

# Equivariant Eisenstein classes, critical values of Hecke $L$ -functions and $p$ -adic interpolation. (joint with J. Sprang)

§1. Main results

§2. Equivariant motivic Eisenstein classes

§3.  $p$ -adic interpolation

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(paper: [arXiv:1912.03657v2](https://arxiv.org/abs/1912.03657v2))

## §1 Main results

Let  $L$  be an algebraic number field,  $f \subset \mathcal{O}_L$  integral ideal,  $I(f) :=$  fractional ideals of  $L$  prime to  $f$

$$\chi: I(f) \longrightarrow \overline{\mathbb{Q}}^\times$$

an algebraic Hecke character of conductor  $f$ . Choose  $\overline{\mathbb{Q}} \subset \mathbb{C}$  and let

$$L(\chi, s) = \sum_{(\mathcal{O}_L, f) = 1} \frac{\chi(\mathcal{O}_L)}{N \mathcal{O}_L^s}$$

be its  $L$ -function.

Expectation:

critical values of  $L(\chi, s)$  divided by suitable periods are algebraic numbers (Deligne conjecture) and can be  $p$ -adically interpolated.

Remark:

- $L(\chi, \tau) = L(\chi \cdot N_{L/\mathbb{Q}}^{-\tau}, 0)$ ,  $\tau \in \mathbb{Z}$ .
- $L(\chi, 0)$  critical if  $\Gamma$ -factors in fct-eq are finite and  $\neq 0$  at  $s=0$ .

Two cases

$L/\mathbb{Q}$  totally real  
 $\chi = \mathfrak{s} \cdot N_{L/\mathbb{Q}}^{-\tau}$   
 $\mathfrak{s}$  Dirichlet character

$L \supset K$ ,  $K$  CM  
field + conditions  
on  $\infty$ -type of  $\chi$

## Known results (for critical values)

### $L$ totally real

Klingen - Siegel :  $L(\chi, 0)$  algebraic

Deligne - Ribet, Barsky, Cassou-Nogues :  
construct  $p$ -adic  $L$ -fct.

### $L = K$ CM field

Damerell  $[K:\mathbb{Q}]=2$ , Shimura :  $\frac{L(\chi, 0)}{\text{periods}}$  alg

Katz : constructs  $p$ -adic  $L$ -fct.

Blasius : Deligne conjecture is true

### $L \supset K$ CM field

Colmez - Schneider :  $K$  imag. quadratic and certain  $L$

$p$ -adic interpolation in these cases

Harder : announced Deligne conjecture 1984  
(no integrality or  $p$ -adic interpolation)

Theorem (K. - Sprang) If  $L \supset K$  ( $M$  field,  $\chi$  critical at 0) (of  $\infty$ -type  $\beta$ - $\alpha$  with respect to a CM type  $\Sigma$  of  $L$ )  
 $e \in \mathbb{Q}$ ,  $(e, f) = 1$  ( $f \neq 0$  for simplicity), then

$$\frac{(\alpha - 1)! (2\pi i)^{|\beta|}}{\Omega^\alpha \cdot \Omega^\vee{}^\beta} (\chi(e) N_e - 1) L(\chi, 0) \in \mathbb{Q}_k \left[ \frac{1}{\text{d.f.} \cdot N_e} \right]$$

where:  $A$  abelian variety with CM by  $\mathbb{Q}_L$

$\omega(A)$ ,  $\omega(A^\vee)$  bases of  $\omega_{A/\mathbb{Q}}$ ,  $\omega_{A^\vee/\mathbb{Q}}$ ,  $\Omega$ ,  $\Omega^\vee$  assoc periods

$x$   $f$ -torsion point on  $A$

$k$  field of definition of  $(A/\mathbb{Q}, \omega(A), \omega(A^\vee), x)$

( $\leadsto$  generalises Katz/Shimura to arbitrary critical  $\chi$ )

Remark similar results were announced by Bergeron-Charollais-Garcia (completely different method and relying on Katz's result.)

Corollary (pointed out by Blasius):

$$\frac{L(\chi, 0)}{c^+(R_{L/\mathbb{Q}} M(\chi))} \in \overline{\mathbb{Q}}^\times$$

weak Deligne conjecture

Period of the motive  $M(\chi)$  of  $\chi$

if  $F \in K$  max. tot. real  
all primes of  $F$  over  $p$   
split completely in  $K$

## Theorem (K-Sprang) ( $p$ -adic interpolation)

Let  $p$  a prime number,  $(p, d_L) = 1$ , and assume that the CM-type  $\Sigma$  of  $L$  is **ordinary at  $p$** .

Then there exists a **measure  $\mu_f$**  on  $\text{Gal}(L^{\infty, f}/L)$  such that:

$$\frac{1}{\Omega_p^\alpha \Omega_p^{\vee\beta}} \int_{\text{Gal}(L^{\infty, f}/L)} \chi(g) d\mu_f(g) = \frac{(\alpha-1)!(2\pi i)^{|\beta|}}{\Omega^\alpha \cdot \Omega^{\vee\beta}} \text{ (explicit local factor) } L(\chi, 0)$$

for all  $\chi$  of conductor dividing  $p^{\infty, f}$ , **critical at 0**,  $\infty$ -type  $\beta - \alpha$

( $\leadsto$  generalises Katz to arbitrary critical  $\chi$ )

## Why does the Shimura-Katz approach not generalise?

- 1) Shimura-Katz use **algebraic Eisenstein series** on **Kilbert modular varieties**. The Eisenstein series for  $L \neq K$  live on  $GL_n(\mathcal{O}_K)$  if  $n = [L:K]$  whose locally symmetric space is **not** algebraic.
- 2) They use algebraicity properties of **Maaß-Shimura operators**, which are also not available.

## Our approach:

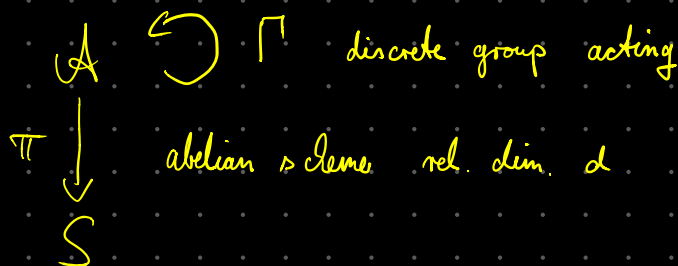
- work with a **single abelian variety**  $A$  with CM by  $\mathbb{Q}_L$
- construct **equivariant** (for  $\mathbb{Q}_L^\times$ -action) integral coherent cohomology classes which are related to **Eisenstein-Kronecker series**
- construction uses completed Poincaré bundle (inspired by work of Banai-Kobayashi)  
 $\rightsquigarrow$  integrality **without  $q$ -expansion**
- the formal cohomology class gives a  **$p$ -adic measure**  $\rightsquigarrow$   **$p$ -adic  $L$ -fct.**

inspired by  
equivariant motivic  
Eisenstein classes

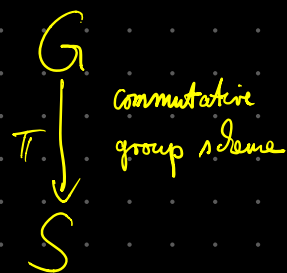
Remark: Our approach is even new for  $L=K$ , no  $q$ -expansion necessary.

## §2. Equivariant motivic Eisenstein classes

Situation:



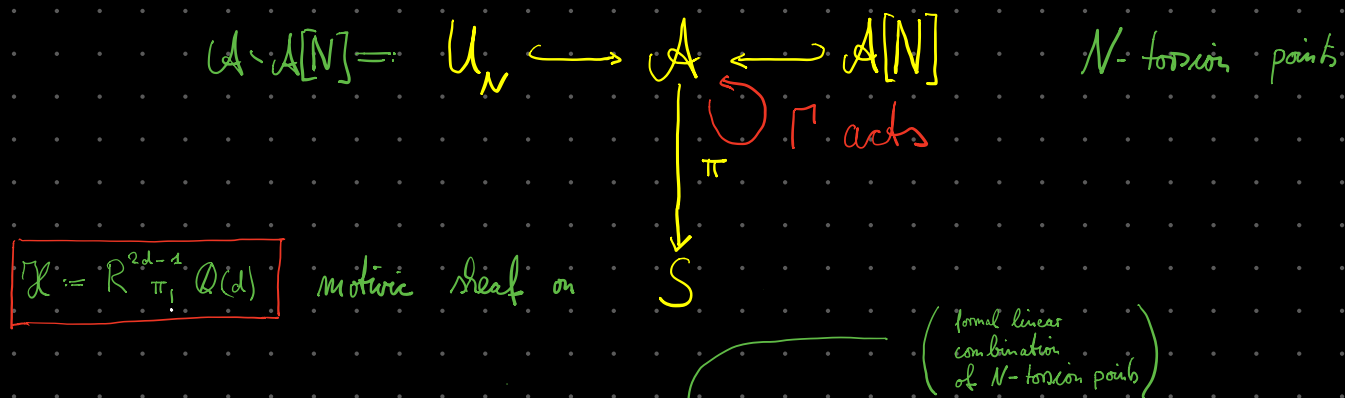
more generally:



Examples:

- $A \otimes_{\mathbb{Z}} \mathbb{Z}^n$  with  $GL_n(\mathbb{Z})$  action
- $A$  CM by  $\mathcal{O}_L$ ,  $A$  has  $\mathcal{O}_L^\times$ -action
- $A$  CM by  $\mathcal{O}_K$ ,  $A \otimes_{\mathcal{O}_K} \mathcal{O}_K^{\oplus n}$  has  $GL_n(\mathcal{O}_K)$ -action
- also  $G_m^n$  with  $GL_n(\mathbb{Z})$ -action

Consider



Theorem (K.) For  $f: A[N](S) \rightarrow \mathbb{Q}$  exists equivariant motivic Eisenstein class

$$\text{Eis}_{\text{mot}, \Gamma}^b(f) \in H_{\text{mot}}^{2d-1}(U_N, \Gamma, \text{Log}^b(d))$$

where:  $U_N := A \setminus A[N]$

$\left( \begin{array}{l} \text{equivariant} \\ \text{motivic cohom.} \end{array} \right)$

$\text{Log}^b$  filtered unipotent motivic sheaf with assoc. graded

$$\text{gr } \text{Log}^b \cong \bigoplus_{i=0}^b \text{Sym}^i \mathcal{X}$$

## Application in our situation:

$L \supset K$ ,  $K$  CM field,  $m = [L:K]$



Consider de Rham realization

$$H_{dR}^{2d-1}(U_N, \Gamma, (\text{Log}^b, \nabla))$$

$$(\text{Log}, \nabla): \text{Log}^b \otimes \mathcal{O}_{\mathcal{A}} \xrightarrow{\nabla} \text{Log}^b \otimes \Omega_{\mathcal{A}}^1 \rightarrow \dots \rightarrow \text{Log}^b \otimes \Omega_{\mathcal{A}}^d$$

$\leadsto$  "Hodge filtration"

## Fact

$$\text{Ein}_{dR, \Gamma}^b(\tfrac{1}{t}) \in H_{\text{coh}}^{d-1}(U_N, \Gamma, \text{Log}^b \otimes \Omega_{\mathcal{A}}^d) \subset H_{dR}^{2d-1}(U_N, \Gamma, (\text{Log}^b, \nabla))$$

(follows also from "integral refinement" later)

To produce more classes use  $\nabla$  differentiation in fibre direction as substitute for Maaß-Shimura operators.

Using  $\nabla$  and a torsion section  $t \in U_N$

$$\nabla^a \text{Eis}_{dR, \Gamma}^b(f) \in H_{\text{c\acute{e}t}^1}^{d-1}(U_N, \Gamma, \text{Sym}^a \Omega_{\mathcal{A}}^1 \otimes \text{Log}^b \otimes \Omega_{\mathcal{A}}^d)$$

uses  
 $t_{\text{Log}}^* \cong \bigoplus_{i=0}^b \text{Sym}^i \mathcal{L}$   
 Splitting principle

$$\downarrow t^*$$

$$H_{\text{c\acute{e}t}^1}^{d-1}(\mathbb{A}, \Gamma, \text{Sym}^a t^* \Omega_{\mathcal{A}}^1 \otimes \text{Sym}^b \mathcal{L} \otimes \Omega_{\mathcal{A}}^d)$$

$\omega_{\mathcal{A}} := e^* \Omega_{\mathcal{A}}^1$   
 $\omega_{\mathcal{A}}^d := e^* \Omega_{\mathcal{A}}^d$

$$\parallel$$

$$t^* \nabla^a \text{Eis}_{dR, \Gamma}^b(f) \in H_{\text{c\acute{e}t}^1}^{d-1}(GL_n(\mathcal{O}_K), H_{\text{c\acute{e}t}^1}^0(\mathbb{A}, \text{Sym}^a \omega_{\mathcal{A}} \otimes \text{Sym}^b \mathcal{L} \otimes \omega_{\mathcal{A}}^d))$$

cohomology of  $GL_n(\mathcal{O}_K)$   
 with values in sections  
 of an algebraic bundle

Restrict to  $\mathcal{O}_L^* \subset GL_n(\mathcal{O}_K)$  and cap with fundamental class

$\xi \in H_{d-1}(\mathcal{O}_L^*, \mathbb{Z})$  gives  $\cap \mathcal{O}_L^* = d-1$   $d = \dim \mathcal{A}$

$$t^* \nabla^a \text{Eis}_{dR, \Gamma}^b(f) \cap \xi \in H_{\text{c\acute{e}t}^1}^0(\mathbb{A}, \text{Sym}^a \omega_{\mathcal{A}} \otimes \text{Sym}^b \mathcal{L} \otimes \omega_{\mathcal{A}}^d)$$

evaluation at basis  $\omega(\mathcal{A}), \omega(\mathcal{A}^\vee)$  gives value in  $\mathbb{A}$ .

- Remarks:
- similar constructions give classes related to work of Bergeron-Charollois-Garcia, Sharifi-Venkatesh, Bergeron-Venkatesh
  - Beilinson-K.-Levin used  $GL_n(\mathcal{O}_F)$ ,  $\mathcal{O}_F$  tot. real to show Klingen-Siegel and p-adic interpolation
  - Graf constructed the Eisenstein cohomology for Hilbert modular varieties in this way.

### §3 p-adic interpolation

Slogan: for integrality and p-adic interpolation use refinement via the Poincaré bundle.

$P \rightarrow A \times_S A^\vee$   
and its pull-back  $P^\sharp$  on universal vector extension  $A^\sharp$  of  $A^\vee$

$$0 \rightarrow \omega_{A/S} \rightarrow A^\sharp \xrightarrow{P} A^\vee \rightarrow 0$$

Fact:

$P^\sharp := p^* P$  classifies line bdl. with integrable connection

$$\downarrow$$

$$A \times_S A^\sharp$$

$$\downarrow \widehat{(\omega, e^\sharp)}$$

$$A$$

$\leadsto P^\sharp$  has connection  $\nabla^\sharp$

completion along  $A \times \{e^\sharp\} \leadsto$

$$\widehat{P}^\sharp = \varprojlim_n P^{\sharp(b)}$$

$$\downarrow$$

$$A$$

$\uparrow$   
b-th inf. mbr. of  $A \times \{e^\sharp\}$

Theorem (Schneider)

$$(P^{\sharp(b)}, \nabla^{\sharp(b)}) \cong (\text{Log}^b, \nabla)$$

Main point: construct refinement with values in the Poincaré bundle

$$\begin{array}{ccc} \text{Eis}_{\text{coh}, \Gamma}^{\pm b}(f) \in H_{\text{coh}}^{d-1}(U_N, \Gamma, \mathcal{P}^{(b)} \otimes \Omega_{\mathcal{A}}^d) & & \\ \downarrow & & \downarrow \\ \text{Eis}_{\text{coh}, \Gamma}^b(f) \in H_{\text{coh}}^{d-1}(U_N, \Gamma, \mathcal{P}^{\pm(b)} \otimes \Omega_{\mathcal{A}}^d) & & \end{array}$$

Fact:  $\Sigma$  ordinary at  $p \Rightarrow \widehat{\mathcal{P}}|_{\widehat{\mathcal{A}}} \cong \widehat{\mathcal{O}_{\mathcal{A} \times \mathcal{A}}}$  completion along  $\{c \times c^v\} \subset \mathcal{A} \times \mathcal{A}^v$

and  $H^0(\widehat{\mathcal{A} \times \mathcal{A}}, \widehat{\mathcal{O}_{\mathcal{A} \times \mathcal{A}}}) \cong \text{Meas}(\mathcal{O}_{\mathcal{L}} \otimes \mathbb{Z}_p, \mathcal{O}_{\mathcal{C}_p})$  (depends on base change to  $\mathcal{O}_{\mathcal{C}_p}$ )

Hence  $\text{Eis}_{\text{coh}, \Gamma}^{\pm b}(f) \rightsquigarrow$  class in  $H^{d-1}(\Gamma, H^0(\widehat{\mathcal{A} \times \mathcal{A}}, \widehat{\mathcal{O}_{\mathcal{A} \times \mathcal{A}}} \otimes \omega_{\mathcal{A}}^d))$   
 $\cong$   
 $\text{Meas}(\mathcal{O}_{\mathcal{L}} \otimes \mathbb{Z}_p, \mathcal{O}_{\mathcal{C}_p})^{\Gamma}$

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