On the mod p cohomology for GL_2 (I)

(joint with Yongquan Hu)

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Plan of the talk

Notations

Let p be a prime number.

$F/\mathbb{Q}$ a totally real extension in which p is unramified.

D a quaternion algebra over F which is split at all places above p and at exactly one infinite place.

$\mathbb{F}$ a sufficiently large finite extension of $\mathbb{F}_p$.

$\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ a continuous absolutely irreducible totally odd modular representation.

Notations

$K \subset (D \otimes_F \mathbb{A}_{F,f})^\times$ open compact subgroup.

$X_K/F =$ the associated Shimura curve of level K. It is a smooth projective algebraic curve over F.

$H^1_{et}(X_K \times_F \overline{F}, \mathbb{F})$ mod p étale cohomology of X_K.

$$\pi^D(\overline{r}) = \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, \varinjlim_K H^1_{et}(X_K \times_F \overline{F}, \mathbb{F}(1))\right).$$

$\pi^D(\overline{r})$ is a smooth representation of $(D \otimes_F \mathbb{A}_{F,f})^\times$ over $\mathbb{F}$.

The local factor at v

Conjecture (Buzzard-Diamond-Jarvis)

There is a decomposition "à la Flath"

$$\pi^D(\overline{r}) = \otimes'_w \pi^D_w(\overline{r}),$$

where the restricted tensor product is taken over all finite places of F, $\pi^D_w(\overline{r})$ is a smooth admissible $(D \otimes_F F_w)^\times$-representation over $\mathbb{F}$.

We fix a place $v|p$.

Goal

Understand the smooth admissible $\mathrm{GL}_2(F_v)$-representation $\pi^D_v(\overline{r})$.

The local factor at v

Expectation

We expect $\pi_v^D(\overline{r})$ should only depend on $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ (shouldn't depend on $F, D, \overline{r}$ etc.)

Breuil-Diamond, Emerton-Gee-Savitt defined the "local factor" $\pi_v^D(\overline{r})$ at v in an "ad hoc" way under the following assumptions on $\overline{r}$.

Assumptions on $\overline{r}$

- $p \geq 7$, $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible;
- $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is non-scalar if D ramifies at w;
- $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ satisfies some weak generic condition for $w|p$.

Gelfand-Kirillov dimension

Let $f := [F_v : \mathbb{Q}_p]$. $K := \mathrm{GL}_2(\mathcal{O}_{F_v})$, $K_1 := \ker(\mathrm{GL}_2(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_{p^f}))$.

Definition/Theorem

Let π be a smooth admissible representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$. There is a unique integer $0 \leq \mathrm{GK}(\pi) \leq \dim_{\mathbb{Z}_p} K_1$ such that there are real numbers $a \leq b$ with

$$a \leq \frac{\dim_{\mathbb{F}} \pi^{K_1^{p^n}}}{p^{n\mathrm{GK}(\pi)}} \leq b, \quad \forall n \geq 1.$$

Example

$f = 1$, GK(Principal Series)= GK(Supersingular) $= 1$ (Paškūnas, Morra)

General f, GK(Principal Series) $= f$.

Some known results

We write $\overline{\rho} := \overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ and $\pi(\overline{\rho}) := \pi_v^D(\overline{r})$. $\pi(\overline{\rho})$ is of central character $\omega^{-1}\det(\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)})$.

Theorem (Breuil, Colmez, Emerton, Kisin, Paškūnas...)

Assume $F = \mathbb{Q}$ and $D = \mathrm{GL}_2$, then $\pi(\overline{\rho})$ is explicitly known. In particular,

(1) $\mathrm{GK}(\pi(\overline{\rho})) = 1$.

(2) $\pi(\overline{\rho})$ depends only on $\overline{\rho}$.

(3) $\pi(\overline{\rho})$ is of finite length.

Some known results

Recall $F_v/\mathbb{Q}_p$ unramified of degree f.

Let $W(\overline{\rho})$ be the set of Serre weights of $\overline{\rho}$.

The weight part of Serre's conjecture:

Theorem. (Gee, Gee-Kisin, Gee-Liu-Savitt, Emerton-Gee-Savitt)

$\mathrm{soc}_K \pi(\overline{\rho})$ depends only on $\overline{\rho}$. In fact,

$$\mathrm{soc}_K\left(\pi(\overline{\rho})\right) = \oplus_{\sigma \in W(\overline{\rho})} \sigma.$$

Some known results

$I := \{g \in K,\ g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p\} =$ Iwahori subgroup

$I_1 := \{g \in K,\ g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod p\} =$ pro-p-Iwahori subgroup

$Z =$ center of $\mathrm{GL}_2(F_v)$, $Z_1 = Z \cap K_1$. Z_1 acts trivially on $\pi(\overline{\rho})$.

$\mathfrak{m}_{I_1} =$ the maximal ideal of Iwasawa algebra $\mathbb{F}[[I_1/Z_1]]$.

Theorem (Emerton-Gee-Savitt)

The I/I_1-representation $\pi(\overline{\rho})^{I_1} = \pi(\overline{\rho})[\mathfrak{m}_{I_1}]$ is explicitly known and depends only on $\overline{\rho}$.

Definition of $D_0(\overline{\rho})$

Let $\Gamma = K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_{p^f})$.

Definition/Theorem (Breuil-Paškūnas)

There exists a unique finite dimensional representation $D_0(\overline{\rho})$ of Γ over $\mathbb{F}$ such that

(1) $\mathrm{soc}_\Gamma D_0(\overline{\rho}) \cong \oplus_{\sigma \in W(\overline{\rho})} \sigma$

(2) each $\sigma \in W(\overline{\rho})$ only occurs once as a Jordan-Hölder factor of $D_0(\overline{\rho})$

(3) $D_0(\overline{\rho})$ is maximal for the properties (1) and (2).

Moreover, $D_0(\overline{\rho})$ is explicit and is multiplicity free.

Example of $D_0(\overline{\rho})$

Example ($f = 1$)

Let $\overline{\rho}$ be reducible and $\overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$, $* \neq 0$. Then

$$D_0(\overline{\rho}) = \mathrm{Sym}^r\,\mathbb{F}^2 - \left(\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det{}^r \oplus \mathrm{Sym}^{p-3-r}\,\mathbb{F}^2 \otimes \det{}^{r+1}\right)$$

Some known results

$\mathfrak{m}_{K_1}$ = the maximal ideal of Iwasawa algebra $\mathbb{F}[[K_1/Z_1]]$.

Theorem (Hu-W., Le-Morra-Schraen, Le)

The finite dimensional Γ-representation

$$\pi(\overline{\rho})^{K_1} = \pi(\overline{\rho})[\mathfrak{m}_{K_1}]$$

depends only on $\overline{\rho}$. In fact,

$$\pi(\overline{\rho})[\mathfrak{m}_{K_1}] = D_0(\overline{\rho}).$$

Hence it is multiplicity free.

Main results: assumptions on $\overline{\rho}$

Let I_{F_v} be the inertia subgroup of $\mathrm{Gal}(\overline{F}_v/F_v)$.

ω_f is Serre's fundamental character of I_{F_v} of level f.

Definition

If $\overline{\rho}$ is reducible, say $\overline{\rho}$ is *strongly generic* if

$$\overline{\rho}|_{I_{F_v}} \sim \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & 1 \end{pmatrix} \otimes \eta,$$

with $2 \leq r_i \leq p-5$ for each i.

From now on we assume $\overline{\rho}$ is **reducible non-split strongly generic**.

Definition of $\widetilde{D}_0(\overline{\rho})$

Let $\widetilde{\Gamma} = \mathbb{F}[[K]]/\mathfrak{m}_{K_1}^2$.

Definition/Theorem (Hu-W. 2020)

There exists a unique finite dimensional representation $\widetilde{D}_0(\overline{\rho})$ of $\widetilde{\Gamma}$ over $\mathbb{F}$ such that

(1) $\mathrm{soc}_{\widetilde{\Gamma}} \widetilde{D}_0(\overline{\rho}) = \mathrm{soc}_{\Gamma} D_0(\overline{\rho}) = \oplus_{\sigma \in W(\overline{\rho})} \sigma$

(2) each $\sigma \in W(\overline{\rho})$ only occurs once as a Jordan-Hölder factor of $\widetilde{D}_0(\overline{\rho})$

(3) $\widetilde{D}_0(\overline{\rho})$ is maximal for the properties (1) and (2).

Moreover, $\widetilde{D}_0(\overline{\rho})$ is multiplicity free.

Example of $\widetilde{D}_0(\overline{\rho})$

Remark

When $\overline{\rho}$ is semi-simple, similar result is also obtained by Breuil-Herzig-Hu-Morra-Schraen 2020.

Example. ($f = 1$)

Let $\overline{\rho}$ be reducible and $\overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$, $* \neq 0$. Then

$$\widetilde{D}_0(\overline{\rho}) = \mathrm{Sym}^r - (\mathrm{Sym}^{p-1-r} \otimes \det{}^r \oplus \mathrm{Sym}^{p-3-r} \otimes \det{}^{r+1})$$
$$- (\mathrm{Sym}^{r-2} \otimes \det \oplus \mathrm{Sym}^{r+2} \otimes \det{}^{-1}) - (\mathrm{Sym}^{p+1-r} \otimes \det{}^{r-1} \oplus \mathrm{Sym}^{p-5-r} \otimes \det{}^{r+2})$$

Main results

Maintain the assumptions we have made on $\overline{r}$ and $\overline{\rho}$. Assume further

- the framed deformation ring of $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ over the Witt vectors $W(\mathbb{F})$ is formally smooth for $w \nmid p$ such that either D or $\overline{r}$ ramifies.

Theorem (Hu-W.2020)

(i) $\mathrm{GK}(\pi(\overline{\rho})) = f$.

(ii) The finite dimensional $\widetilde{\Gamma}$-representation $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ depends only on $\overline{\rho}$. In fact,

$$\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2] = \widetilde{D}_0(\overline{\rho}).$$

Hence it is multiplicity free.

Remark.

(1) Breuil-Herzig-Hu-Morra-Schraen (2020) proved similar results for $\overline{\rho}$ semi-simple, even for "non-minimal case".

(2) Gee-Newton proved $\mathrm{GK}(\pi(\overline{\rho})) \geq f$ (without the assumptions on $\overline{\rho}$).

(3) To prove $\mathrm{GK}(\pi(\overline{\rho})) \leq f$, we apply the "Control Theorem" obtained by Breuil-Herzig-Hu-Morra-Schraen (2020)

(4) BHHMS need to compute the potentially crystalline deformation ring of $\overline{\rho}$ of **Hodge-Tate weights** $(2, -1)$, while we need the results of Le on the potentially crystalline deformation ring of $\overline{\rho}$ of **Hodge-Tate weights** $(1, 0)$.

Sketch of proof: the first reduction

Theorem A

For any $\sigma \in W(\overline{\rho})$, σ occurs only once as a Jordan-Hölder factor of $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$.

Theorem A is equivalent to:

Theorem B

For any $\sigma \in W(\overline{\rho})$,

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma), \pi(\overline{\rho})) = 1,$$

where $\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma)$ is a projective envelope of σ in the category of $\widetilde{\Gamma}$-modules.

Thm. A $\Rightarrow$ (ii):

We assume Thm. A.

By the definition of $\widetilde{D}_0(\overline{\rho})$ (it is the largest representation with the properties (1) and (2))

$$\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2] \subseteq \widetilde{D}_0(\overline{\rho}).$$

The projectivity of $M_\infty|_K$ and the multiplicity freeness of $\widetilde{D}_0(\overline{\rho})$ gives

$$\widetilde{D}_0(\overline{\rho}) \subseteq \pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$$

Thm. A $\Rightarrow$ (i):

Thanks to Gee-Newton, it suffices to prove $\mathrm{GK}(\pi(\overline{\rho})) \leq f$.

Breuil-Herzig-Hu-Morra-Schraen: Pass to the Iwahori subgroup representation.

Proposition A

If $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ is multiplicity free, then $\pi(\overline{\rho})[\mathfrak{m}_{I_1}^3]$ is multiplicity free (as I-representation).

The Control Theorem of [BHHMS]

Control Theorem (Breuil-Herzig-Hu-Morra-Schraen, 2020)

Let π be a smooth admissible representation of I/Z_1. If $\pi[\mathfrak{m}_{I_1}^3]$ is multiplicity free, then $\mathrm{GK}(\pi) \leq f$.

Thm. A + Prop. A + Control Thm. $\Rightarrow$ (i).

A word on the proof of the Control Theorem: (see Prop. 6 of Prof. Breuil's talk in this conference)

The condition "$\pi[\mathfrak{m}_{I_1}^3]$ is multiplicity free" implies that $\mathrm{gr}_{\mathfrak{m}_{I_1}} \pi^\vee$ is a finitely generated module over $\mathbb{F}[(X_i, Y_i)_{1\leq i\leq f}]/(X_i Y_i)$.

Thm. A $\Leftrightarrow$ Thm. B $\Leftrightarrow$ Thm. C

Recall: to prove Thm. A, it is equivalent to prove Thm. B: For any $\sigma \in W(\overline{\rho})$,

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma), \pi(\overline{\rho})) = 1.$$

When $\overline{\rho}$ is **indecomposable**, the Diamond diagrams associated to $\overline{\rho}$ are indecomposable, then Thm. B is equivalent to

Theorem C

There exists $\sigma \in W(\overline{\rho})$ such that

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma), \pi(\overline{\rho})) = 1.$$

The patching module M_∞

Using Taylor-Wiles-Kisin patching method,
Caraiani-Emerton-Gee-Geraghty-Paškūnas-Shin constructed:

$$\begin{aligned} R_\infty &= \text{ a power series ring over a tensor product of local deformation rings} \\ &= \text{ a power series ring over } W(\mathbb{F}) \text{ (under the assumptions on } \overline{r}). \end{aligned}$$

$\mathfrak{m}_\infty =$ the maximal ideal of R_∞.

$M_\infty =$ the patched module, a finitely generated $R_\infty[\mathrm{GL}_2(F_v)]$-module, $M_\infty|_K$ is projective. We use the "minimal" version of M_∞ (Dotto-Le).

$\pi(\overline{\rho})^\vee = M_\infty/\mathfrak{m}_\infty$.

Thm. C $\Leftrightarrow$ Thm. D

By the projectivity of $M_\infty|_K$, M_∞ defines an exact functor on K-representations over $\mathbb{F}$:

$$V \mapsto M_\infty(V) := \mathrm{Hom}_K(M_\infty, V^\vee)^\vee.$$

We have

$$\mathrm{Hom}_{\mathbb{F}}(M_\infty(V)/\mathfrak{m}_\infty, \mathbb{F}) \cong \mathrm{Hom}_K(V, \pi(\overline{\rho})).$$

Hence Thm. C is equivalent to

Theorem D

There exists $\sigma \in W(\overline{\rho})$ such that $M_\infty(\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma))$ is a cyclic R_∞-module, (equivalently a quotient of R_∞).

Proof of Thm. D

We prove Thm. D for $\sigma_0 = (r_0, r_1, \ldots, r_{f-1})$ the "ordinary Serre weight".

Proposition

There is a quotient V of $\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma_0)$ such that $M_\infty(\mathrm{Proj}_{\widetilde{\Gamma}}(\sigma_0))$ is R_∞-cyclic if and only if $M_\infty(V)$ is R_∞-cyclic.

There is an exact sequence

$$0 \to M_\infty(V) \to M_\infty(V^{\mathrm{ord}}) \oplus M_\infty(\mathrm{Proj}_\Gamma(\sigma_0)) \to M_\infty(\sigma_0) \to 0,$$

where V^{ord} is a K-representation which captures the information of Emerton's ordinary part of a smooth $\mathrm{GL}_2(F_v)$-representation over $\mathbb{F}$.

Proof of Thm. D

Let $\overline{R}_\infty = R_\infty \otimes_{W(\mathbb{F})} \mathbb{F}$.

• Gee+Emerton-Gee-Savitt: $M_\infty(\sigma_0)$ is a cyclic $\overline{R}_\infty$-module; $I^{\sigma_0} := \mathrm{Ann}_{\overline{R}_\infty}(M_\infty(\sigma_0))$ = the reduction mod p of the ideal of R_∞ giving the crystalline deformation ring associated to σ_0 at v.

• Hu, Breuil-Ding: $M_\infty(V^{\mathrm{ord}})$ is a cyclic $\overline{R}_\infty$-module. Let $I^{\mathrm{ord}} := \mathrm{Ann}_{\overline{R}_\infty}(M_\infty(V^{\mathrm{ord}}))$. $I^{\mathrm{ord}} \supseteq I^{\mathrm{red}}$ = the ideal of $\overline{R}_\infty$ which at v gives the reducible deformations of $\overline{\rho}$.

• Le: $M_\infty(\mathrm{Proj}_\Gamma(\sigma_0))$ is a cyclic $\overline{R}_\infty$-module. $I^{\mathrm{tame},\sigma_0} := \mathrm{Ann}_{\overline{R}_\infty}(M_\infty(\mathrm{Proj}_\Gamma(\sigma_0))$ = the reduction mod p of the ideal of R_∞ which at v gives the multi-tame type deformations of $\overline{\rho}$.

Proof of Thm. D

Hence we have an exact sequence

$$0 \to M_\infty(V) \to \overline{R}_\infty/I^{\mathrm{ord}} \oplus \overline{R}_\infty/I^{\mathrm{tame},\sigma_0} \to \overline{R}_\infty/I^{\sigma_0} \to 0.$$

Proposition

$I^{\mathrm{ord}} + I^{\mathrm{tame},\sigma_0} = I^{\sigma_0}$.

Theorem D then follows from the following easy lemma:

Lemma

Let $(R, \mathfrak{m})$ be a noetherian local ring, $I_1, I_2 \subseteq I_0 \subseteq \mathfrak{m}$. Then

$$\ker(R/I_1 \oplus R/I_2 \twoheadrightarrow R/I_0) \text{ is cyclic over } R \text{ iff } I_0 = I_1 + I_2$$

Consequence

Theorem (Gee-Newton)

Since R_∞ is a power series ring over $W(\mathbb{F})$,

$$\mathrm{GK}(\pi(\overline{\rho})) = f \text{ is equivalent to } M_\infty \text{ is flat over } R_\infty.$$

Corollary

Let $x : R_\infty \to \mathcal{O}'$ be a local morphism of $W(\mathbb{F})$-algebras. Set

$$\Pi(x)^0 := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}'}(M_\infty \otimes_{R_\infty, x} \mathcal{O}', \mathcal{O}')$$

and $\Pi(x) := \Pi(x)^0 \otimes_{\mathcal{O}'} E'$. Then $\Pi(x)$ is a **nonzero** admissible unitary Banach representation of $\mathrm{GL}_2(F_v)$ over E' with $\mathrm{GL}_2(F_v)$-invariant unit ball $\Pi(x)^0$ which lifts $\pi(\overline{\rho}) \otimes_{\mathbb{F}} \mathbb{F}'$.