Equivariant Eisenstein classes, critical values of Hecke L-functions and p-adic interpolation (joint with J. Sprang)

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§1 Main results

Let L be an algebraic number field, $\mathfrak{f} \subset \mathcal{O}_L$ integral ideal, $I(\mathfrak{f}) :=$ fractional ideals of L prime to $\mathfrak{f}$

$$\chi : I(\mathfrak{f}) \longrightarrow \overline{\mathbb{Q}}^{\times}$$

an algebraic Hecke character of conductor $\mathfrak{f}$. Choose $\overline{\mathbb{Q}} \subset \mathbb{C}$ and let

$$L(\chi, s) = \sum_{(\mathfrak{a}, \mathfrak{f}) = 1} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s}}$$

be its L-function.

Expectation: $\boxed{\text{critical values of } L(\chi,s) \text{ divided by suitable periods are algebraic numbers (Deligne conjecture) and can be } p\text{-adically interpolated.}}$

Remark:

- $L(\chi, r) = L(\chi \cdot N_{L/\mathbb{Q}}^{-r}, 0)$, $r \in \mathbb{Z}$.
- $L(\chi, 0)$ critical if Γ-factors in fct-eq are finite and $\neq 0$ at $s=0$.

Two cases

$L/\mathbb{Q}$ totally real $\chi = \rho \cdot N_{L/\mathbb{Q}}^{r}$ ρ Dirichlet character	$L \supset K$, K CM field + conditions on ∞-type of χ

Known results (for critical values)

L totally real

Klingen - Siegel : $L(\chi, 0)$ algebraic

Deligne - Ribet, Barsky, Cassou-Nogues : construct p-adic L-fct.

L = K CM field

Damerell $[K:\mathbb{Q}]=2$, Shimura : $\frac{L(\chi,0)}{\text{periods}}$ alg

Katz : constructs p-adic L-fct.

Blasius : Deligne conjecture is true

$L \supset K$ CM field

Colmez-Schneps : K imag. quadratic and certain L

p-adic interpolation in these cases

Harder : announced Deligne conjecture 1984

(no integrality or p-adic interpolation)

Theorem (K.-Sprang) If $L \supset K$ CM field, χ critical at 0 (of ∞-type $\beta - \alpha$ with respect to a CM type Σ of L) $\mathfrak{c} \subset \mathcal{O}_L$, $(\mathfrak{c}, \mathfrak{f}) = 1$ ($\mathfrak{f} \neq \mathcal{O}_L$ for simplicity), then

$$\frac{(\alpha - \underline{1})!\,(2\pi i)^{|\beta|}}{\Omega^{\alpha} \cdot \Omega^{\vee\,\beta}} \left(\chi(\mathfrak{c}) N\mathfrak{c} - 1\right) L(\chi, 0) \in \mathcal{O}_k\left[\frac{1}{d_L \mathfrak{f} \cdot N\mathfrak{c}}\right]$$

where: A abelian variety with CM by $\mathcal{O}_L$
$\omega(A)$, $\omega(A^\vee)$ bases of $\omega_{A/k}$, $\omega_{A^\vee/k}$, Ω, $\Omega^\vee$ assoc. periods
x $\mathfrak{f}$-torsion point on A
k field of definition of $(A/k, \omega(A), \omega(A^\vee), x)$

($\rightsquigarrow$ generalizes Katz/Shimura to arbitrary critical χ)

Remark similar results were announced by Bergeron-Charollois-Garcia (completely different method and relying on Katz's result)

Corollary (pointed out by Blasius):

$$\frac{L(\chi, 0)}{c^+(R_{L/\mathbb{Q}} M(\chi))} \in \overline{\mathbb{Q}}^\times$$

weak Deligne conjecture.

Period of the motive $M(\chi)$ of χ

if $F \subset K$ max. tot. real
all primes of F over p
split completely in K

Theorem (K.-Sprang) (p-adic interpolation)

Let p a prime number, $(p, d_L) = 1$, and assume that the CM-type Σ of L is ordinary at p.

Then there exists a measure μ_f on $Gal(L(p^\infty f)/L)$ such that:

$$\frac{1}{\Omega_p^\alpha \, \Omega_p^{\vee\beta}} \int_{Gal(L(p^\infty f)/L)} \chi(g) \, d\mu_f(g) = \frac{(\alpha-1)! \, (2\pi i)^{|\beta|}}{\Omega^\alpha \cdot \Omega^{\vee\beta}} \left(\begin{smallmatrix}\text{explicit} \\ \text{local factors}\end{smallmatrix}\right) L(\chi, 0)$$

for all χ of conductor dividing $p^\infty f$, critical at 0, ∞-type $\beta - \alpha$

($\rightsquigarrow$ generalizes Katz to arbitrary critical χ)

Why does the Shimura-Katz approach not generalize?

1) Shimura-Katz use algebraic Eisenstein series on Hilbert modular varieties. The Eisenstein series for $L \neq K$ live on $GL_n(\mathcal{O}_K)$ if $n = [L:K]$ whose locally symmetric space is not algebraic.
2) They use algebraicity properties of Maaß-Shimura operators, which are also not available.

Our approach:

- work with a single abelian variety $\mathcal{A}$ with CM by $\mathcal{O}_L$
- construct equivariant (for $\mathcal{O}_L^\times$-action) integral coherent cohomology classes which are related to Eisenstein-Kronecker series
- construction uses completed Poincaré bundle (inspired by work of Bannai-Kobayashi)
  $\rightsquigarrow$ integrality without q-expansion
- the formal cohomology class gives a p-adic measure $\rightsquigarrow$ p-adic L-fct.

inspired by $\boxed{\text{equivariant motivic Eisenstein classes}}$

Remark: Our approach is even new for $L=K$, no q-expansion necessary.

§2 Equivariant motivic Eisenstein classes

Situation:

$$\begin{array}{ccc} \mathcal{A} & \circlearrowleft \Gamma & \text{discrete group acting} \\ \pi \big\downarrow & & \text{abelian scheme rel. dim. } d \\ S & & \end{array}$$

more generally:

$$\begin{array}{cc} G & \\ \pi \big\downarrow & \text{commutative group scheme} \\ S & \end{array}$$

Examples:

- $\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}^n$ with $GL_n(\mathbb{Z})$ action
- $\mathcal{A}$ CM by $\mathcal{O}_L$, $\mathcal{A}$ has $\mathcal{O}_L^\times$ - action
- $\mathcal{A}$ CM by $\mathcal{O}_K$, $\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_K^{\oplus n}$ has $GL_n(\mathcal{O}_K)$ - action
- also $\mathbb{G}_m^n$ with $GL_n(\mathbb{Z})$ - action

Consider

$$\begin{tikzcd} \mathcal{A} \setminus \mathcal{A}[N] =: U_N \arrow[r, hook] & \mathcal{A} \arrow[d, "\pi"] \arrow[loop right, "\Gamma \text{ acts}"] & \mathcal{A}[N] \arrow[l, hook'] \\ & S & \end{tikzcd}$$

N- torsion points

$\boxed{\mathcal{H} := R^{2d-1}\pi_! \mathbb{Q}(d)}$ motivic sheaf on S

(formal linear combination of N- torsion points)

Theorem (K.) For $f : \mathcal{A}[N](S) \longrightarrow \mathbb{Q}$ exists equivariant motivic Eisenstein class

$$\mathrm{Eis}^b_{mot, \Gamma}(f) \in H^{2d-1}_{mot}(U_N, \Gamma; \mathcal{L}og^b(d))$$

(equivariant motivic cohom.)

where: $U_N := \mathcal{A} \setminus \mathcal{A}[N]$

$\mathcal{L}og^b$ filtered unipotent motivic sheaf with assoc. graded

$$\mathrm{gr}\, \mathcal{L}og^b \cong \bigoplus_{i=0}^{b} \mathrm{Sym}^i \mathcal{H}$$

Application in our situation:

$L \supset K$, K CM field, $n = [L:K]$

$$\begin{array}{ccc} \mathcal{A} & & \\ \downarrow & \text{CM by } \mathcal{O}_L & \rightsquigarrow \\ \operatorname{Spec} k & & \end{array} \qquad \begin{array}{c} \mathcal{A} \\ \\ \text{isogeneous to} \end{array} \qquad \begin{array}{c} \mathcal{B}^n \\ \downarrow \\ \operatorname{Spec} k \end{array} \qquad \begin{array}{l} \mathcal{B} \text{ CM by } \mathcal{O}_K \\ \Gamma = GL_n(\mathcal{O}_K) \\ \text{acts} \end{array}$$

Consider de Rham realization

$$H^{2d-1}_{dR}(U_N, \Gamma, (\mathcal{L}og^b, \nabla))$$

$$(\mathcal{L}og, \nabla): \quad \mathcal{L}og^b \otimes \mathcal{O}_{\mathcal{A}} \xrightarrow{\nabla} \mathcal{L}og^b \otimes \Omega^1_{\mathcal{A}} \to \cdots \to \mathcal{L}og^b \otimes \Omega^d_{\mathcal{A}}$$

$\rightsquigarrow$ „Hodge filtration"

Fact

$$\mathcal{E}is^b_{dR,\Gamma}(f) \in H^{d-1}_{coh}(U_N, \Gamma, \mathcal{L}og^b \otimes \Omega^d_{\mathcal{A}}) \subset H^{2d-1}_{dR}(U_N, \Gamma, (\mathcal{L}og^b, \nabla))$$

(follows also from „integral refinement" later)

To produce more classes use ∇ differentiation in fibre direction as substitute for Maaß-Shimura operators.

Using ∇ and a torsion section $t \in \mathcal{U}_N$

uses $t^*\mathcal{L}og^b \cong \bigoplus_{i=0}^{b} \mathrm{Sym}^i \mathcal{X}$ Splitting principle

$$\nabla^a \mathcal{E}is^b_{dR,\Gamma}(f) \in H^{d-1}_{coh}(\mathcal{U}_N, \Gamma, \mathrm{Sym}^a \Omega^1_{\mathcal{A}} \otimes \mathcal{L}og^b \otimes \Omega^d_{\mathcal{A}})$$

$$\Big\downarrow t^*$$

$$H^{d-1}_{coh}(\mathcal{R}, \Gamma, \mathrm{Sym}^a t^*\Omega^1_{\mathcal{A}} \otimes \mathrm{Sym}^b \mathcal{X} \otimes \Omega^d_{\mathcal{A}})$$

$$\wr\|$$

$$t^*\nabla^a \mathcal{E}is^b_{dR,\Gamma}(f) \in H^{d-1}(GL_n(\mathcal{O}_K), H^0_{coh}(\mathcal{R}, \mathrm{Sym}^a \omega_{\mathcal{A}} \otimes \mathrm{Sym}^b \mathcal{X} \otimes \omega^d_{\mathcal{A}}))$$

$\omega_{\mathcal{A}} := e^*\Omega^1_{\mathcal{A}}$
$\omega^d_{\mathcal{A}} := e^*\Omega^d_{\mathcal{A}}$

cohomology of $GL_n(\mathcal{O}_K)$ with values in sections of an algebraic bundle

Restrict to $\mathcal{O}_L^\times \subset GL_n(\mathcal{O}_K)$ and cap with fundamental class $\xi \in H_{d-1}(\mathcal{O}_L^\times, \mathbb{Z})$ gives $\quad \mathrm{rk}\, \mathcal{O}_L^\times = d-1 \quad d = \dim \mathcal{A}$

$$t^*\nabla^a \mathcal{E}is^b_{dR,\Gamma}(f) \cap \xi \in H^0_{coh}(\mathcal{R}, \mathrm{Sym}^a \omega_{\mathcal{A}} \otimes \mathrm{Sym}^b \mathcal{X} \otimes \omega^d_{\mathcal{A}})$$

evaluation at basis $\omega(\mathcal{A}), \omega(\mathcal{A}^\vee)$ gives value in $\mathcal{R}$.

Remarks:

- similar constructions give classes related to work of Bergeron-Charollois-Garcia, Sharifi-Venkatesh, Bergeron-Venkatesh
- Beilinson-K.-Levin used $GL_n(\mathcal{O}_F)$, $\mathcal{O}_F$ tot. real to show Klingen-Siegel and p-adic interpolation
- Graf constructed the Eisenstein cohomology for Hilbert modular varieties in this way.

§3 p-adic interpolation

Slogan: for integrality and p-adic interpolation use refinement via the Poincaré bundle.

$$\mathcal{P} \longrightarrow \mathcal{A} \times_S \mathcal{A}^\vee$$

and its pull-back $\mathcal{P}^\natural$ on universal vector extension $\mathcal{A}^\natural$ of $\mathcal{A}^\vee$

$$0 \longrightarrow \omega_{\mathcal{A}/S} \longrightarrow \mathcal{A}^\natural \xrightarrow{p} \mathcal{A}^\vee \longrightarrow 0$$

Fact:

$$\begin{tikzcd} \mathcal{P}^\natural := p^* \mathcal{P} \arrow[d] \\ \mathcal{A} \times_S \mathcal{A}^\natural \arrow[d, shift right] \\ \mathcal{A} \arrow[u, shift right, "(id, e^\natural)"'] \end{tikzcd}$$

classifies line bdl. with integrable connection

$\rightsquigarrow$ $\mathcal{P}^\natural$ has connection $\nabla^\natural$

completion along $\mathcal{A} \times \{e^\natural\}$ $\rightsquigarrow$

$$\boxed{\begin{tikzcd} \widehat{\mathcal{P}}^\natural = \varprojlim_n \mathcal{P}^{\natural (b)} \arrow[d] \\ \mathcal{A} \end{tikzcd}}$$

b-th inf. nbh. of $\mathcal{A} \times \{e^\natural\}$

Theorem (Scheider)

$$\left(\mathcal{P}^{\natural (b)}, \nabla^{\natural (b)}\right) \cong \left(\mathcal{L}og^b, \nabla\right)$$

Main point: construct refinement with values in the Poincaré bundle

$$\begin{array}{ccc} \mathrm{Eis}^{\natural b}_{\mathrm{coh},\Gamma}(f) & \in & H^{d-1}_{\mathrm{coh}}(U_N, \Gamma, \mathcal{P}^{(b)} \otimes \Omega^d_{\mathcal{A}}) \\ \downarrow & & \downarrow \\ \mathrm{Eis}^{b}_{\mathrm{coh},\Gamma}(f) & \in & H^{d-1}_{\mathrm{coh}}(U_N, \Gamma, \mathcal{P}^{\natural(b)} \otimes \Omega^d_{\mathcal{A}}) \end{array}$$

Fact: Σ ordinary at $p \Rightarrow \widehat{\mathcal{P}}|_{\widehat{\mathcal{A}}} \cong \mathcal{O}_{\widehat{\mathcal{A}\times\mathcal{A}^\vee}}$ $\leftarrow$ $\boxed{\text{completion along } \{e\times e^\vee\} \subset \mathcal{A} \times_S \mathcal{A}^\vee}$

and $H^0(\widehat{\mathcal{A}\times\mathcal{A}^\vee}, \mathcal{O}_{\widehat{\mathcal{A}\times\mathcal{A}^\vee}}) \cong \mathrm{Meas}(\mathcal{O}_L \otimes \mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p})$ (depends on base change to $\mathcal{O}_{\mathbb{C}_p}$)

Hence $\mathrm{Eis}^{\natural b}_{\mathrm{coh},\Gamma}(f) \rightsquigarrow$ class in $H^{d-1}(\Gamma, H^0(\widehat{\mathcal{A}\times\mathcal{A}^\vee}, \mathcal{O}_{\widehat{\mathcal{A}\times\mathcal{A}^\vee}} \otimes \omega^d_{\mathcal{A}}))$

$$\cong \mathrm{Meas}(\mathcal{O}_L \otimes \mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p})^{\Gamma}$$