mathrm{Ind}_P^G \chi_2\omega^{-1} \otimes \chi_1$$

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  and π_i *supersingular for* $1 \leq i \leq f-1$.
- *If* $\bar{\rho} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ *is nonsplit, then* $\pi(\bar{\rho})$ *has a Jordan-Hölder filtration :*

  $$\pi_0 \text{ --- } \pi_1 \text{ --- } \cdots \text{ --- } \pi_{f-1} \text{ --- } \pi_f$$

  and $\pi(\bar{\rho}^{\mathrm{ss}}) = \pi(\bar{\rho})^{\mathrm{ss}}$.

Example. $L = \mathbb{Q}_p$, $\bar{\rho}|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$ with $1 \leq r \leq p-3$ and $* \neq 0$, then

- $W(\bar{\rho}) = \{\mathrm{Sym}^r \mathbb{F}^2\}$
- $D_0(\bar{\rho}) = \left(\mathrm{Sym}^r \mathbb{F}^2 \text{ --- } \mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det^r \bigoplus \mathrm{Sym}^{p-3-r} \mathbb{F}^2 \otimes \det^{r+1}\right)$
- $\pi(\bar{\rho}) = (\pi_0 - \pi_1)$.

Main results

Keep the global hypotheses in Haoran's talk. Assume $\bar{\rho} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ is reducible non-split and strongly generic.

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Theorem B (H.-Wang, 2020)

If $f = 2$, then $\pi(\bar{\rho})$ has length 3, of the form

$$\pi_0 — \pi_1 — \pi_2$$

with π_0, π_2 principal series and π_1 supersingular.

Corollary

We have $\mathrm{End}_G(\pi(\bar{\rho})) = \mathbb{F}$.

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Proof. To $\pi(\bar{\rho})$ we associate the Diamond diagram

$$D(\bar{\rho}) := \big(D_0(\bar{\rho})^{I_1} \hookrightarrow D_0(\bar{\rho})\big).$$

Taking restriction gives

$$\mathrm{End}_G(\pi(\bar{\rho})) \to \mathrm{End}_{\mathcal{DIAG}}(D(\bar{\rho}))$$

which is injective because of Theorem A. By Breuil-Paškūnas, one knows $\mathrm{End}_{\mathcal{DIAG}}(D(\bar{\rho})) = \mathbb{F}$. □

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  Proof : By Frobenius, $\langle G.\sigma_0 \rangle$ is a quotient of c-$\mathrm{Ind}_{KZ}^G \sigma_0$ which carries an action of $\mathbb{F}[T]$ (Hecke algebra of Barthel-Livné).
  Choose $v_0 \in \sigma_0^{I_1}$ of character χ_0. If $T(v_0) = 0$, then some JH factor of $\mathrm{Ind}_I^K \chi_0^s$ **other than** σ_0 will be in $\mathrm{soc}_K \pi(\bar{\rho})$, hence in $W(\bar{\rho})$, but it is not the case.
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- Compute ordinary part $\mathrm{Ord}_P(\pi(\bar{\rho}))$ (H., Breuil-Ding : semisimplicity)

Step 2. Show $\pi(\bar{\rho})^\vee$ is essentially self-dual, w.r.t. $\mathrm{Ext}^{2f}_{\Lambda}(-,\Lambda)$, where $\Lambda := \mathbb{F}[\![K_1/Z_1]\!]$, i.e.

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Proof : M_∞ is flat over R_∞ which is a regular local ring. Get a Koszul complex resolution of $\pi(\bar{\rho})^\vee$. Also need self-duality of Emerton's completed cohomology. □

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  (**Rk** : the cosocle of a smooth admissible rep. could be 0.)
- A subspace $\tau \subset \pi(\bar{\rho})$ generates $\pi(\bar{\rho})$ as G-representation **iff**

  $$\tau \hookrightarrow \pi(\bar{\rho}) \twoheadrightarrow \pi_f$$

  is non-zero.

Step 2 (cont'd).

Criterion

Let $\tau \subset \pi(\bar{\rho})|_I$. If for some i, some character $\chi : I \to \mathbb{F}^\times$, the composition

$$\mathrm{Ext}^i_I(\chi, \tau) \xrightarrow{\beta_i} \mathrm{Ext}^i_I(\chi, \pi(\bar{\rho})) \xrightarrow{\gamma_i} \mathrm{Ext}^i_I(\chi, \pi_f)$$

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We will find an explicit $\tau(\bar{\rho})$ which is multiplicity free (as I-rep.) satisfying Criterion, so that $\pi(\bar{\rho}) = \langle G.\tau(\bar{\rho})\rangle$.

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I-rep. vs K-rep. : difficult to write down a minimal injective resolution of $D_0(\bar{\rho})$.
For I-rep., can first construct a resolution for the *graded* module, then lift.

Step 3.

Lemma

Take $i = 2f$, $\chi = \chi_0^s$ where $\chi_0 := \sigma_0^{I_1}$, then γ_{2f} is isomorphism.

Proof • Construct isomorphisms

$$\mathrm{Ext}_I^{2f}(\chi_0^s, \pi(\bar{\rho})) \overset{(1)}{\xrightarrow{\sim}} \mathrm{Ext}_K^{2f}(\mathrm{Ind}_I^K \chi_0^s, \pi(\bar{\rho})) \overset{(2)}{\twoheadrightarrow} \mathrm{Ext}_K^{2f}(\sigma_0, \pi(\bar{\rho})) \overset{(3)}{\twoheadrightarrow} \mathrm{Ext}_G^{2f+1}(\pi_0, \pi(\bar{\rho}))$$

where (1) induced by Frobenius; (2) by $\sigma_0 \hookrightarrow \mathrm{Ind}_I^K \chi_0^s$; (3) by (Barthel-Livné)

$$0 \to \text{c-Ind}_{KZ}^G \sigma_0 \overset{T-\lambda_0}{\to} \text{c-Ind}_{KZ}^G \sigma_0 \to \pi_0 \to 0.$$

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- $R^f\mathrm{Ord}_P\pi(\bar\rho) \to R^f\mathrm{Ord}_P\pi_f$ is surjective, as $R^{f+1}\mathrm{Ord}_P = 0$. Again an isom.

□

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Consider the following situation :

- $(R, \mathfrak{m})$ = noetherian local ring, $\underline{x} := (x_1, \ldots, x_n)$ with $x_i \in \mathfrak{m}$
- $K_\bullet(\underline{x}, R)$ = Koszul complex : $\cdots \to K_2 \overset{d_2}{\to} K_1 \overset{d_1}{\to} K_0 \to 0$, with $K_i \cong R^{\binom{n}{i}}$
- $\beta_\bullet : K_\bullet(\underline{x}, R) \to F_\bullet$, with $F_\bullet$ = complex of free R-modules.

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Lemma (Serre, 1956)

Assume

(a) $x_1, \ldots, x_n$ are linearly independent mod $\mathfrak{m}^2$;

(b) $\beta_0 : K_0 \to F_0$ is a direct summand.

Then $\beta_i : K_i \to F_i$ is a direct summand for all $0 \le i \le n$.

Recall M_∞ is flat over R_∞ (regular). Get Koszul resolution $P_\bullet$ of $\pi(\bar{\rho})^\vee$. Also can construct a minimal projective resolution $Q_\bullet$ of $\tau(\bar{\rho})^\vee$. The inclusion $\tau(\bar{\rho}) \hookrightarrow \pi(\bar{\rho})$ induces a projection $\epsilon : \pi(\bar{\rho})^\vee \twoheadrightarrow \tau(\bar{\rho})^\vee$ and

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To apply Lemma of Serre, find some finite dimensional I-representation λ such that

$$R := \mathrm{End}_I(\lambda) \cong \mathbb{F}[\![x_1, \ldots, x_{2f}]\!]/(x_i, 1 \leq i \leq 2f)^2.$$

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- (a) holds because $\pi(\bar{\rho})[\mathfrak{m}_{I_1}^3]$ is multiplicity free : "full tangent space"
- (b) holds by construction of $\tau(\bar{\rho})$ (i.e. $\tau(\bar{\rho})$ can not be too *small*).

Proof of Theorem B ($f = 2$!)

Already know : $\pi(\bar{\rho}) = \pi_0$ — V — π_2. Need to show V is irreducible.

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$\Longrightarrow$ obtain (π_0 — π_1) $\hookrightarrow \pi(\bar{\rho})$.

Step 2. Denote $Q := \pi(\bar{\rho})/(\pi_0 — \pi_1)$. The explicit structure of $D_0(\bar{\rho})$ shows

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By Theorem **A**, Q can be generated by σ_2, i.e.

$$\text{c-Ind}_{KZ}^{G}\,\sigma_2 \twoheadrightarrow Q.$$

Step 3.

Lemma

Let Q be a quotient of $\text{c-Ind}_{KZ}^{G}\,\sigma_2$. Assume the G-cosocle of Q is irreducible and isomorphic to $\text{c-Ind}_{KZ}^{G}\,\sigma_2/(T-\lambda)$ for some $\lambda \in \mathbb{F}^{\times}$. Then

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To finish the proof, we need to show $n = 1$. If not, the self-duality of $\pi(\bar{\rho})$ would imply $\pi_0 — \cdots — \pi_0$ (n copies) embeds in $\pi(\bar{\rho})$, a contradiction.

Thank you !