Bibliography:

Breuil, Paskunas, "Toward a modulo p Langlands correspondance for GL_2"

Paskunas, "Coefficient systems and supersingular representations of $GL_2(F)$"

Thm (Emerton)

$\bar{e} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{F})$ modular

$\bar{e}_p = \bar{e}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ absolutely irreducible

Assume $p > 2$ and $\bar{e} \not\cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ up to twist

Then $\widetilde{H}^1(N, \mathbb{F})[\bar{e}] \simeq \left(\pi(\bar{e}_p)^{\oplus r}\right) \otimes (\omega \circ \det)$

$$\Rightarrow V(\widetilde{H}^1(N,\mathbb{F})[\bar{e}]) \simeq \left(\bar{e}_p \otimes (\omega \circ \det)\right)^{\oplus r}$$

$\boxed{r \geq 1}$

depends on N and $\bar{e}_\ell$: $\ell \neq p$.

Link with Serre's conjecture

$$\sigma \in \mathrm{soc}_{GL_2(\mathbb{Z}_p)}\left(\pi(\bar{\rho}_p) \otimes \omega \circ \det\right) \iff \mathrm{Hom}_K\left(\sigma, \tilde{H}^1(N, \mathbb{F})[\bar{\rho}]\right) \neq 0$$

$$\mathrm{Hom}_K\left(\sigma, \tilde{H}^1(N, \mathbb{F})[\bar{\rho}]\right) \cong H^1(X(N), \sigma^\vee)[\bar{\rho}]$$

$$\iff \sigma \in W(\bar{\rho}_p)$$

$$JH\left(\mathrm{soc}_{GL_2(\mathbb{Z}_p)} \pi(\bar{\rho}_p) \otimes (\omega \circ \det)\right) = W(\bar{\rho}_p)$$

<u>Dimension</u> $\left(\mathrm{Ind}_B^G \chi\right)$ has dimension 1

π supersingular has dimension 1 (Morra).

$\Rightarrow$ $\pi(\bar{e}_p)$ has dimension 1

$\Rightarrow$ $\tilde{H}^1(N, \mathbb{F})[\bar{e}]$ ————

. . .

$\Rightarrow$ $\tilde{H}^1(N, W(\mathbb{F}))^\vee_{\bar{e}}$ is a <u>flat</u> $R_s(\bar{e})$-module

3. Representations of $GL_2(L)$, $L/\mathbb{Q}_p$ unramified

$$G = GL_2(L) \quad K := GL_2(\mathcal{O}_L) \quad \mathbb{F}_q = \mathcal{O}_L / p\mathcal{O}_L \subset \overline{\mathbb{F}}$$

$$K_1 = \mathrm{Ker}(K \to GL_2(\mathbb{F}_q))$$

Irreducible smooth representations of G over $\overline{\mathbb{F}}$

* Principal series : $JH(\mathrm{Ind}_B^G \chi)$
  Same classification than for $GL_2(\mathbb{Q}_p)$.
* Supersingular representations
  not classified for $L \neq \mathbb{Q}_p$.

Example of supersingular representation (Paškūnas)

σ weight : $\sigma \in \mathrm{Irr}_{\mathbb{F}}(K) = \mathrm{Irr}_{\mathbb{F}}(GL_2(\mathbb{F}_q))$

$\exists$ supersingular representation π st $\mathrm{soc}_K(\pi) = \sigma \oplus \sigma^s$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s : \quad \bigotimes \left(\mathrm{Sym}^{r_i}(\mathbb{F}^2)\right)^{(i)} \longleftrightarrow \bigotimes \left(\mathrm{Sym}^{p-1-r_i}(\mathbb{F}^2) \otimes \det^{r_i}\right)^{(i)}$

Problem: $JH(\mathrm{soc}_K(\pi)) \neq W(\bar{\rho})$ for $\bar{\rho}$ irreducible
$L \neq \mathbb{Q}_p$

$[L : \mathbb{Q}_p] = 2 \qquad \bar{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \longrightarrow GL_2(\mathbb{F})$

irreducible (generic)

then $\sigma \in W(\bar{\rho}) \Longrightarrow \sigma^s \notin W(\bar{\rho})$.

Reducible semisimple case $([L:\mathbb{Q}_p]=2)$

$$\bar{\rho} = \chi_1 \oplus \chi_2 \quad (\text{generic})$$

$$W(\bar{\rho}) = JH\left(\mathrm{soc}_K \operatorname{Ind}_B^G(\chi_1 \omega \otimes \chi_2)\right)$$
$$\sqcup\ JH\left(\mathrm{soc}_K \operatorname{Ind}_B^G(\chi_2 \omega \otimes \chi_1)\right)$$
$$\sqcup\ JH\left(\mathrm{soc}_K \pi\right)$$

with π supersingular (from Paškūnas).

Expectation: $([L:\mathbb{Q}_p]=2)$

$$\pi(\chi_1 \oplus \chi_2) \longleftrightarrow \operatorname{Ind}_B^G(\chi_1 \omega \otimes \chi_2) \oplus SS \oplus \operatorname{Ind}_B^G(\chi_2 \omega \otimes \chi_1)$$

$$\cap$$

$$\mathrm{Rep}_{\mathbb{F}}\, GL_2(L)$$

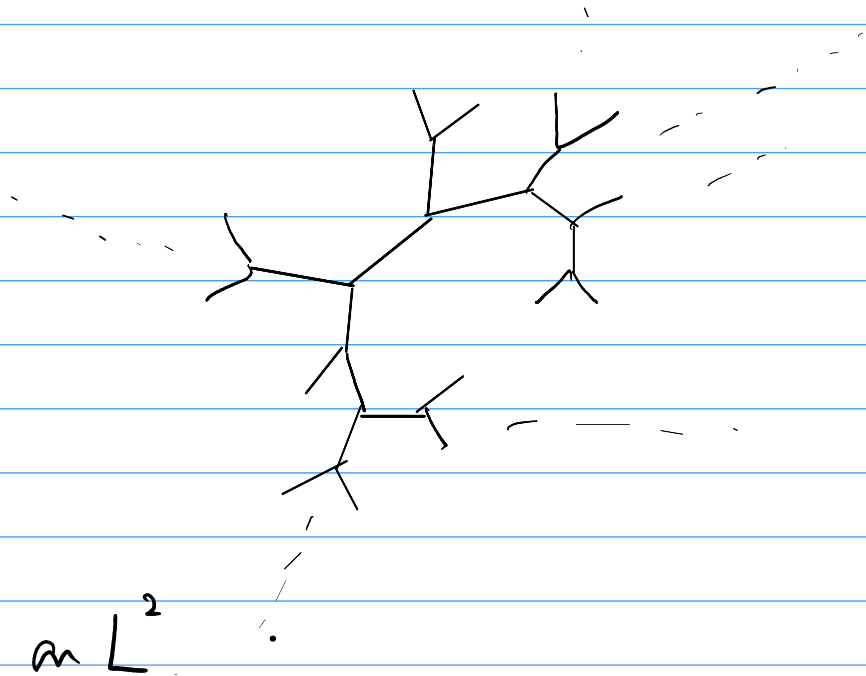
Construction of supersingular representations with prescribed socle. (Breuil - Paškūnas)

Bruhat - Tits tree $\mathcal{T}$

Vertices: homothety classes of lattices $R \subset L^2$

Edges: $\{[R_0],[R_1]\}$ st $pR_0 \subsetneq R_1 \subsetneq R_0$

$$[R_0 : R_1] = q$$



G acts on $\mathcal{T}$ by its action on L^2.

Coefficient system on $\mathcal{T}$ $\quad (V_\sigma)$

σ vertice or edge $\rightsquigarrow V_\sigma$ $\quad \dim_F V_\sigma < \infty$.

$r_{\sigma,\sigma'} : V_\sigma \longrightarrow V_{\sigma'}$ if $\sigma \supset \sigma'$

G - equivariant coefficient system $\mathcal{V} = (V_\sigma)$.

$V_\sigma \xrightarrow[c_{g,\sigma}]{} V_{g\sigma}$ satisfying cocycle relation

$g \in G$ $\qquad c_{gh,\sigma} = c_{g,h\sigma} \circ c_{h,\sigma}, \quad c_{1,\sigma} = Id_{V_\sigma}$

+ obvious commutation conditions

$$\rightsquigarrow H_i(\mathcal{V}) = H_i(C_\bullet^{or}(\mathcal{T},\mathcal{V}))$$

smooth representation of G.

$$C_\bullet^{or}(\mathcal{T},\mathcal{V}) = [\, C_1 \longrightarrow C_0 \,]$$

Fix $v_0 \in e_0$

G acts transitively on vertices and edges.

G - equivariant coeff system $\longleftrightarrow$ (D_0, D_1, r)

$V_{v_0} = D_0 \in \mathrm{Rep}_{\mathbb{F}}(\mathrm{Stab}(v_0))$

$V_{e_0} = D_1 \in \mathrm{Rep}_{\mathbb{F}}(\mathrm{Stab}(e_0))$

$r : D_1 \to D_0$

$\mathrm{Stab}(v_0) \cap \mathrm{Stab}(e_0)$ - equivariant

$\boxed{v_0 = [\mathcal{O}_L^2]}$ $\boxed{e_0 = \{[\mathcal{O}_L \oplus p\mathcal{O}_L], [\mathcal{O}_L^2]\}}$

$\mathrm{Stab}(v_0) = KZ$

$Z \subset G$ center

$\mathrm{Stab}(e_0) = N =$ normalizer of $I = \begin{pmatrix} \mathcal{O}_L^\times & \mathcal{O}_L \\ p\mathcal{O}_L & \mathcal{O}_L^\times \end{pmatrix}$

$KZ \cap N = IZ$.

Diagram : (D_0, D_1, r) with

$D_0 \in \mathrm{Rep}_{\mathbb{F}}(KZ)$

$D_1 \in \mathrm{Rep}_{\mathbb{F}}(N)$

$r : D_1|_{IZ} \to D_0|_{IZ}$

$[N : IZ] = 2$

Theorem (Breuil-Paškūnas)

Let $D = (D_0, D_1, r)$ be a diagram such that

$D_1 \underset{r}{\xrightarrow{\sim}} D_0^{I_1}$ and $D_0^{K_1} = D_0$ ($I_1 \subset I$ pro-p-Sylow)

$\Rightarrow \exists\, \pi$ smooth representation of G such that

(i) $D_0 \hookrightarrow \pi|_{KZ}$

$D_0 \in \mathrm{Rep}(GL_2(\mathbb{F}_q))$

(ii) D_0 generates π as G-rep.

(iii) $\boxed{\mathrm{soc}_K(\pi) = \mathrm{soc}_K(D_0).}$

(iv)
$$\begin{tikzcd} D_1 \arrow[r, hook, "r"] \arrow[dr, "N\text{-equivariant}"'] & D_0 \arrow[d, hook] \\ & \pi \end{tikzcd} \qquad \pi \circlearrowleft G \supset N$$

$H_1(\mathcal{V}_D) = 0.$

Construction of π $\quad \mathcal{V}_D$ corresponding coefficient system

$H_0(\mathcal{V}_D)$ is a smooth representation of G.

$\mathrm{inj}_K(D_0)|_I \simeq \bigoplus_{\chi \subset D_0^{I_1} \subseteq D_1} \mathrm{inj}_I \chi \quad \circlearrowleft N$ many choices

$\mathrm{soc}_K(\mathrm{inj}_K D_0) = \mathrm{soc}_K(D_0) \qquad G = K *_I N.$

$$\begin{array}{ccc} & \pi & \\ \nearrow & & \\ H_0(\mathcal{V}_D) & \xrightarrow[G\text{-equivariant}]{} & \mathrm{inj}_K(D_0) \circlearrowleft G \end{array}$$

admissible

finitely presented not adm

admissible not fg over G.

Consequences Bad: can construct many irreducible representations of G not related to Galois (if $[L:\mathbb{Q}_p] > 1$)

Good: can construct smooth admissible finitely generated representations of G with socles in $W(\bar{\rho})$

Not so good: it can exist infinitely many representations satisfying the assumptions of the thm (Hu).

How to choose a diagram D associated to $\bar{e}$?

Look at the cohomology!

$\cancel{\left(\bigoplus_{\sigma \in W(\bar{e})} \sigma \right)^{I_1}}$ $\quad \sigma \mapsto \sigma^S$

F totally real st $F_p \simeq L$ (K^p tame level)

completed cohomology of a tower of Shimura curves$/F$

$\bar{r} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_2(\mathbb{F})$ modular

+ abs irreducible

Expectation: $\tilde{H}^1(K^p, \mathbb{F})[\bar{r}]$ depends only on $\bar{e} = \bar{r}|_{\mathrm{Gal}(\bar{L}/L)}$ and $\simeq \pi(\bar{e})^{\oplus r}$

$$Z_1 \cap Z \qquad \bigoplus_{\sigma \in W(\bar{e})} \sigma \underset{\text{socle}}{\subset} \tilde{H}^1(K^p, \mathbb{F})[\bar{r}] \qquad (Z_1 \text{ center of } K_1)$$

$$\bigoplus_{\sigma} \mathrm{inj}_{GL_2(\mathbb{F}_q)} \sigma \hookrightarrow \dashrightarrow \tilde{H}^1(K^p, \mathbb{F})^{Z_1}_{\bar{r}}$$

injective representation of K/Z_1

<u>Fact</u> $\mathrm{soc}_K \left(\tilde{H}^1(K^p, \mathbb{F})_{\bar{r}} \big/ \tilde{H}^1(K^p, \mathbb{F})[\bar{r}] \right)$

contains only weights of $W(\bar{e})$

Def: $D_0(\bar{e}) \subset \bigoplus_{\sigma \in W(\bar{e})} \mathrm{inj}_{GL_2(\mathbb{F}_q)}(\sigma)$ largest sub such that

(i) $\mathrm{soc}_K D_0(\bar{e}) = \bigoplus_{\sigma \in W(\bar{e})} \sigma$

(ii) $[D_0(\bar{e}) : \sigma] = 1$ if $\sigma \in W(\bar{e})$.

Thm (Breuil - Paškūnas) Assume $\bar{e} : \mathrm{Gal}(\bar{L}/L) \to GL_2(\mathbb{F})$ generic

$\to D_1(\bar{e}) = D_0(\bar{e})^{I_1} \simeq \bigoplus_{\chi \neq \chi^s} (\chi \oplus \chi^s)$ $\quad \chi : I \twoheadrightarrow I/I_1 \to \mathbb{F}^\times$

the action of I extends to N.

$\rightsquigarrow \boxed{D(\bar{e}) = (D_0(\bar{e}), D_1(\bar{e}), r)}$ Diamond diagram associated to $\bar{e}$.

$\bar{\rho} : \mathrm{Gal}(\bar{L}/L) \longrightarrow GL_2(\mathbb{F})$ generic

<u>Def</u> A smooth representation π of G is associated to $\bar{\rho}$ if, π has a central character,

$D_0(\bar{\rho}) \hookrightarrow \pi|_K$ and $D_0(\bar{\rho})$ generates π.

$$\mathrm{soc}_K(\pi) = \mathrm{soc}_K D_0(\bar{\rho}) = \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$$

<u>Questions</u>: * Is $\tilde{H}^1(K^p, \mathbb{F})[\bar{r}]$ associated to $\bar{r}|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)}$? $\quad \bar{r} : \mathrm{Gal}(\bar{F}/F) \to GL_2(\mathbb{F})$.

* Does it depend only on $\bar{r}|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)}$?

* What is its Gelfand-Kirillov dimension?

$[L : \mathbb{Q}_p]$ $\qquad GL_2(L)/B$ $\qquad [L : \mathbb{Q}_p]$

Known facts $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_2(\mathbb{F})$

such that $\bar{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_p))}$ abs irreducible (TW hyp)

and $\bar{\rho} = \bar{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)}$ is generic

Let $\pi(\bar{r}) = \widetilde{H}^1(K^p, \mathbb{F})[\bar{r}]$. $\circlearrowleft GL_2(L)$.

1) $\pi(\bar{r})^{I_1} = D_1(\bar{\rho})^{\oplus r}$ (Dembele, Emerton-Gee-Savitt)

$\Rightarrow \pi(\bar{r})$ contains a (Breuil) representation associated to $\bar{\rho}$

$GL_2(\mathbb{F}_q) \circlearrowleft$

2) $\pi(\bar{r})^{K_1} = D_0(\bar{\rho})^{\oplus r}$ (Hu-Wang, Le-Morra-S., Le)

$\Rightarrow \exists\, \pi \longleftrightarrow \bar{\rho}$ st $\underbrace{\pi^{K_1} = D_0(\bar{\rho})}$;

$K \curvearrowleft$ $\qquad\qquad\qquad\qquad \mathfrak{m}_{K_1} \subset \mathbb{F}[[K_1]]$.

3) $\tilde{H}^1(K^p, \mathbb{F})[\bar{r}][\mathfrak{m}_{K_1}^2]$ can be described similarly (Hu-Wang, Breuil-Herzig-Hu-Morra-S.)

(more restrictive genericity hypotheses)

4) the isomorphism class of

$(\pi(\bar{e})^{K_1}, \pi(\bar{e})^{I_1}, i)$ depends only

$D_0(\bar{e}) \quad\quad D_1(\bar{e})$

on $\bar{e}|_{\mathrm{Gal}(\bar{L}/L)}$ (local at p).

(Dotto-Le)

Idea of proof (Emerton - Gee - Savitt)

Reduce to the case where $r = 1$ (minimal case)

$\chi \subset \longrightarrow \tilde{H}^1(K^p, \mathbb{F})[\bar{r}]$ prove that it appears with multiplicity 1.

$$\mathrm{Hom}_I(\chi, \tilde{H}^1(K^p, \mathbb{F})[\bar{r}])$$
$$\simeq \mathrm{Hom}_K(\underbrace{\mathrm{Ind}_I^K(\chi)}, \tilde{H}^1(K^p, \mathbb{F}))[\bar{r}].$$

reduction modulo p of $\Theta = \mathrm{Ind}_I^K[\chi]$

$[\chi] : I \longrightarrow W(\mathbb{F})^\times$ lattice $\circlearrowleft K$

$$\mathrm{Hom}_K\left(\Theta, \tilde{H}^1(K^p, \mathbb{F})\right)[\bar{\pi}]^\vee$$

$$\simeq \quad \left(H^1(X(K^pK), \Theta^\vee)^{\psi}\right)^\vee \big/ \mathfrak{m}_{\bar{\pi}}$$

$\mathfrak{m}_{\bar{\pi}} \subset R_S(\bar{\pi})^{\psi}$ universal deformation ring of $\bar{\pi}$ with fixed determinant

$\Theta[\frac{1}{p}]$ is a smooth rep of K (in char 0)

$$R_S(\bar{\pi})^{\psi} \circlearrowright H^1(X(K^pK), \Theta^\vee)^{\psi}$$

$$\downarrow$$

$$R_S(\bar{e})^{pBT, 0, \psi} \circlearrowright$$

free of rk 1 after $[\frac{1}{p}]$.

pot BT condition at p.

Reduced to prove that

$$H^1(Sh(K^pK), \mathcal{O}^\vee)_{\bar{r}}^\vee$$

is a flat $R_S(\bar{r})^{pBT,\mathcal{O},\psi}$ - module
($\Leftrightarrow$ cyclic)

Use patching

$$M_\infty(\mathcal{O}) \circlearrowleft \underbrace{R_{\bar{e}}^{pBT,\mathcal{O}}}$$ - module

pot BT deformation ring of $\bar{e}$
with descent data given by $\mathcal{O}$.

These rings have been calculated
by Breuil - Mézard.

Use the functoriality of $\Theta \longmapsto M_\infty(\Theta)$

$+$ description of $\mathrm{Spec}(R_{\bar{e}}^{\rho BT, \mathcal{O}}/p)$

to reduce to the case where $R_{\bar{e}}^{\rho BT, \mathcal{O}}$ is

regular

$\rightarrow$ "miracle flatness"

π^{K_1} have to prove that

$\mathrm{Hom}_K(\mathrm{inj}_{\delta K}(\sigma), \tilde{H}^1)[\bar{e}]$ is 1-dim

$\hookrightarrow$ lift it in char 0 and

compute $R_{\bar{e}}^{\rho BT}$ with "multi" descent data

(Le). $\square$