(The weight part of) Serre's Conjecture for GL_2 over totally really fields

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Outline

Lecture 1(-2?): The weight in Serre's Conjecture over $\mathbb{Q}$

Lecture 2(-3?): The algebraic Serre weight conjecture over F

Lectures 3-4: The geometric Serre weight conjecture over F

Main references

Part I: The weight in Serre's Conjecture over $\mathbb{Q}$

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Part II: The algebraic Serre weight conjecture over F

- Kevin Buzzard, Fred Diamond, Frazer Jarvis, *On Serre's conjecture for mod* ℓ *Galois representations over totally real fields*, Duke Math. J., 2010.
- Toby Gee, *Companion forms over totally real fields, II*, Duke Math. J., 2007.
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Part III: The geometric Serre weight conjecture over F

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- Fred Diamond, Shu Sasaki, *A Serre weight conjecture for geometric Hilbert modular forms in characteristic p*, preprint, 2017.
- Fred Diamond, *Geometric weight-shifting operators on Hilbert modular forms in characteristic p*, preprint, 2020.

Serre's Conjecture over $\mathbb{Q}$

Theorem (Khare–Wintenberger)

Suppose that

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is odd, continuous and irreducible.
Then ρ is modular of level $N(\rho)$ and weight $k(\rho)$.

- $N(\rho) =$ prime-to-p Artin conductor;
- $k(\rho)$ depends only on $\rho|_{I_p}$;
- $N(\rho)$ and $k(\rho)$ are (in a sense) minimal;
- the equivalence between "weak" and refined versions (for $p > 2$) was proved first (Mazur, Ribet, Carayol, Gross, Coleman–Voloch, Edixhoven) and used in the proof of the "weak" version.

Serre's recipe for $k(\rho)$

Suppose $2 \leq k \leq p+1$. Then $k(\rho) = k$ if and only if either

$$I) \ \rho|_{I_p} \simeq \begin{pmatrix} \chi^{k-1} & * \\ 0 & 1 \end{pmatrix} \text{ and } * \text{ is } \begin{cases} \text{peu ramifée if } k = 2 \\ \text{très ramifiée if } k = p+1 \end{cases}$$

$$\text{OR} \qquad II) \ \rho|_{I_p} \simeq \omega_2^{k-1} \oplus \omega_2^{p(k-1)} \text{ and } k \leq p,$$

where χ is the cyclotomic character
and ω_2 is a fundamental character of niveau 2,
i.e., $\omega_2(g) = g(\pi)/\pi$, where $\pi^{p^2-1} = p$ (and $\chi = \omega_2^{p+1}$).

Case I (resp. II) occurs only if $\rho|_{G_{\mathbb{Q}_p}}$ is reducible
(resp. irreducible).

Three obvious questions:

Q1: Where does this recipe come from?

Q2: Why assume $k \geq 2$?

Q3: Why assume $k \leq p + 1$?

Some (preliminary) answers:

Q1: Deligne and Fontaine proved that if $2 \leq k \leq p + 1$ and ρ is modular of weight k, then it's of the form above. Can view this as a consequence of p-adic Hodge theory — more on this later.

Q2: (Why $k \geq 2$?) Three (related) answers:

Answer 1: $k(\rho) \geq 2$ in Serre's recipe

Answer 2: There are two notions of modularity
(fix a level N prime to p):

- $\rho = \overline{\rho}_f$ for an eigenform $f \in M_k(N; \mathbb{C}) := H^0(X_1(N), \omega^k)$;
- $\rho = \rho_f$ for an eigenform $f \in M_k(N; \overline{\mathbb{F}}_p) := H^0(X_1(N)_{\overline{\mathbb{F}}_p}, \omega^k)$.

For $k \geq 2$, the notions are equivalent, not for $k = 1$
—more on this later.

Answer 3: Another interpretation of modularity for $k \geq 2$:
The Eichler–Shimura isomorphism implies these are $\Leftrightarrow$

- $\exists$ eigenform $f \in H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}(\overline{\mathbb{F}}_p^2))^\dagger$ such that $T_v f = a_v f$ (and $\langle v \rangle f = d_v f$) for (almost) all $v \nmid pN$, where

$$X^2 - a_v X + d_v v^{k-1}$$

is the characteristic polynomial of $\rho(\mathrm{Frob}_v)$.

† - or $H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathcal{F})$ where $\mathcal{F}$ is the rank two lisse/locally constant $\overline{\mathbb{F}}_p$ sheaf $R^1 s_* \overline{\mathbb{F}}_p$,
where $s : E \to Y_1(N)$ the universal elliptic curve.

Q3: (Why $k \leq p+1$?) Again three (related) answers:

Answer 1: So I could fit the recipe on one slide

Answer 2: For every ρ, there are m such that

$$k(\rho \otimes \chi^{-m}) \leq p+1$$

Answer 3: The full recipe can be reduced to this case.
—More on this next.

Serre weights

Eichler–Shimura suggests another notion of weight (Ash–Stevens, Khare):

Consider the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$:

$$\sigma_{m,n} = \det{}^m \otimes \mathrm{Sym}^n \overline{\mathbb{F}}_p^2, \qquad m \in \mathbb{Z}/(p-1)\mathbb{Z},\ 0 \le n \le p-1.$$

Say ρ is *modular* (of level N) and weight σ if the corresponding system of Hecke eigenvalues arises in $H^1(\Gamma_1(N), \sigma)$

So for $k \geq 2$, the following are equivalent:

- ρ is (algebraically) modular of weight k
- ρ is modular of weight $\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2$
- $\rho \otimes \chi^m$ is modular of weight $\det^m \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2$
- ρ is modular some weight in $JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2)$

Define the *set of Serre weights of ρ* to be:

$$W(\rho) = \{\sigma_{m,n} \,|\, k(\rho \otimes \chi^{-m}) = n+2, \text{ or } 2 \text{ if } n = p-1\,\}$$

(where $m \in \mathbb{Z}/(p-1)\mathbb{Z}$, $0 \leq n \leq p-1$).

Examples:

- $\rho|_{I_p} = \begin{pmatrix} \chi^{n+1} & * \\ 0 & 1 \end{pmatrix}$, non-split, $0 < n < p-1$
  $\Rightarrow W(\rho) = \{\sigma_{0,n}\}$;
- $\rho|_{I_p} = \chi^{n+1} \oplus 1$, $0 < n < p-3$
  $\Rightarrow W(\rho) = \{\sigma_{0,n}, \sigma_{n+1,p-3-n}\}$
- $\rho|_{I_p} = \omega_2^{n+1} \oplus \omega_2^{p(n+1)}$, $0 < n < p-1$
  $\Rightarrow W(\rho) = \{\sigma_{0,n}, \sigma_{n,p-1-n}\}$.

Then $W(\rho)$ determines $k(\rho)$ as follows:

For $\sigma = \sigma_{m,n}$, let $k_\sigma = \min\{\, k \geq 2 \mid \sigma \in JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2) \,\}$.

Theorem (Wiersema - direct proof)

If $0 \leq m \leq p-2$ *and* $0 \leq n \leq p-1$*, then*

$$k_{\sigma_{m,n}} = \begin{cases} m(p+1)+n+2, & \text{if } m+n < p-1; \\ m(p+1)+(n+2)p+1-p^2, & \text{if } m+n \geq p-1. \end{cases}$$

Therefore Serre's $k(\rho) = \min\{\, k_\sigma \mid \sigma \in W(\rho) \,\}$

$$= \min\{\, k \geq 2 \mid JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2) \cap W(\rho) \neq \emptyset \,\}.$$

This reduces the weight part of Serre's Conjecture to the case $2 \leq k(\rho) \leq p+1$.

(Alternatively, use θ-cycles — more on this later.)

Low weight cases are treated using "companion forms" theorems and Mazur's Principle.

p-adic Hodge theory

Returning to Q1 (where does the recipe come from?):

Suppose $\rho : G_{\mathbb{Q}_p} \to \mathrm{Aut}_E(V)$ of dimension d, $\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p$.

$$D_{\mathrm{crys}}(V) := (V \otimes B_{\mathrm{crys}})^{G_{\mathbb{Q}_p}}$$

is a filtered E-vector space of dimension $\leq d$
(where B_{crys} is Fontaine's crystalline period ring).

Say V is *crystalline* if $\dim_E D_{\mathrm{crys}}(V) = d$, and
its *Hodge-Tate (HT) weights* are the i such that
$\mathrm{gr}^{-i} D_{\mathrm{crys}}(V) \neq 0$.

Theorem (Fontaine-Laffaille, Berger-Li-Zhu)

Suppose that $2 \leq k \leq p+1$.
Then $\rho|_{G_{\mathbb{Q}_p}}$ *has a crystalline lift with HT weights* $\{0, k-1\}$ *if and only if either* $k = k(\rho)$, *or* $k = p+1$ *and* $k(\rho) = 2$.

Corollary

$W(\rho) =$
$\{\, \sigma_{m,n} \mid \rho|_{G_{\mathbb{Q}_p}}$ *has a crystalline lift with HT weights* $\{m, m+n+1\}.\,\}$.

The (algebraic) Serre Weight Conjecture becomes:

The following are equivalent:

- ρ *is modular of weight* $\sigma_{m,n}$ *and level prime to* p
- $\rho|_{G_{\mathbb{Q}_p}}$ *has a crystalline lift with HT weights* $\{m, m+n+1\}$.

Combining the corollary with (a corollary of) the Breuil-Mezard Conjecture gives:

Theorem (Kisin, Paskunas, Hu-Tan, Tung)

Suppose that $k \geq 2$. Then $\rho|_{G_{\mathbb{Q}_p}}$ has a crystalline lift of with HT weights $\{0, k-1\}$ if and only $W(\rho) \cap JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2) \neq \emptyset$.

Combining this with Wiersema's formula gives (a purely local proof) of:

Corollary

$k(\rho) =$

$\min\{k \geq 2 \,|\, \rho|_{G_{\mathbb{Q}_p}}$ *has a crystalline lift with HT weights* $\{0, k-1\}\}$

The geometric variant

Returning to Q2: What about $k = 1$?

Recall we had two notions of modularity
(both equivalent to algebraic modularity if $k \geq 2$):

- ρ arises from $M_k(N;\mathbb{C}) = M_k(N;\mathbb{Z}[1/N]) \otimes \mathbb{C}$;
- ρ arises from $M_k(N;\overline{\mathbb{F}}_p)(\hookleftarrow M_k(N;\mathbb{Z}[1/N]) \otimes \overline{\mathbb{F}}_p)$.

For $k = 1$, the first notion isn't characterized by $\rho|_{I_p}$,
so Edixhoven uses the second;
call this *geometric modularity* of weight k (and level N).

Define:

$$k_{\mathrm{geom}}(\rho) = \begin{cases} 1, & \text{if } \rho \text{ is unramified at } p; \\ k(\rho), & \text{otherwise.} \end{cases}$$

$$= \min\{\, k \geq 1 \mid \rho|_{G_{\mathbb{Q}_p}} \text{ has a crystalline lift with HT weights } \{0, k-1\}\}$$

The Geometric Serre Weight Conjecture is then:

Suppose $k \geq 1$. Then the following are equivalent:

- *ρ is geometrically modular of weight k and level prime to p*
- *$\rho|_{G_{\mathbb{Q}_p}}$ has a crystalline lift with HT weights $\{0, k-1\}$*
- *$k \in k_{\mathrm{geom}}(\rho) + t(p-1)$ for some $t \in \mathbb{Z}_{\geq 0}$.*

Geometric weight-shifting

The Hasse invariant:

Verschiebung on the universal E over $\overline{Y}_1(N) = Y_1(N)_{\overline{\mathbb{F}}_p}$ induces $\omega \to \omega^p$, or equivalently

$$H \in M_{p-1}(\Gamma_1(N); \overline{\mathbb{F}}_p).$$

Multiplication by H defines Hecke-equivariant:

$$M_k(N; \overline{\mathbb{F}}_p) \to M_{k+p-1}(N; \overline{\mathbb{F}}_p).$$

So ρ geometrically modular of weight k
$\Rightarrow \rho$ geometrically modular of weight $k + p - 1$.

Katz's qd/dq-operator:

The Gauss–Manin connection $\omega \to \Omega^1_{\overline{Y}_1(N)/\overline{\mathbb{F}}_p} \otimes \omega^{-1}$ induces $KS : \omega^2 \cong \Omega^1_{\overline{X}_1(N)/\overline{\mathbb{F}}_p}(\mathrm{cusps})$.

Use this to define:

$$\Theta : M_k(N, \overline{\mathbb{F}}_p) \to M_{k+p+1}(N; \overline{\mathbb{F}}_p).$$

with the following properties:

- twists the action of T_v by v;
- has image in $H \cdot M_{k+2}(N; \overline{\mathbb{F}}_p)$ if $p|k$;
- $\Theta^p = H^{p+1}\Theta$.

So ρ geometrically modular of weight k
$\Rightarrow \chi \otimes \rho$ geometrically modular of weight $k + p + 1$
(in fact $k + 2$ if $p|k$).

Recall $\rho \otimes \chi^{-m}$ is modular of weight $\leq p + 1$ for some m
(for which Edixhoven also gives a geometric proof).

An elementary analysis of possible "Θ-cycles"
reduces the proof of the (geometric) Serre weight
conjecture to the case $1 \leq k_{\mathrm{geom}}(\rho) \leq p$,
which is then completed by an extension
of the companion forms theorem to $k_{\mathrm{geom}}(\rho) = 1$.

Hilbert modular forms

Let F be a totally real field, $d = [F : \mathbb{Q}] > 1$, ring of integers $\mathcal{O}_F$.
Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$,
so $\Sigma := \{ F \hookrightarrow \overline{\mathbb{Q}} \} := \coprod_{v|p} \Sigma_v$.
For open compact $U \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$, let

$$Y_U = \mathrm{GL}_2(F)_+ \backslash (\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U$$

denote the Hilbert modular variety of level U.

In particular, let $Y_1(\mathfrak{n}) = Y_{U_1(\mathfrak{n})}$ and $Y(\mathfrak{n}) = Y_{U(\mathfrak{n})}$.

- coarse moduli space for HBAV's with additional structure;
- Y_U has dimension d, smooth for sufficiently small U;
- canonical model over $\mathbb{Q}$, action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ on $\varprojlim_U Y_U$;
- components of $Y_{U_1(\mathfrak{n})} \leftrightarrow$ strict class group fo F.

Suppose $\vec{k}, \vec{m} \in \mathbb{Z}^{\Sigma}$ and $w = k_\theta + 2m_\theta$ is independent of θ.
(In particular $\vec{k}$ is paritious.)
Then can define an automorphic line bundle $\mathcal{A}_{\vec{k},\vec{m}}$ on Y_U.

Define the space of Hilbert modular forms of weight $(\vec{k}, \vec{m})$ and level U:

$$M_{\vec{k},\vec{m}}(U, \mathbb{C}) = H^0(Y_U, \mathcal{A}_{\vec{k},\vec{m}})$$

and of level $\mathfrak{n}$:

$$M_{\vec{k},\vec{m}}(\mathfrak{n}, \mathbb{C}) = M_{\vec{k},\vec{m}}(U_1(\mathfrak{n}), \mathbb{C})$$

Equipped with a Hecke action, in particular T_v, S_v for $v \nmid \mathfrak{n}$.

Theorem (many people)

Suppose that $f \in M_{\vec{k},\vec{m}}(\mathfrak{n},\mathbb{C})$ *is such that* $T_v f = a_v f, S_v f = d_v f$ *for all* $v \nmid \mathfrak{n}$*. Then there exists unique semisimple (irreducible* $\Leftrightarrow$ f *cuspidal)*

$$\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that for all $v \nmid p\mathfrak{n}$*,* ρ_f *is unramified at* v*, and* $\rho_f(\mathrm{Frob}_v)$ *has char. poly.*

$$X^2 - a_v X + d_v \mathrm{Nm}_{F/\mathbb{Q}}(v).$$

Furthermore if $k_\theta \geq 2$ *for all* $\theta \in \Sigma$ *and* $v|p$*, then* $\rho_f|_{G_{F_v}}$ *is de Rham (crystalline* $\Leftrightarrow$ $v \nmid \mathfrak{n}$*) with* θ*-labelled*[†] *HT weights* $\{m_\theta, k_\theta + m_\theta - 1\}$ *for* $\theta \in \Sigma_v$*.*

† - $D_{HT}(V) = (V \otimes B_{HT})^{G_{F_v}}$, where $B_{HT} = \oplus \mathbb{C}_p(i)$, is free rank 2 over $F_v \otimes \overline{\mathbb{Q}}_p = \oplus_{\theta\in\Sigma_v}\overline{\mathbb{Q}}_p$.

Conjecture (Fontaine–Mazur–Langlands)

Every totally odd, irreducible, geometric $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *is isomorphic to* ρ_f *for some* f *as above.*

Conjecture ("Weak" Serre)

Every totally odd, irreducible $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *is isomorphic to* $\overline{\rho}_f$ *for some* f *as above.*

Refined Serre Conjecture:
What can we say about $\mathfrak{n}$, $\vec{k}$ (and $\vec{m}$)?

Minimal prime-to-p part of $\mathfrak{n}$ should be Artin conductor of ρ.
(Known, at least under Taylor–Wiles hypothesis.)

What about $\vec{k}$?

Again this should be determined by $\rho|_{I_v}$ for $v|p$, but some significant differences:

Not all ρ can arise from forms f of level prime to p. (Such ρ necessarily have $\det \rho|_{I_v} = \chi^{w-1}$ for all $v|p$.)

No obvious notion of minimality (since $\vec{k} \in \mathbb{Z}^{\Sigma}$).

Two approaches:

- Algebraic: Make sense of "modularity of weight σ" for arbitrary irreducible $\overline{\mathbb{F}}_p$-representations σ of $\mathrm{GL}_2(\mathcal{O}_F/p)$, and describe $W(\rho)$ in terms of $\rho|_{I_v}$ for $v|p$.
- Geometric: Interpret modularity in terms of geometrically defined mod p Hilbert modular forms —more on this in later talks.

The algebraic Serre weight Conjecture

For simplicity, assume there is a unique $\mathfrak{p}|p$ in $\mathcal{O}_F$.
Let $k = \mathcal{O}_F/\mathfrak{p}$, $f = [k : \mathbb{F}_p]$,
K_0 maximal unramified subextension of $K = F_v$,
$e = [K : K_0]$, so $d = ef$. Let

$$\Sigma_0 = \{ K_0 \hookrightarrow \overline{\mathbb{Q}}_p \} \leftrightarrow \{ k \hookrightarrow \overline{\mathbb{F}}_p \} = \{ \tau_0, \ldots, \tau_{f-1} \}$$

where $\tau_i = \tau \circ \phi^i$ for $i \in \mathbb{Z}/f\mathbb{Z}$, and arbitrarily choose an ordering $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,e}$ of the extensions of τ_i to K, so

$$\Sigma = \{ \theta_{i,j} \,|\, i = 0, \ldots, f-1, \ j = 1, \ldots, e \}.$$

Recall the irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$ (or equivalently $\mathrm{GL}_2(\mathcal{O}_F/p)$ are:

$$\sigma_{\vec{m},\vec{n}} = \bigotimes_{i=1}^{f} \det{}^{m_i} \mathrm{Sym}^{n_i} k^2 \otimes_{k,\tau_i} \overline{\mathbb{F}}_p,$$

where $\vec{m}, \vec{n} \in \mathbb{Z}^f = \mathbb{Z}^{\Sigma_0}$, $0 \leq n_i \leq p-1$ for all i.

$$\sigma_{\vec{m},\vec{n}} \sim \sigma_{\vec{m}',\vec{n}'} \qquad \iff$$

$$\vec{n} = \vec{n}' \quad \text{and} \quad \textstyle\sum_i m_i p^i \equiv \sum_i m_i' p^i \bmod p^f - 1.$$

Could consider $H^i(Y_1(\mathfrak{n}), \mathcal{F})$ for suitable locally constant $\overline{\mathbb{F}}_p$-sheaves $\mathcal{F}$, but interesting degree is $i = d > 1$, which introduces complications, so use Jacquet–Langlands to reinterpret modularity (in characteristic zero):

Let D be a quaternion algebra over F unramified at $\mathfrak{p}$ and exactly one infinite place.

Let Y_U^D be the associated Shimura curve
(for sufficiently small $U = U^{\mathfrak{p}} U_{\mathfrak{p}}$, $U_{\mathfrak{p}} \cong \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{p}})$).

(Could just as well work with totally definite D
and 0-dimensional Y_U^D.)

For each $\sigma = \sigma_{\vec{m},\vec{n}}$, can define a locally constant $\overline{\mathbb{F}}_p$-sheaf $\mathcal{F}_\sigma = \mathcal{F}_{\vec{m},\vec{n}}$ on Y_U^D, and an action of a Hecke algebra $\mathbb{T}$ (generated by T_v and S_v for all but finitely many $v \neq \mathfrak{p}$) on

$$H^1(Y_U^D, \mathcal{F}_\sigma).$$

Given $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, say ρ is modular of weight σ (and level U, w.r.t. D) if $\mathfrak{m}_\rho$ is in its support, i.e.,

$$H^1(Y_U^D, \mathcal{F}_\sigma)_{\mathfrak{m}_\rho} \cong \mathrm{Hom}_{\mathrm{GL}_2(k)}(\sigma, H^1(Y_{U \cap U(p)}^D, \overline{\mathbb{F}}_p)_{\mathfrak{m}_\rho}) \neq 0$$

where $\mathfrak{m}_\rho \subset \mathbb{T}$ is generated by

$$T_v - \mathrm{tr}(\rho(\mathrm{Frob}_v)), \qquad \mathrm{Nm}_{F/\mathbb{Q}}(v) S_v - \det(\rho(\mathrm{Frob}_v))$$

for all but finitely many v. Equivalently $H^1(Y_U^D, \mathcal{F}_\sigma)[\mathfrak{m}_\rho] \neq 0$.

Let

$$W^D_{\mathrm{mod}}(\rho) = \{\, \sigma = \sigma_{\vec{m},\vec{n}} \mid \rho \text{ is modular of weight } \sigma \text{ w.r.t. } D \,\}$$

Then $W^D_{\mathrm{mod}}(\rho)$

- should depend only on $\rho|_{I_K}$ (unless $\mathrm{Disc}(D)$ is incompatible with ρ, in which case it's $\emptyset$);
- is the set of isomorphism classes of irreducibles appearing in the $U_{\mathfrak{p}}$-socle of $\varinjlim_V H^1(Y_V^D, \overline{\mathbb{F}}_p)[\mathfrak{m}_\rho]$;
- determines the possible $U_{\mathfrak{p}}$-types of local factors $\pi_{\mathfrak{p}}$ of $\pi \leftrightarrow f$ giving rise to ρ for each weight $(\vec{k}, \vec{m})$ with all $k_\theta \geq 2$.

Let $W(\rho) =$

$$\left\{ \sigma_{\vec{m},\vec{n}} \;\middle|\; \begin{array}{c} \rho|_{G_K} \text{ has a crystalline lift with } \theta_{i,j}\text{-labeled HT-weights} \\ \{m_i, m_i+n_i+1\} \text{ if } j=1, \text{ and } \{0,1\} \text{ if } j>1. \end{array} \right\}.$$

Conjecture

If D is compatible with ρ, then $W^D_{\mathrm{mod}}(\rho) = W(\rho)$.

- Formulation is due to Gee, generalizing more explicit definitions of Buzzard-D-Jarvis (for p unramified in F) and Schein (for $\rho|_{G_K}$ semisimple).
- $W(\rho)$ can be made more explicit in general (D-Dembélé-Roberts, Steinmetz).
- The conjecture is proved by Gee + collaborators, assuming ρ is modular and satisfies TW-hypothesis.

Example: $d = f = 2$

$$\rho|_{G_K} = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix},$$

$\psi|_{I_K} = \omega^{a_0 + a_1 p}$, $1 \leq a_0, a_1 \leq p$, not both 1,
where $\omega : I_K \xrightarrow{\omega_2} k^\times \xrightarrow{\tau_0} \overline{\mathbb{F}}_p^\times$.

Then $\sigma_{\vec{0},\vec{n}} \in W(\rho)$, where $\vec{n} = (a_0 - 1, a_1 - 1)$.
But there may be more, depending on $*$.

Suppose for simplicity $\psi|_{I_K} \neq 1 = \omega^{p^2-1}$ ($\leftrightarrow a_0 = a_1 = p - 1$)
and $\psi|_{I_K} \neq \chi = \omega^{p+1}$ ($\leftrightarrow a_0 = a_1 = p$).

Then $* \leftrightarrow c_\rho \in H^1(G_K, \overline{\mathbb{F}}_p(\psi))$, dimension $2 = [K : \mathbb{Q}_p]$

Can rewrite:

$$\rho|_{I_K} = \omega^{-a_0'} \otimes \begin{pmatrix} \omega^{pa_1'} & * \\ 0 & \omega^{a_0'} \end{pmatrix}$$

for a unique (a_0', a_1') with $1 \leq a_0', a_1' \leq p$,
$(= (p - a_0, a_1 + 1)$ if $a_0, a_1 < p)$.

Analysis of crystalline liftability shows:

$$\sigma_{\vec{m}', \vec{n}'} \in W(\rho) \iff c_\rho \in L'$$

where $\vec{m}' = (-a_0', 0)$, $\vec{n}' = (a_0' - 1, a_1' - 1)$, and L' is a one-dimensional subspace of $H^1(G_K, \overline{\mathbb{F}}_p(\psi))$.

Similarly get another weight $\sigma_{\vec{m}'', \vec{n}''} \in W(\rho) \iff c_\rho \in L''$ for a one-dimensional L'' (which $= L' \Leftrightarrow a_0$ or $a_1 = p$),

and yet another weight if $\rho|_{G_K}$ splits (i.e., $c_\rho = 0$).

Strategy of Gee, et al for proving $W(\rho) = W^D_{\mathrm{mod}}(\rho)$:

- Use automorphy lifting theorems to prove existence and automorphy of potentially Barsotti–Tate lifts (i.e. $\leftrightarrow \vec{k} = \vec{2}$) with prescribed local behavior at p.
- Play off the relation between weights and types implicit in the Breuil–Mézard Conjecture to get an equality $W^D_{\mathrm{mod}}(\rho) = W_{\mathrm{BT}}(\rho) \subset W(\rho)$.
- Weight elimination: use integral p-adic Hodge theory to prove $W(\rho) \subset W_{\mathrm{BT}}(\rho)$.

Toy example: Companion forms for $F = \mathbb{Q}$

Suppose $\rho|_{I_p} \sim \chi^{k-1} \oplus 1$, $3 \le k \le p-1$.

Then $W(\rho) = \{ \sigma_{0,k-2}, \sigma_{k-1,p-1-k} \}$ (and $\sigma_{p-2,p-1}$ if $k = p-1$).

Automorphy lifting theorems
$\Rightarrow \rho$ modular of weight 2, level p, character $\tilde{\chi}^{k-2}$,
i.e., type $\mathrm{Ind}_{\mathrm{Iw}}^{\mathrm{GL}_2(\mathbb{Z}_p)}(1 \otimes \tilde{\chi}^{k-2})$.

$\Rightarrow \rho$ modular of some weight in

$$JH(\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes \chi^{k-2})) = \{ \sigma_{0,k-2}, \sigma_{k-2,p+1-k} \}.$$

But $\sigma_{k-2,p+1-k} \notin W(\rho)$,
so ρ is modular of weight of $\sigma_{0,k-2}$.

The geometric setting

Take $L \subset \overline{\mathbb{Q}}_p$ is sufficiently large (containing $\theta(F)$ for all $\theta \in \Sigma$), and let $\mathcal{O} = \mathcal{O}_L$, residue field $\mathbb{F}$.

Recall Y_U is a moduli space for HBAV's,
i.e., abelian varieties of dimension d with $\mathcal{O}_F$-action
and level U-structure.

Pappas–Rapoport define a smooth model for Y_U over $\mathcal{O}$.

To ease notation, continue to assume there's a unique $\mathfrak{p}|p$.

Pappas–Rapoport filtrations

Consider the functor on locally Noetherian $\mathcal{O}$-schemes:
$S \rightsquigarrow$ isomorphism classes of $(A, \iota, \lambda, \eta, \mathcal{F}^\bullet)/S$, where:

- $s : A \to S$ is an abelian scheme of dimension d;
- $\iota : \mathcal{O}_F \to \mathrm{End}_S(A)$;
- λ is an $\mathcal{O}_F$-quasi-polarization of degree prime-to-p;
- η is a level U-structure;
- for each $\tau = \tau_i \in \Sigma_0$, a filtration

$$0 = \mathcal{F}_i^{(0)} \subset \mathcal{F}_i^{(1)} \subset \cdots \subset \mathcal{F}_i^{(e-1)} \subset \mathcal{F}_i^{(e)} = (s_*\Omega^1_{A/S})_{\tau_i}$$

such that for $j = 1, \ldots, e$, the quotient

$$\mathcal{L}_{i,j} := \mathcal{F}_i^{(j)} / \mathcal{F}_i^{(j-1)}$$

is a line bundle on S on which $\mathcal{O}_F$ acts via $\theta_{i,j}$.

This is representable by a scheme $\widetilde{\mathcal{Y}}_U$:

- smooth of relative dimension d over $\mathcal{O}$;
- complex points $\mathrm{SL}_2(\mathcal{O}_{F,(p)})\backslash(\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})/U^p$;
- components $\longleftrightarrow (\mathbb{A}^{(p)}_{F,\mathbf{f}})^\times / \det(U^p)$ (infinite);
- $\mathcal{O}^\times_{F,(p),+}/(U \cap \mathcal{O}^\times_F)^2$ acts freely (via polarization).

The quotient $\mathcal{Y}_U$ is a smooth model for Y_U.

$\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ acts on $\varprojlim_U \mathcal{Y}_U$.

We're interested in $\overline{Y}_U := \mathcal{Y}_{U,\mathbb{F}}$.

Mod p Hilbert modular forms

Recall we have (universal) line bundles $\mathcal{L}_{\theta_{i,j}} = \mathcal{L}_{i,j}$ on $\widetilde{\mathcal{Y}}_U$.

Can also define (trivial) line bundles $\mathcal{N}_\theta$ on $\widetilde{\mathcal{Y}}_U$ so that

$$\bigotimes_{\theta\in\Sigma} \mathcal{L}_\theta^{k_\theta} \mathcal{N}_\theta^{m_\theta}$$

identifies with the pull-back of $\mathcal{A}_{\vec{k},\vec{m}}$ (over $\widetilde{\mathcal{Y}}_{U,\mathbb{C}}$).

More generally if R is an $\mathcal{O}$-algebra in which

$$\prod_\theta \theta(\mu)^{k_\theta+2m_\theta} = 1$$

for all $\mu \in U \cap \mathcal{O}_F^\times$, then $\bigotimes_{\theta\in\Sigma} \mathcal{L}_\theta^{k_\theta} \mathcal{N}_\theta^{m_\theta}$ descends canonically to a line bundle $\mathcal{A}_{\vec{k},\vec{m},R}$ on $\mathcal{Y}_{U,R}$.

In particular for all sufficiently small U have:

$$\overline{\mathcal{A}}_{\vec{k},\vec{m}} := \mathcal{A}_{\vec{k},\vec{m},\mathbb{F}}$$

on $\overline{Y}_U$ for all $\vec{k}, \vec{m}$.

Define the space of mod p Hilbert modular forms of weight $(\vec{k}, \vec{m})$ and level U to be:

$$M_{\vec{k},\vec{m}}(U;\mathbb{F}) = H^0(\overline{Y}_U, \overline{\mathcal{A}}_{\vec{k},\vec{m}})$$

Get a natural action on $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ on

$$\varinjlim_{U} M_{\vec{k},\vec{m}}(U;\mathbb{F}).$$

In particular operators T_v and S_v for all but finitely many v.

Partial Hasse invariants

Generalizations of the classical Hasse invariant.
Defined in this setting by Reduzzi–Xiao
(buliding on Goren, Andreatta–Goren).

Choose a uniformizer ϖ of $K = F_{\mathfrak{p}}$.
Then $\iota(\varpi) : A \to A$ over $\widetilde{\mathcal{Y}}_{U,\mathbb{F}}$ induces

$$\overline{\mathcal{F}}_i^{(j)} \to \overline{\mathcal{F}}_i^{(j-1)}$$

for $j = 1, \ldots, e$, and hence

$$\overline{\mathcal{L}}_{i,j} \to \overline{\mathcal{L}}_{i,j-1}$$

for $j = 2, \ldots, e$, i.e. a section $H_{i,j}$ of $\overline{\mathcal{L}}_{i,j}^{-1}\overline{\mathcal{L}}_{i,j-1}$.

On the other hand if $j = 1$, then $\mathrm{Ver} : A^{(p)} \to A$ induces

$$\overline{\mathcal{L}}_{i,e} \longrightarrow \overline{\mathcal{L}}_{i-1,e}^{p},$$

which factors uniquely as $H_{i,1} \circ H_{i,2} \circ \cdots \circ H_{i,e}$,
where $H_{i,1}$ is a section of $\overline{\mathcal{L}}_{i,1}^{-1}\overline{\mathcal{L}}_{i-1,e}^{p}$.

The $H_{i,j}$ descend to $\overline{Y}_U$, so define elements

$$H_\theta \in M_{\vec{h}_\theta,\vec{0}}(U;\mathbb{F}),$$

where $\vec{h}_\theta = n_\theta \vec{e}_{\sigma^{-1}\theta} - \vec{e}_\theta$, (i.e., $(\cdots, 0, p \,\text{or}\, 1, -1, 0, \cdots)$)
σ is the "right-shift" cyclic permutation of Σ:

$$(1,1) \mapsto (1,2) \mapsto \cdots \mapsto (1,e) \mapsto (2,1) \mapsto \cdots$$

and $n_{\theta_{i,j}} = n_{i,j} = \begin{cases} p & \text{if } j = 1; \\ 1 & \text{if } j > 1. \end{cases}$

Furthermore the H_θ are $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$-invariant.

Associated Galois representations

Theorem (Goldring–Koskivirta, Emerton–Reduzzi–Xiao, D–Sasaki)

Suppose that $f \in M_{\vec{k},\vec{m}}(U(\mathfrak{n}), \mathbb{F})$ *is such that* $T_v f = a_v f, S_v f = d_v f$ *for all* $v \nmid \mathfrak{p}\mathfrak{n}$.
Then there exists a unique semisimple

$$\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

such that for all $v \nmid p\mathfrak{n}$, ρ_f *is unramified at* v, *and* $\rho_f(\mathrm{Frob}_v)$ *has characteristic polynomial*

$$X^2 - a_v X + d_v \mathrm{Nm}_{F/\mathbb{Q}}(v).$$

- The strategy:
  Multiply by partial Hasse invariants to shift to a weight $(\vec{k}', \vec{m}')$ such that ampleness implies forms lift to characteristic zero.
- The obstacle:
  It might not be possible to do this so that $\vec{k}'$ paritious.
- The idea:
  Lift to a form with paritious weight and level $U \cap U_1(\mathfrak{p})$.

Sketch of proof:

Suppose for now p is unramified (for simplicity).

Twist to reduce to the case $\vec{m} = \vec{0}$,
and multiply by H_θ's so that $\vec{k} = \vec{N} - \vec{\delta}$,
where $N >> 0$ and $0 \leq \delta_\theta \leq p - 1$ for all θ (but $\vec{\delta} \neq \vec{p} - \vec{1}$).

Let $\mathcal{Y} = \mathcal{Y}_U$, and consider the models $\mathcal{Y}_i$ defined
by Pappas for HMV's of level $U \cap U_i(\mathfrak{p})$, $i = 0, 1$. So

$$\pi : \mathcal{Y}_1 \xrightarrow{\pi_1} \mathcal{Y}_0 \xrightarrow{\pi_0} \mathcal{Y},$$

$\mathcal{Y}_0$ is flat l.c.i. over $\mathcal{O}$ and π_1 is finite flat (but π_0 is neither).

In particular the $\mathcal{Y}_i$ are Cohen–Macaulay,
so have relative dualizing sheaves $\mathcal{K}_i$,
isomorphic to $\mathcal{A}_{\vec{2},-\vec{1}}$ over L (by Kodaira–Spencer).

Use the canonical section $\overline{Y} \hookrightarrow \overline{Y}_0$ and isomorphism

$$\overline{\pi}_{1,*}\overline{\mathcal{K}}_1 \cong \mathcal{H}om_{\mathcal{O}_{\overline{Y}_0}}(\overline{\pi}_{1,*}\mathcal{O}_{\overline{Y}_1}, \overline{\mathcal{K}}_0)$$

to get $\overline{\mathcal{A}}_{-\vec{\delta}+\vec{2},-\vec{1}} \hookrightarrow \overline{\pi}_*\overline{\mathcal{K}}_1$, and so

$$M_{\vec{k},\vec{0}}(U;\mathbb{F}) \hookrightarrow H^0(\overline{Y}_1, \overline{\mathcal{K}}_1 \otimes \overline{\pi}^*\overline{\mathcal{A}}_{\vec{N}-\vec{2},\vec{1}}).$$

Use ampleness of $\mathcal{A}_{\vec{N},\vec{0}}$ on the minimal compactification of Y_U, and the vanishing of $R^1\pi_*\mathcal{K}_1$ (D–Kassaei–Sasaki) to prove the image is contained in that of reduction:

$$\begin{array}{rcl} M_{\vec{N},\vec{0}}(U_1(\mathfrak{p}),\mathcal{O}) & := & H^0(Y_1, \mathcal{K}_1 \otimes \pi^*\mathcal{A}_{\vec{N}-\vec{2},\vec{1}}) \\ & \longrightarrow & H^0(\overline{Y}_1, \overline{\mathcal{K}}_1 \otimes \overline{\pi}^*\overline{\mathcal{A}}_{\vec{N}-\vec{2},\vec{1}}). \end{array}$$

All the maps are Hecke-equivariant, so Deligne–Serre lifting lemma gives a characteristic zero eigenform whose associated Galois representation has the desired reduction.

In p is ramified, the main changes are:

- (Emerton–Reduzzi–Xiao) Shift $(\vec{N}, \vec{0})$ by $M(2\vec{\epsilon}, -\vec{\epsilon})$ with $M >> 0$ and $\epsilon_{i,j} = j$ to get an ample bundle.
- It's the (powers of) $\theta_{i,e}$ that appear in $\pi_* \overline{\mathcal{K}}_1$.
- Only get that the image of $S_{\vec{k},\vec{m}}(U; \mathbb{F})$ is contained in the image of reduction, so need to argue separately for the contribution from cusps.

Geometric modularity

We say $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is geometrically modular of weight $(\vec{k}, \vec{m})$ if $\rho \sim \rho_f$ for some level U prime to p and Hecke eigenform $f \in M_{\vec{k},\vec{m}}(U; \mathbb{F})$.

Given ρ, what is the set of weights for which ρ is modular?

Expect the answer to be related to:

- set of $(\vec{k}, \vec{m})$ such that $\rho|_{G_K}$ has a crystalline lift with θ-labeled HT weights $(m_\theta, m_\theta + k_\theta - 1)$ for each $\theta \in \Sigma$.
- set of $(\vec{k}, \vec{m})$ such that ρ is algebraically modular of weight $(\vec{k}, \vec{m})$,i.e., of some weight in

$$JH\left(\bigotimes_{i,j} \det{}^{m_{i,j}} \mathrm{Sym}^{k_{i,j}-2} k^2 \otimes_{k,\tau_i} \overline{\mathbb{F}}_p\right).$$

Warning: The naive conjectures are false!

Recall (multiplication by) $H_\theta \in M_{\vec{h}_\theta,\vec{0}}(U;\mathbb{F})$ defines a Hecke-equivariant injective map:

$$M_{\vec{k},\vec{m}}(U;\mathbb{F}) \longrightarrow M_{\vec{k}+\vec{h}_\theta,\vec{m}}(U;\mathbb{F})$$

Write $\vec{k} \leq_{\mathrm{Ha}} \vec{k}'$ if $\vec{k}' = \vec{k} + \sum_\theta b_\theta \vec{h}_\theta$ for some $\vec{b} \in \mathbb{Z}^{\Sigma}_{\geq 0}$.

So if ρ geometrically modular of weight $(\vec{k},\vec{m})$ and $\vec{k} \leq_{\mathrm{Ha}} \vec{k}'$, then ρ geometrically modular of weight $(\vec{k}',\vec{m})$.

Find that ρ may be geometrically modular of weight $(\vec{k},\vec{m})$

- with all $k_\theta \geq 2$, but not algebraically modular of some JH-factor of the corresponding weight (shifting by $(\vec{h}_\theta,\vec{0})$ can lose JH-factors).
- with some $k_\theta = 1$, but $\rho|_{G_K}$ has no crystalline lift with the corresponding labeled weights (Bartlett)

Minimal weights

For non-zero $f \in M_{\vec{k},\vec{m}}(U;\mathbb{F})$, can define its minimal weight (Adreatta–Goren, Deo–Dimitrov-Wiese, D–Kassaei):

The divisors of the partial Hasse invariants H_θ:

- meet every irreducible component of $\overline{Y}_U$,
- have no common irreducible components, so

$$\left\{ \vec{r} \in \mathbb{Z}^\Sigma \;\middle|\; \prod_{\theta \in \Sigma} H_\theta^{-r_\theta} f \in M_{\vec{k}-\sum_\theta r_\theta \vec{h}_\theta,\vec{m}}(U;\mathbb{F}) \right\}$$

has a unique maximal element $\vec{r}$, and let

$$\vec{k}_{\min}(f) = \vec{k} - \sum_\theta r_\theta \vec{h}_\theta.$$

Define the minimal cone by:

$$\Xi^{\min} = \{\, \vec{x} \in \mathbb{Z}^{\Sigma} \mid x_{\sigma^{-1}\theta} \leq n_\theta x_\theta \,\}.$$

Then $\Xi^{\min} \subset \mathbb{Z}^{\Sigma}_{\geq 0}$, e.g.,

- if $d = f = 2$, then $\Xi^{\min}$ is spanned by $(1, p)$ and $(p, 1)$;
- if $d = f = 3$, then by $(1, p, p^2)$, $(p, p^2, 1)$ and $(p^2, 1, p)$.
- if $d = e = 2$, then by $(1, 1)$ and $(1, p)$;
- if $d = e = 3$, then by $(1, 1, 1)$, $(1, 1, p)$ and $(1, p, p)$;

Theorem (D–Kassaei)

If $f \neq 0$, then $\vec{k}_{\min}(f) \in \Xi^{\min}$.

A geometric Serre weight Conjecture

Conjecture (D–Sasaki)

If $\vec{m} \in \mathbb{Z}^{\Sigma}$ and $\rho : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible, then there is a unique $\vec{k}_{\min} = \vec{k}_{\min}(\rho, \vec{m}) \in \Xi^{\min}_{\geq 1}$ such that the following are equivalent for all $\vec{k} \in \Xi^{\min}_{\geq 1}$:

1. $\vec{k} \geq_{\mathrm{Ha}} \vec{k}_{\min}$
2. ρ *is geometrically modular of weight* $(\vec{k}, \vec{m})$
3. $\rho|_{G_K}$ *has a crystalline lift with* θ*-labeled weights* $\{m_\theta, m_\theta + k_\theta - 1\}$ *for all* $\theta \in \Sigma$.

- (1) $\Leftrightarrow$ (2) should hold for all $\vec{k} \in \mathbb{Z}^{\Sigma}$.
- If ρ_f is irreducible and $p > 3$, then $k_{\min}(f) \in \Xi^{\min}_{\geq 1}$. (D-Kassaei)
- Existence of a $\vec{k}_{\min}$ such that (1) $\Leftrightarrow$ (3) is a purely p-adic Hodge theoretic conjecture (and doesn't extend to $\mathbb{Z}^{\Sigma}_{\geq 1}$).

Relation between algebraic and geometric modularity

Conjecture (D–Sasaki)

If $\rho : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and $\vec{k} \in \Xi^{\min}_{\geq 2}$, then ρ is geometrically modular of weight $(\vec{k}, \vec{m})$ if and only if ρ is algebraically modular of weight $(\vec{k}, \vec{m})$.

- *If* ρ is geometrically modular of some weight, then ρ is algebraically modular of some weight.
- $\Longleftarrow$ should hold on $\mathbb{Z}_{\geq 2}$, is easy if $\vec{k}$ is paritious (and maybe not so hard in general)
- This conjecture follows from:
  Algebraic Serre weight conjecture
  + Geometric Serre weight conjecture
  + Breuil–Mézard Conjecture.

Partial Θ-operators

How should $\vec{k}_{\min}(\rho, \vec{m})$ depend on $\vec{m}$?

- If ρ is modular of weight $(\vec{k}, \vec{m})$, then

$$\det \rho|_{I_K} = \prod_\theta \omega_\theta^{k_\theta + 2m_\theta - 1} = \omega^{\sum_{i,j}(k_{i,j} + 2m_{i,j} - 1)(p^i)},$$

  so ρ and $\vec{m}$ determine $\sum_{i,j} k_{i,j} p^i \bmod (p^f - 1)$, i.e., $\vec{k} \bmod \Lambda := \oplus_\theta \mathbb{Z}\vec{h}_\theta$.
- If $\vec{m} \equiv \vec{m}' \bmod \Lambda$, then ρ is modular of weight $(\vec{k}, \vec{m})$ if and only if ρ is modular of weight $(\vec{k}, \vec{m}')$.
  So $\vec{k}_{\min}(\rho, \vec{m})$ depends only on $\vec{m} \bmod \Lambda$.
- If $\xi : G_F \to \overline{\mathbb{F}}_p^\times$ is such that $\xi|_{I_K} = \prod_\theta \omega_\theta^{b_\theta}$, then ρ is modular of weight $(\vec{k}, \vec{m})$ if and only if $\xi \otimes \rho$ is modular of weight $(\vec{k}, \vec{m} + \vec{b})$.

Fixing ρ and varying $\vec{m}$ (mod Λ)?

Use partial Θ-operators, defined/refined by Andreatta–Goren, D–Sasaki, Deo–Dimitrov–Wiese and D:

Theorem

Let $\tau = \tau_i \in \Sigma_0$ and $\theta = \theta_{i,e}$. Then there is a Hecke-equivariant:

$$\Theta_\tau : M_{\vec{k},\vec{m}}(U;\mathbb{F}) \longrightarrow M_{\vec{k}',\vec{m}'}(U;\mathbb{F}),$$

where $\vec{k}' = \vec{k} + \vec{h}_\theta + 2\vec{e}_\theta$ and $\vec{m}' = \vec{m} - \vec{e}_\theta$.
Furthermore $\Theta_\tau(f)$ is divisible by H_θ if and only if either f is divisible by H_θ or k_θ is divisible by p.

Note in particular that $\vec{k}' = \begin{cases} \vec{k} + \vec{e}_{i,e-1} + \vec{e}_{i,e}, & \text{if } e > 1; \\ \vec{k} + p\vec{e}_{i-1,1} + \vec{e}_{i,1}, & \text{if } e = 1. \end{cases}$

Idea of proof/construction:

The bundles $\mathcal{A}_{\vec{e}_\theta,\vec{0}}$ have tautological sections h_θ over the Igusa cover of $\overline{Y}_U$.

- Divide by $\prod_\theta h_\theta^{k_\theta}$ to get a rational function,
- differentiate and apply Kodaira–Spencer,
- multiply by $H_{\theta_{i,e}} \prod_\theta h_\theta^{k_\theta}$.

Can also describe the kernel of Θ_τ in terms of the image of a partial Frobenius operator V.

Corollary

If ρ is geometrically modular of weight $(\vec{k}, \vec{m})$, then ρ is geometrically modular of weight $(\vec{k} + \vec{h}_\theta + 2\vec{e}_\theta, \vec{m} - \vec{e}_\theta)$ (and of weight $(\vec{k} + 2\vec{e}_\theta, \vec{m} - \vec{e}_\theta)$ if $p|k_\theta$).

Partial weight one

As a special case of the geometric Serre weight conjecture, ρ should be geometrically modular of weight $(\vec{1}, \vec{0})$ if and only if ρ is unramified at (all primes over) p.

$\Rightarrow$ is known (Emerton–Reduzzi–Xiao, Dimitrov–Wiese)

$\Leftarrow$ under technical hypotheses (Gee–Kassaei)

What about partial weight one?

If $\{\mathfrak{p}\} \subsetneq S_p$, then the conjecture implies:

ρ is unramified at $\mathfrak{p}$ $\Leftrightarrow$ ρ is geometrically modular of some weight of the form $(\vec{k}, \vec{m})$ with $k_\theta = 1$, $m_\theta = 0$ for all $\theta \in \Sigma_\mathfrak{p}$.

$\Leftarrow$ is known for paritious $\vec{k}$ (Deo–Dimitrov–Wiese, De Maria)

What about $k_\theta = 1$ for some but not all $\theta \in \Sigma_\mathfrak{p}$?

Suppose for example $d = 2$ and p is inert. (D–Sasaki)

Up to exchanging τ_0, τ_1, such values of $\vec{k} \in \Xi^{\min}_{\geq 1}$ are $(1, k_1)$ with $2 \leq k_1 \leq p$. For simplicity assume $k_1 \neq 2$.

Suppose $\rho|_{G_K}$ has a crystalline lift with labeled HT weights $\{0, 0\}_0$ and $\{0, k_1 - 1\}_1$.

If $\rho|_{G_K}$ is reducible, then

$$\rho \sim \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix},$$

with $\psi|_{I_K} = \omega^{p(k_1-1)} = \omega^{a_0 + a_1 p}$ where $a_0 = p$ and $a_1 = k_1 - 2$, so $L' = L''$.

One finds that $\rho|_{G_K}$ has a crystalline lift with these labeled HT-weights $\Leftrightarrow c_\rho \in L' = L''$, so

1. $\rho|_{G_K}$ has a crystalline lift with labeled HT weights $\{0, p\}_0$ and $\{0, k_1 - 2\}_1$, and
2. $\rho|_{G_K}$ has a crystalline lift with labeled HT weights $\{0, p\}_0$ and $\{-1, k_1 - 1\}_1$.

Furthermore the converse holds.

And in fact the equivalence also holds if $\rho|_{G_K}$ is irreducible.

On the other hand, suppose $\rho \sim \rho_f$ for some f of weight $(\vec{k}, \vec{0}) = ((1, k_1), (0, 0))$.

1. Multiplying by H_1 implies ρ is geometrically modular of weight $(\vec{k'}, \vec{0}) = ((p + 1, k_1 - 1), (0, 0)$.
2. Applying Θ_1 shows ρ is geometrically modular of weight $(\vec{k}'', -\vec{e}_1) = ((p + 1, k_1 + 1), (0, -1)$.

Conversely suppose $\rho \sim \rho_{f'}$ for some f' of weight $(\vec{k'}, \vec{0})$, and $\rho \sim \rho_{f''}$ for some f'' of weight $(\vec{k}'', -\vec{e}_1)$.

Then $\Theta_1(f')$ has weight $((2p + 1, k_1), (0, -1)) = (\vec{k}'' + \vec{h}_1, -\vec{e}_1)$.

Choosing suitable normalized eigenforms and using the q-expansion principle, can ensure $\Theta_1(f') = H_1 f''$.
Since $k_1 - 1$ is not divisible by p, it follows that $H_1 | f'$, so ρ is modular of weight $(\vec{k}, \vec{0}) = (\vec{k}' - \vec{h}_1, \vec{0})$.

So if we assume:

- algebraic Serre weight conjecture for ρ,
- equivalence of algebraic and geometric modularity for $\vec{k} \in \Xi^{\text{min}}_{\geq 2}$,

then we get:

ρ is geometrically modular of weight $(\vec{k}, \vec{0})$ if and only if $\rho|_{G_K}$ has a crystalline lift with labeled weights $\{0, 0\}_0$ and $\{0, k_1 - 1\}_1$.

Since algebraic $\Rightarrow$ geometric modularity for paritious weights, and the algebraic Serre weight conjecture is known (under mild hypotheses, and ad hoc arguments work when they fail), we get the following:

Theorem (D–Sasaki)

Suppose F is a quadratic extension of $\mathbb{Q}$ in which p is inert or ramified. If ρ is modular and $\rho|_{G_K}$ has a crystalline lift with labeled HT-weights $\{0,0\}_1$ and $\{0,k-1\}_2$, where k is odd and $3 \le k \le p$, then ρ is geometrically modular of weight $((0,0),(1,k))$.

The ramified quadratic case is proved by a similar argument.

The possible weights are again $(1,k)$ with $2 \le k \le p$, but now θ_1 and θ_2 are not interchangeable.

The relevant algebraic weights (for $k \ge 3$) become:

- $(\vec{k}', \vec{0}) = ((2, k-1), (0,0))$.
- $(\vec{k}'', -\vec{e}_2) = ((2, k+1), (0,-1))$.

Use Gee–Liu–Savitt for the p-adic Hodge theory.

Note Θ still sends weight $(\vec{k}', \vec{0})$ to weight $(\vec{k}'' + \vec{h}_2, -\vec{e}_2)$.