On Sharifi's conjectures and generalizations

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- Today: review of Sharifi's conjectures, known results about these, and generalizations to Bianchi 3-manifolds.

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- Need generators and relations for $H_1(X_1(N), C_1(N), \mathbb{Z}')^+$.

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- Complex conjugation : $[u,v] \to [-u,v]$.

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- Conjecture (Sharifi, 2010'):
  1. (Eisenstein quotient) ϖ_N is killed by I_N.
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- Necessary to restrict to absolute homology to get the isomorphism.

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- Take $N = p^n$ prime power ($p \geq 3$). Then

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- We see that Sharifi's conjecture implies the IMC.

Known results

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- Today we briefly survey the works of Fukaya–Kato, L.–Wang and Sharifi–Venkatesh.

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- Roughly, their strategy is to define a Hecke equivariant map $z_N : H_1(X_1(N), C_1(N), \mathbb{Z}_p) \to K_2(Y_1(N)_{\mathbb{Z}[\frac{1}{N}]}) \otimes \mathbb{Z}_p$ sending $[u, v]$ to $\{g_{0,\frac{u}{N}}, g_{0,\frac{v}{N}}\}$, and then "specialize at ∞" to get ϖ_N.

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- Here, $g_{0,\frac{u}{N}}$ is the usual Siegel unit in $\mathcal{O}(Y_1(N))^\times \otimes \mathbb{Z}[\frac{1}{6N}]$. The "leading term" at infinity is $1 - \zeta_N^u$.

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- They need to go to the p-adic tower, take ordinary parts and apply some p-adic regulator.

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- The assumption is that $\prod_{k=1}^{\frac{N-1}{2}} k^k$ vanishes in $(\mathbb{Z}/N\mathbb{Z})^\times \otimes \mathbb{Z}_p$. This quantity was discovered by Merel (1996).

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- Contrary to Fukaya–Kato, their proof do not use Siegel units. It uses the motivic cohomology of $\mathbb{G}_m^2$.
- They also (independently) get results for the map z_N about Siegel units (again, they do not consider U_ℓ^*). They use the motivic cohomology of $\mathcal{E}^2$ where $\mathcal{E}$ is the universal elliptic curve over $Y_1(N)$.

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- $N \in \mathbb{Z}_{>0}$, $\mathfrak{X}_1(N) = \Gamma_1(N\mathcal{O}_K)\backslash\mathfrak{h}_3^*$.
- Cremona gave a presentation by Manin symbols $[u, v]$ of $H_1(\mathfrak{X}_1(N), C_1(N), \mathbb{Z})$, where u, $v \in \mathcal{O}_K/N\mathcal{O}_K$, $\gcd(u, v, N) = 1$ and u, v non-divisible by N.

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- The method of Sharifi–Venkatesh should prove the conjecture for K of class number 1.
- For arbitrary K, there is no explicit presentation of the homology, and Hecke operators are more complicated. This is work in progress with Venkatesh.

Thanks for your attention!