

p-adic automorphic forms & (big) Igusa varieties.

Goal: Explain a natural space of p-adic automorphic forms to the Katz-Serre of p-adic modular forms + some consequences (classical + philosophical)

Partial reference: "A unipotent circle action on p-adic modular forms". math.UTah.edu/~howe

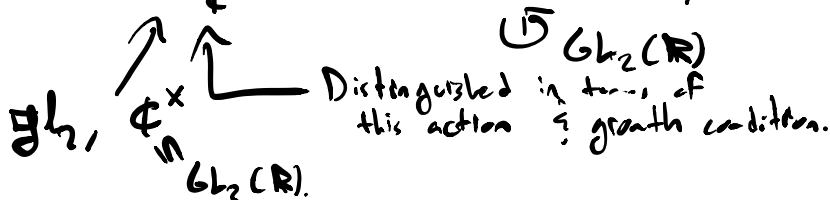
Notation: $K \subseteq GL_2(\mathbb{A}_p)$ compact open subgroup

Y_K modular curve of level K , $\pi: E_{univ} \rightarrow Y_K$

$\omega := \pi_* \Omega_{E_{univ}/Y_K}$ modular line bundle.

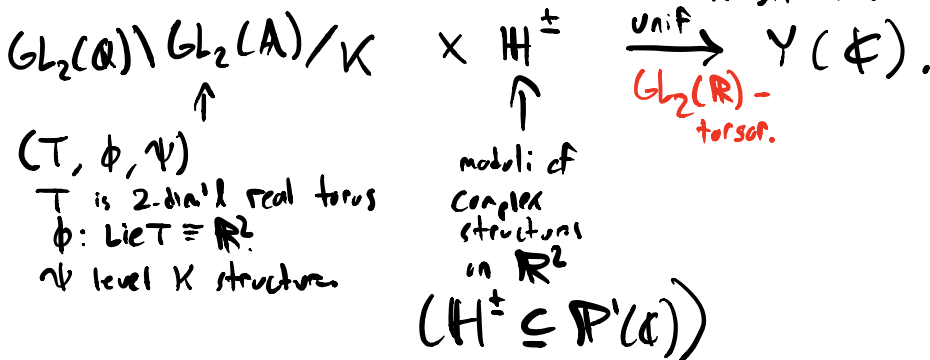
The archimedean story

Automorphic forms: $\Omega_K \subseteq C^\infty(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K, \mathbb{C})$



Modular forms: $M_K \subseteq H^0(Y_K(\mathbb{C}), \omega^K)$
 growth at cusps.

The relation: $Y(\mathbb{C}) = (E, \psi)$, E/\mathbb{C} elliptic curve $\psi =$ level K structure.



$$\text{unif}^* \omega = \pi_2^* \mathcal{O}(-1)$$

$GL_2(\mathbb{R})$ - equivariant identification

Restrict unif to

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K \times \{i\} \xrightarrow{\text{unif}} Y(\Phi).$$

Φ^x -torsion
 \uparrow
 $\text{Stab}(i) \subseteq GL_2(\mathbb{R})$
 action on $\mathcal{O}(-1)|_i$

Choose \dagger trivializing $\mathcal{O}(-1)|_i$

\rightarrow Evaluation map

$$\begin{array}{ccc} M_K & \longrightarrow & \mathcal{A}_\dagger \\ \uparrow & \longmapsto & \frac{\text{unif}^* \dagger}{\dagger^K} \end{array}$$

\leftarrow The image of this map consists exactly of functions which transform via $z \rightarrow z^k$ under $\Phi^x \in GL_2(\mathbb{R})$
 \dagger annihilated by $\mathfrak{n}_i \subseteq \mathfrak{g}_{\mathbb{Z}, \Phi}$
 $(i \in H^{\pm} \subseteq \mathbb{P}^1(\mathbb{F}))$

The p-adic story: $K = GL_2(\mathbb{Z}_p) K^p$ (take $K^p = \Gamma_1(N)$).

2 definitions of p-adic modular forms $M_{\mathbb{Q}_p}$.

Serre: $\sum_K q(L^0(Y(\mathbb{Q}), \omega^K)) \cap \{[c, \dagger]\}$ (in $\mathbb{C}[[T]]$)

power series w/ p-adically bounded coefficients
 \downarrow
 Complete for the sup p-adic norm
 to get $M_{\mathbb{Q}_p}$.

\mathbb{Q}_p -Banach spaces

Katz: $\mathcal{I}_{\mathbb{Q}_p, \text{Katz}} \xrightarrow{\mathbb{Z}_p^x} \hat{Y}_{K, \text{ord}} (\mathbb{C} / S_p \mathbb{F} \mathbb{Z}_p)$

$\mathcal{I}_{\mathbb{Q}_p, \text{Katz}}(\mathbb{R}) = (E, \phi, \psi)$, E/\mathbb{R} elliptic curve
 ψ level K structure.

$\phi: \hat{E} \rightarrow \hat{G}_m$

Comodification \downarrow
 $\mathbb{Z}_p^x \uparrow \mathfrak{g}$

$M_{K, \mathbb{Q}_p} = \mathcal{O}(\mathcal{I}_{\mathbb{Q}_p, \text{Katz}}^*) \left[\frac{1}{n} \right]$

$$M_K \rightarrow M_{\mathbb{Q}_p}$$

$$f \mapsto \frac{f}{\left(\frac{q^x + 1}{t}\right)^k} \leftarrow \text{transform via } z \rightarrow z^k \text{ under } \mathbb{Z}_p^\times.$$

Features of $M_{\mathbb{Q}_p}$:

- Encodes congruences between q -expansions \leftrightarrow Hecke eigenvalues
- Weights: characters of \mathbb{Z}_p^\times .
- Weight families (e.g. Eisenstein, Hida)
- U_p , Frob
- $\theta = q \frac{d}{dq}$ \leftarrow Related to Gauss-Manin

Claim: Like if you only looked at local wt. vectors in $\mathcal{A}_{\mathbb{Q}_p}$.

How to get something better:

Serre perspective: Instead of interpolating spherical vectors, interpolate the whole aut. representation — interpolate full Kirillov models.

Katz perspective: Use a better moduli problem.

$$I_{\theta, \text{CS}}(\mathbb{R}) = (E, \phi, \psi) \quad \phi: E \times \mathbb{P}^1 \rightarrow \widehat{\mathbb{G}_m} \times \mathbb{A}^1 / \mathbb{Z}_p$$

$$\mathcal{A}_{\mathbb{Q}_p} = \mathcal{O}(I_{\theta, \text{CS}, \sigma_{\mathbb{Q}_p}}^\times) \left[\frac{1}{p} \right].$$

\uparrow
can give Dwork moduli interpretation.

$$I_{\theta, \text{CS}} \hookrightarrow \begin{pmatrix} \mathbb{Q}_p^\times & X \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

$$X(\mathbb{R}) = \text{Cont. characters of } \mathbb{Q}_p \text{ in } \mathbb{R}^\times$$

$$X(\mathbb{Z}_p) = 1$$

$$X(\mathcal{O}_{\mathbb{Q}_p}) = \lim_{\leftarrow} 1 + m_{\mathbb{Q}_p}$$

$$X = \tilde{\mathbb{G}}_n \stackrel{x \mapsto x^m}{=} \text{Hom}(\mathbb{Q}_p, \mathbb{C}_n)$$

$$\mathcal{A}_{\mathbb{Q}_p} \hookrightarrow \begin{pmatrix} \mathbb{Q}_p^{\times} & X(\mathbb{C}_p) \\ 0 & \mathbb{Q}_p^{\times} \end{pmatrix}$$

$$\mathbb{I}_{\mathbb{Q}_p}^{\text{Katz}} = \mathbb{I}_{\mathbb{Q}_p}^{\text{CS}} / \begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & \mathbb{Z}_p^{\times} \end{pmatrix}$$

$$\text{so } \mathcal{M}_{\mathbb{Q}_p} = \mathcal{A}_{\mathbb{Q}_p} / \begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & \mathbb{Z}_p^{\times} \end{pmatrix}$$

Get \mathbb{Z}_p^{\times} -action, U_p , Frob $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Hecke operators.

$$\mathbb{Z}_p(1) \backslash X = \hat{\mathbb{G}}_n \text{ } \mathbb{G}^{\times} \mathbb{I}_{\mathbb{Q}_p}^{\text{Katz}}$$

$$\mathbb{I}_{\mathbb{Q}_p}^{\text{Katz}} = \mathbb{I}_{\mathbb{Q}_p}^{\text{CS}} / \begin{pmatrix} 1 & \mathbb{Z}_p(1) \\ 0 & 1 \end{pmatrix}$$

\rightarrow integrator θ .

$$\mathcal{A}_{\mathbb{Q}_p}[\mathbb{Z}_p] \hookrightarrow \begin{pmatrix} \mathbb{Q}_p^{\times} & X(\mathbb{C}_p) \\ 0 & \mathbb{Q}_p^{\times} \end{pmatrix}$$

If $p|G_n$ is irreducible.

then

$$\mathcal{A}_{\mathbb{Q}_p}[\mathbb{Z}_p] \subseteq C^{\text{bdd}}(\mathbb{Q}_p^{\times}, \mathbb{C}_p) \hookrightarrow \begin{pmatrix} \mathbb{Q}_p^{\times} & X \\ 0 & 1 \end{pmatrix}$$

functions which go to 0 at 0 & ∞ .

If $p|G_n$ reducible.

get some $\neq \mathbb{F}$ true at ∞ .

\uparrow
 1) characters of \mathbb{Q}_p^{\times}
 corresponding to
 wt. and 12-e.v.

of the order for.

$$\text{Ordering matrix of } \cong A_{(p \times (q))}$$