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Dimension of admissible rep of p-adic Lie group G.

(π, V) $\qquad H \subset G$ open pro-p-subgroup.

$$\mathrm{gr}(\mathbb{F}[[H]]) \simeq \mathbb{F}[x_1, \dots, x_d] \quad (\text{Lazard})$$

$$d = \dim(G)$$

$$\dim_G(\pi, V) = \dim(\underbrace{\mathrm{Supp}(\mathrm{gr}(V^\vee))})$$

(Gelfand-Kirillov dimension)

$$\cap$$

$$\mathrm{Spec}(\mathrm{gr}(\mathbb{F}[[H]]))$$

$$\| $$

$$\mathrm{Spec}(\mathbb{F}[x_1, \dots, x_d]).$$

$\mathfrak{m}_H$-adic filtration on $V^\vee$.

Hilbert-Samuel: $\exists\, e \geq 1$ s.t.

$$\dim_{\mathbb{F}}(V[\mathfrak{m}_H^n]) \underset{n \to +\infty}{\sim} \frac{e}{\dim_G(\pi)!}\, n^{\dim_G(\pi, V)}$$

2. Mod p Langlands correspondence for $GL_2(\mathbb{Q}_p)$

Irreducible smooth rep of $GL_2(\mathbb{Q}_p)$ (over $\mathbb{F}$)

(Barthel–Livne, Breuil, Berger).

Principal series: $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ loc constant

$$\begin{array}{ccc} \mathbb{Q}_p^\times & \longrightarrow & \mathbb{F}^\times \\ \wr\| & & \uparrow \\ \mathbb{Z}_p^\times \times p^{\mathbb{Z}} & \twoheadrightarrow & \mathbb{F}_p^\times \times p^{\mathbb{Z}} \end{array}$$

$\omega : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ induced $\mathbb{F}_p^\times \subset \mathbb{F}^\times$

trivial on $p^{\mathbb{Z}}$.

$\lambda \in \mathbb{F}^\times$ $\quad \mathrm{ur}(\lambda) : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ trivial on $\mathbb{Z}_p^\times$.

$p \longmapsto \lambda$

$\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ $\qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$(\chi_1 \otimes \chi_2)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \chi_1(a)\chi_2(d)$

$\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ smooth rep of $GL_2(\mathbb{Q}_p)$.

$G = GL_2(\mathbb{Q}_p)$ $\hookrightarrow$ reducible iff $\chi_1 = \chi_2$

$\chi_1 \chi_2^{-1} = \omega(p^{-2})$

$= | \ |^2$

$\chi_1 = \chi_2 = 1$.

$$0 \to 1 \to \mathrm{Ind}_B^G(1) \to \underbrace{Sp}_{\text{irreducible}} \to 0$$

$$JH(\mathrm{Ind}_B^G(\chi \otimes \chi)) = \{\chi \circ \det, \ Sp \otimes (\chi \circ \det)\}.$$

$$= JH(\mathrm{Ind}_B^G(\chi | \ | \otimes \chi | \ |^{-1}))$$

- Supersingular representations = irred rep which are not subquotients of $\mathrm{Ind}_B^G(\ )$.

σ weight of $K = GL_2(\mathbb{Z}_p)$ $\qquad \sigma = \mathrm{Sym}^r(\mathbb{F}^2) \otimes (\det)^m$

" irreducible rep of K $\qquad \downarrow$ $\qquad \boxed{GL_2(\mathbb{F}_p)}$ $\qquad 0 \leq r \leq p-1$

$0 \leq m \leq p-2$.

$$\sigma^s = \mathrm{Sym}^{p-1-r}(\mathbb{F}^2) \otimes \det^{m+r}. \qquad \left(s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$K_1 = \mathrm{Ker}(GL_2(\mathbb{Z}_p) \longrightarrow GL_2(\mathbb{F}_p))$.

<u>Thm</u> (Breuil) 1) $\exists!$ supersingular representation $\pi(\sigma)$ of $GL_2(\mathbb{Q}_p)$ such that

$$\mathrm{soc}_K(\pi(\sigma)) = \sigma \oplus \sigma^s.$$

" largest semisimple K-subrep of π.

2) A supersingular representation of $GL_2(\mathbb{Q}_p)$ is isomorphic $\pi(\sigma) \otimes (\chi \circ (\det))$ for some $\chi : \mathbb{Q}_p^\times \longrightarrow \mathbb{F}^\times$.

<u>Corollary</u>: All irreducible representations (smooth) of $GL_2(\mathbb{Q}_p)$ are admissible.

Proof: - $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ are admissible.

- supersingular representations

$\dim(\mathrm{soc}_K(\Pi)) < \infty$. $\quad K_1 = \mathrm{Ker}(GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p))$.

pro-p group

$\dim_{\mathbb{F}}(\Pi^{K_1}) < \infty$, $\quad \Pi^{K_1} \circlearrowleft K/K_1$ finite group.

$\mathrm{soc}_K(\Pi) = \mathrm{soc}_{K/K_1}(\Pi^{K_1}) \Rightarrow \dim(\Pi^{K_1}) < \infty$. $\square$

Semisimple mod p Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

2-dim rep of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$	"some" semisimple
semisimple $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\mathbb{F})$	rep of $GL_2(\mathbb{Q}_p)$

$0 \le r \le p-1$ / $\mathrm{ind}_{Gal_{\mathbb{Q}_{p^2}}}^{Gal_{\mathbb{Q}_p}} \omega_2^{r+1} \longleftrightarrow \pi(\mathrm{Sym}^r \mathbb{F}^2)$.

$(\omega^{r+1} ur(\lambda) \oplus ur(\mu)) \circ rec^{-1} \longleftrightarrow \mathrm{Ind}_B^G (ur(\mu) \otimes \omega^r ur(\lambda))^{ss}$

$rec : \mathbb{Q}_p^\times \xrightarrow{\sim} W_{\mathbb{Q}_p}^{ab.}$

$\oplus$

$\mathrm{Ind}_B^G (ur(\lambda)\omega^{r+1} \otimes ur(\mu))^{ss}$.

rule $\bar{e} \otimes \chi \longleftrightarrow \pi(\bar{e}) \otimes (\chi \circ \det)$

Remark: "Most of the time"

$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\mathbb{F})$ semisimple

$\pi(\rho)^{ss}$ corresponding ssimple rep of $GL_2(\mathbb{Q}_p)$

$$JH\left(\mathrm{soc}_K\left(\pi(\rho)\otimes(\omega\circ\det)\right)\right) = W(\rho).$$

$$\left(\mathrm{soc}_K(\pi(\sigma)) = \sigma \oplus \sigma^s\right.$$

$$\mathrm{soc}_K\left(\left(\mathrm{Ind}_B^G(\mathrm{un}(\mu)\otimes\omega^r\mathrm{un}(\lambda))\right)\otimes(\omega\circ\det)\right) = \mathrm{Sym}^r(\mathbb{F}^2)$$

This correspondence can be extended to the non semisimple case using Colmez "Montreal functor".

Colmez

$$\begin{tikzcd} \left(\begin{array}{c}\text{smooth adm}\\ \text{fg } G\text{-rep}\end{array}\right) \arrow[rr] \arrow[dr, dashed] & & \left(\begin{array}{c}\text{finite dim}\\ \text{rep of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\end{array}\right) \arrow[dl, "\sim", "V^\vee"'] \\ \pi \arrow[r, mapsto] & \mathcal{D}(\pi) \quad \left(\begin{array}{c}\text{étale}\\ (\varphi,\Gamma)\text{-module}\end{array}\right) & \end{tikzcd}$$

(Fontaine)

Recall A (φ,Γ)-module is a f d $\mathbb{F}((X))$-vs D with $\varphi : D \to D$ $\mathbb{F}$-linear endomorphism

$$\varphi(S(X)m) = S(X^p)\,\varphi(m) \qquad S(X) \in \mathbb{F}((X)).$$

and $\Gamma = \mathbb{Z}_p^\times \to \mathrm{Aut}_{\mathbb{F}}(D)$

$$a \in \mathbb{Z}_p^\times \qquad a\cdot(S(X)\,m) = S\big((1+X)^a - 1\big)(a\cdot m).$$

and actions of Γ and φ have to commute.

A (φ,Γ)-module is étale if $\mathbb{F}((X)) \otimes_{\mathbb{F}((X)),\, X \mapsto X^p} D \xrightarrow{1\otimes\varphi} D$ is an iso.

Construction of the functor $\pi \longmapsto D(\pi)$.

Start with (π, V) adm fg rep of $GL_2(\mathbb{Q}_p)$.

V is an $\mathbb{F}[\![X]\!]$-module

$$\mathbb{F}[\![X]\!] \cong \mathbb{F}[\![\mathbb{Z}_p]\!] \simeq \mathbb{F}\left[\!\left[\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right]\!\right].$$

V is a smooth $\mathbb{F}[\![X]\!]$-module.

$F = \pi\begin{pmatrix} p & \\ & 1 \end{pmatrix} \in \mathrm{End}_{\mathbb{F}}(V)$.

V becomes a module over the non commutative algebra $\mathbb{F}[\![X]\!][F]$ with relation

$$F \cdot X = X^p \cdot F.$$

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & px \\ & 1 \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix}.$$

$\rightsquigarrow$ V is naturally an A-module

$W \subset V$ finite dim $\mathbb{F}$-vs generates (π, V)
as a rep of $GL_2(\mathbb{Q}_p)$.

$M = A \cdot W \subset V$ A-submodule generated by W.

<u>Thm</u> (Colmez, Emerton)

M is admissible as a representation of $\mathbb{Z}_p \simeq \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow M^\vee$ is f.g over $\mathbb{F}[[\mathbb{Z}_p]] \simeq \mathbb{F}[[X]]$.

$D(\pi) = M^\vee[\frac{1}{X}]$ is a finite dim vector space over $\mathbb{F}((X))$.

<u>Define φ and Γ:</u> $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix} \simeq \Gamma$.

$F = \pi(P, 1) : M \longrightarrow M$ is a semilinear map of $\mathbb{F}[[X]]$-modules.

$1 \otimes F : \mathbb{F}[[X]] \otimes_{\mathbb{F}[[X]], X \mapsto X^p} M \longrightarrow M$.

linear map of $\mathbb{F}[[X]]$-modules.

M is fg A-module $\Rightarrow$ Coker$(1 \otimes F)$ is a fg $\mathbb{F}[[X]]$-module + torsion $\Rightarrow$ finite.

$$M^\vee \xrightarrow{(1\otimes F)^\vee} \left(\mathbb{F}[[X]] \otimes_{X \mapsto X^p} M\right)^\vee$$

finite kernel.

$$\begin{array}{ccc} M^\vee[\frac{1}{X}] & \xrightarrow{\sim} & \left(\mathbb{F}[[X]] \otimes_{X \mapsto X^p} M\right)^\vee [\frac{1}{X}] \\ \| & & \wr\| \\ D(\pi) & \xrightarrow[\varphi^{-1}]{\sim} & \mathbb{F}((X)) \otimes_{X \mapsto X^p} D(\pi) \end{array}$$

$\Rightarrow$ étale φ-module.

Action of Γ: choose $W \subset V$ stable under $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$ (possible since (π, V) is smooth)

M will be stable by Γ.

$\rightsquigarrow$ action of Γ on $D(\pi) \rightsquigarrow$ étale (φ, Γ)-module.

<u>Thm</u> (Colmez) D. exact contravariant functor $\begin{pmatrix} \text{adm smooth} \\ \text{fg rep of } GL_2(\mathbb{Q}_p) \end{pmatrix} \longrightarrow \begin{pmatrix} \text{rep of } Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \text{on fd } \mathbb{F}\text{-vs}. \end{pmatrix}$

If $\bar{\rho}$ is a simple 2-dim rep of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ then $V^*(D(\pi(\bar{\rho})^{ss})) \simeq \bar{\rho}$.

<u>Thm</u> (Colmez) If $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\mathbb{F})$.

$\exists!$ admissible finitely generated representation $\pi(\bar{\rho})$ of $GL_2(\mathbb{Q}_p)$ such that $\pi(\bar{\rho})$ has no subobject or quotient which is finite dimensional.

($D(\pi) = 0$ if π is finite dimensional).

Link with completed cohomology of modular curves

$N \geq 5$ prime to p.

$\tilde{H}^1(N, \mathbb{F}) = \tilde{H}^1(N, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F} \circlearrowleft GL_2(\mathbb{Q}_p)$

$\circlearrowleft$

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$\bar{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$ abs irred, odd
unr almost everywhere
$\Rightarrow$ modular (Khare-Wintenberger)

$\bar{\rho}$-isotypic $\tilde{H}^1(N, \mathbb{F})[\bar{\rho}] \simeq \mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}}}(\bar{\rho}, \tilde{H}^1(N, \mathbb{F}))$.

$\circlearrowleft$ $GL_2(\mathbb{Q}_p)$.

Thm (Emerton) $V^*\left(D(\tilde{H}^1(N, \mathbb{F}))[\bar{\rho}] \otimes (\omega \circ \det)\right) \simeq \bar{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}^{\oplus r}$

Hyp: $p > 2$ $\bar{\rho}|_{\mathrm{Gal}_{\mathbb{Q}_p}} \not\simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ up to twist