p-adic automorphic forms & (big) Igusa varieties.

Partial reference:
"A unipotent circle action on p-adic modular forms".
math.utah.edu/~howe

Goal: Explain a natural space of p-adic automorphic forms to the Katz-Serre of p-adic modular forms

+ some consequences (classical + philosophical)

Notation: $K \subseteq GL_2(\mathbb{A}_f)$ compact open subgroup

Y_K modular curve of level K, $\pi: E_{univ} \to Y_K$

$\omega := \pi_* \Omega_{E_{univ}/Y_K}$.

modular line bundle.

The archimedean story

Automorphic forms: $\mathcal{A}^K_{\mathbb{C}} \subseteq C^\infty(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K, \mathbb{C})$

$\circlearrowleft GL_2(\mathbb{R})$

$\mathfrak{gl}_2, \mathbb{C}^\times$

$\cap$

$GL_2(\mathbb{R})$.

Distinguished in terms of this action & growth condition.

Modular forms: $M_K \subseteq H^0(Y_K(\mathbb{C}), \omega^K)$

growth at cusps.

The relation: $Y(\mathbb{C}) = (E, \psi)$, $E/\mathbb{C}$ elliptic curve

ψ = level K structure.

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K \times \mathbb{H}^{\pm} \xrightarrow[GL_2(\mathbb{R})\text{-torsor.}]{unif} Y(\mathbb{C}).$$

(T, ϕ, ψ)

T is 2-dim'l real torus

$\phi: \mathrm{Lie}\, T \cong \mathbb{R}^2$

ψ level K structure

moduli of complex structures on $\mathbb{R}^2$

$(\mathbb{H}^{\pm} \subseteq \mathbb{P}^1(\mathbb{C}))$

$$unif^* \omega = \pi_2^* \mathcal{O}(-1)$$

$GL_2(\mathbb{R})$ - equivariant identification

Restrict unif to

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K \times \{i\} \xrightarrow{\text{unif}} Y(\mathbb{C}).$$

$\mathbb{C}^\times$-torsor

$\cap$

Stab(i) $\subseteq GL_2(\mathbb{R})$

action on $\mathcal{O}(-1)|_i$

Choose t trivializing $\mathcal{O}(-1)|_i$

$\to$ Evaluation map

$$M_K \longrightarrow \mathcal{A}_{\mathbb{C}}$$
$$f \longmapsto \frac{\text{unif}^* f}{t^K}$$

The image of this map consists exactly of functions which transform via $z \mapsto z^K$ under $\mathbb{C}^\times \subseteq GL_2(\mathbb{R})$

& annihilated by $\mathfrak{n}_i \subseteq \mathfrak{gl}_{2,\mathbb{C}}$

$(i \in \mathbb{H}^\pm \subseteq \mathbb{P}^1(\mathbb{C}))$.

__The p-adic story:__ $K = GL_2(\mathbb{Z}_p) K^p$ (take $K^p = \Gamma_1(N)$).

2 definitions of p-adic modular forms $M_{\mathbb{Q}_p}$.

__Serre:__ $\sum_K q(H^0(Y(\mathbb{C}), \omega^K)) \cap \mathbb{Q}[[t]]$ (in $\mathbb{C}[[t]]$)

$\cap$

power series w/ p-adically bounded coefficients

Complete for the sup p-adic norm to get $M_{\mathbb{Q}_p}$.

$\mathbb{Q}_p$-Banach space

__Katz:__ $Ig_{Katz} \xrightarrow{\mathbb{Z}_p^\times} Y^\wedge_{K,ord}$ ($/\operatorname{Spf}\mathbb{Z}_p$).

$Ig_{Katz}(R) = (E, \phi, \psi)$, E/R elliptic curve

ψ level K structure.

$\phi: \hat{E} \to \hat{\mathbb{G}}_m$.

$\mathbb{Z}_p^\times$

Compactification

$$M_{\mathbb{Q}_p} = \mathcal{O}(Ig^*_{Katz})[\tfrac{1}{p}]$$

$$M_\kappa \to M_{\mathbb{Q}_p}$$
$$f \mapsto \frac{f}{\left(\varphi^* \frac{dt}{t}\right)^\kappa} \quad \leftarrow \text{transform via } z \to z^\kappa \text{ under } \mathbb{Z}_p^\times.$$

Features of $M_{\mathbb{Q}_p}$:

- Encodes congruences between q-expansions $\leftrightarrow$ Hecke eigensystems
- Weights: characters of $\mathbb{Z}_p^\times$.
- Weight families (e.g. Eisenstein, Hida)
- U_p, Frob
- $\theta = q\frac{\partial}{\partial q}$ $\leftarrow$ Related to Gauss-Manin

Claim: Like if you only looked at lowest wt. vectors in $\mathcal{A}_{\mathbb{C}}$.

How to get something better?

Serre perspective: Instead of interpolating spherical vectors, interpolate the whole aut. representation — interpolate full Kirillov models.

Katz perspective: Use a better moduli problem.

$$Ig_{CS}(R) = (E, \phi, \psi) \qquad \phi: E[p^\infty] \xrightarrow{\sim} \hat{\mathbb{G}}_m \times \mathbb{Q}_p/\mathbb{Z}_p$$

$$\mathcal{A}_{\mathbb{C}_p} = \mathcal{O}(Ig^*_{CS, \mathcal{O}_{\mathbb{C}_p}})[\tfrac{1}{p}].$$

can give Drinfeld moduli interpretation.

$$Ig_{CS} \circlearrowleft \begin{pmatrix} \mathbb{Q}_p^\times & X \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

$X(R)$ = Cont. characters of $\mathbb{Q}_p$ in $R^\times$

$$X(\mathbb{Z}_p) = 1.$$
$$X(\mathcal{O}_{\mathbb{C}_p}) = \lim 1 + m_{\mathbb{C}}$$

$$X = \widetilde{\widehat{\mathbb{G}}}_m \overset{x \mapsto x^{p}}{=} \mathrm{Hom}(\mathbb{Q}_p, \widehat{\mathbb{G}}_m)$$

$$\Omega_{\mathbb{C}_p} \circlearrowleft \begin{pmatrix} \mathbb{Q}_p^\times & X(\mathbb{C}_p) \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}.$$

$$Ig_{Katz} = Ig_{CS} \Big/ \begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$$

so $M_{\mathbb{C}_p} = \Omega_{\mathbb{C}_p}^{\begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}}$.

Get $\mathbb{Z}_p^\times$-action, U_p, Frob $\begin{pmatrix} p^{\pm 1} & 0 \\ 0 & 1 \end{pmatrix}$.

Hecke operators

$$\mathbb{Z}_p(1) \backslash X = \widehat{\mathbb{G}}_m \circlearrowleft Ig_{Katz}.$$

$$Ig_{Katz} = Ig^{\circ}_{CS} \Big/ \begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & 1 \end{pmatrix}.$$

integrates θ.

$$\Omega_{\mathbb{C}_p}[\rho] \circlearrowleft \begin{pmatrix} \mathbb{Q}_p^\times & X(\mathbb{C}_p) \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

If $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible

then

$$\Omega_{\mathbb{C}_p}[\rho] \subseteq C^{bdd}(\mathbb{Q}_p^\times, \mathbb{C}_p). \circlearrowleft \begin{pmatrix} \mathbb{Q}_p^\times & X \\ 0 & 1 \end{pmatrix}.$$

functions which go to 0 at 0 & ∞.

If $\rho|_{G_{\mathbb{Q}_p}}$ reducible.

get same $+ F$ trunc at ∞.

is character of $\mathbb{Q}_p^\times$ corresponding to unit. root U_p-e.v.

of the ordinary for.

Ordinary p-adic mf $\cong \mathbb{A}_{\mathbb{C}_p} \times C(\mathbb{C}_p)$