Bibliography (Course "Completed cohomology for Shimura curves") | Conference "recent developments around p-adic modular forms 2020" ICTS

Breuil Course of M2 at Orsay 2017-2018

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——— "Locally analytic vectors in representations of loc. p-adic an. groups" Memoirs of AMS [Em2]

Carayol "Sur le mauvaise reduction des courbes de Shimura" Comp. Math. 1986 [Ca]

Deligne "Travaux de Shimura" Sem. Bourbaki 1971 [De]

Schneider Teitelbaum "Banach space representations & Iwasawa theory" Isr. Journal Math. 2002 [ST]

Wiebel "Introduction to homological algebra" CUP 38 [Wie]

"Loose" goal: 1. Introduce $\left(\widetilde{H}^1(Y_{K^p}, V_W)\right)_{sm} \xleftarrow{\sim} H^1(Y_{K^p}, W)$

2. Time permitting: $\mathrm{rk}\left(\widetilde{H}^1(Y_{K^p}, \mathcal{O})_{\mathfrak{m}_{\bar{r}}} \otimes \mathcal{O}\right) \geq 1.$

↳ Thm (Breuil-Herzig-Hu, M.-Schraen, Hu-Wang) $\circlearrowleft$ $\circlearrowleft$ $R^{\square}_{\bar{r}|G_{F_p}} \xrightarrow{x} \mathcal{O}$ $\quad \bar{r}: \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{F})$

technical cond.

F_p unram.

§1 Basic objects & statement of the main theorem

$F/\mathbb{Q}$ totally real (hyp: $\mathfrak{p}$ unique prime $\mid p$) $\quad \mathcal{O} \subseteq E/\mathbb{Q}_p$ $<\infty$ "coefficients"

$$\mathcal{O} \twoheadrightarrow \mathbb{F} = \mathcal{O}/(\varpi)$$

D/F quaternion alg. $\quad D \otimes_F \mathbb{R} \simeq M_2(\mathbb{R}) \times \mathbb{H}^{d-1} \quad d = [F:\mathbb{Q}]$

⇝ $G/\mathbb{Q}$ reductive group $R \mapsto (D \otimes_{\mathbb{Q}} R)^\times$

$K^p K_p = K \underset{c.o}{\overset{\circ}{\leq}} G(\mathbb{A}^\infty)$

$K_p \leq G(\mathbb{Z}_p)$

compact Riemann surface

⇝ $Y_K(\mathbb{C}) := G(\mathbb{Q}) \backslash \left(\mathbb{C} \setminus \mathbb{R} \times G(\mathbb{A}^\infty)/K \right)$

[De]

$\downarrow$

<u>Useful later</u>: Y_K/F smooth proj. curve

$$(Y_K \times_{F, \tau_\infty} \mathbb{C})(\mathbb{C}) = Y_K(\mathbb{C})$$

Assume: $g K g^{-1} \cap \mathrm{Stab}_{G(\mathbb{Q})}(z) = \{1\}$

$g \in G(\mathbb{A}^\infty)$

$z \in \mathbb{C} \setminus \mathbb{R}$

Rmk 1 [Ca, De] $\pi_0(Y_K(\mathbb{C})) \xrightarrow{\sim} F^\times_+ \backslash (\mathbb{A}_F^\infty)^\times / \nu(K)$

$\uparrow$ reduced norm

this construction algebraizes!

2. If $K' \underset{\text{c.o}}{\trianglelefteq} K$ ⇝ $Y_{K'}(\mathbb{C}) \xrightarrow{} Y_K(\mathbb{C})$ ⇝ get a proj. system with $G(\mathbb{A}^\infty)$ action $\{Y_K(\mathbb{C})\}_K$

Galois cover

$$\begin{tikzcd} Y_K(\mathbb{C}) \arrow[d, "\wr"] \\ Y_{gKg^{-1}}(\mathbb{C}) \end{tikzcd}$$

W : (irred.) alg. rep^n Gl_2/E $W = \bigotimes_{\tau_\mathfrak{p} : F_\mathfrak{p} \hookrightarrow E} \left((\mathrm{Sym}^{k_\tau - 2} F_\mathfrak{p}^2) \otimes \det^{m_\tau} \right) \otimes_{\tau_\mathfrak{p}} E$

$\begin{matrix} W \\ \uparrow \\ W^\circ \end{matrix}$ $\mathcal{O}$-lattice, Gl_2 stable

$\downarrow$

W° / ϖ^s ⇝ fibered bundle: $\left[\left(G(\mathbb{Q}) \backslash \mathbb{C}\setminus\mathbb{R} \times G(\mathbb{A}^\infty) \right) \times W^\circ/\varpi^s \right] / K$: $[(x,g), w] \cdot k := [(x, gk), k_p^{-1} \cdot w]$

$\downarrow$

$Y_K(\mathbb{C})$ ⇝ $\mathcal{V}_{W^\circ/\varpi^s, K}$ loc. cst sheaf $/ Y_K(\mathbb{C})$

Rmk (Algebraization): $K_p' \trianglelefteq K_p$ s.t. $K_p' \underset{\text{trivial}}{\circlearrowright} W^\circ/\varpi^s \Rightarrow \mathcal{V}_{W^\circ/\varpi^s, K'} = Y_{K'}(\mathbb{C}) \times W^\circ/(\varpi^s)$

$$\begin{tikzcd} Y_{K'}(\mathbb{C}) \times W^\circ/(\varpi^s) \arrow[d] \\ Y_{K'}(\mathbb{C}) \arrow[d, "\text{Galois, } \frac{K}{K'}"] \\ Y_K(\mathbb{C}) \end{tikzcd}$$

$$\Rightarrow \mathcal{V}_{W^\circ/\varpi^s, K} \simeq \left(\mathcal{V}_{W^\circ/\varpi^s, K'}\right) \Big/ \left(\frac{K}{K'}\right)$$

Set: $\mathcal{V}_{W^\circ,K} := \varprojlim_{s} \mathcal{V}_{W^\circ/\varpi^s, K}$, $\mathcal{V}_{W,K} := \mathcal{V}_{W^\circ,K} \otimes_{\mathcal{O}} E$.

Central objects of study:

$$H^i(Y_{K^p}(\mathbb{C}), \mathcal{V}_{W^\circ}) = \varinjlim_{K_p \leq_{c.o.} G(\mathbb{Q}_p)} \overbrace{H^i(Y_{K^pK_p}(\mathbb{C}), \mathcal{V}_{W^\circ})}^{H^i(K^pK_p,\, W^\circ)}$$

$$= \varinjlim_{K_p \leq_{c.o.} G(\mathbb{Q}_p)} \varprojlim_{s} H^i(K^pK_p, W^\circ/\varpi^s)$$

$$G \quad \hat{H}^i(K^p, W^\circ) = \varprojlim_{s} \left(H^i(K^p, W^\circ) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^s \right)$$

$G(\mathbb{Q}_p)$ continuous

$$G \quad \tilde{H}^i(K^p, W^\circ) = \varprojlim_{s} \left(\varinjlim_{K_p} H^i(K^pK_p, W^\circ/\varpi^s) \right)$$

with discrete topology

Relations?

$$0 \to \mathcal{V}_{W^\circ, K_pK^p} \xrightarrow{\varpi^s} \mathcal{V}_{W^\circ, K_pK^p} \to \mathcal{V}_{W^\circ/\varpi^s, K_pK^p} \to 0$$

gives:

$$0 \to H^i(K^pK_p, W_0)/\varpi^s \to H^i(K^pK_p, W_0/\varpi^s) \to H^{i+1}(K^pK_p, W_0)[\varpi^s] \to 0$$

$\varprojlim_{s} \varinjlim_{K_p}$

$$0 \to \underbrace{\hat{H}^i(K^p, W_0)}_{\text{p-adically complete \& separated}} \xrightarrow{\text{closed emb}} \tilde{H}^i(K^p, W_0) \to \underbrace{T_p H^{i+1}(K^p, W^\circ)}_{\varpi\text{-torsion free}} \to 0$$

$\Rightarrow$ $\tilde{H}^i(K^p, W^\circ)$ is p-adically complete & separated + $\mathcal{C}^0$ action of $G(\mathbb{Q}_p)$

$$\Rightarrow \quad 0 \to \hat{H}^i(K^p, W) \to \tilde{H}^i(K^p, W) \to V_p H^{i+1}(K^p, W) \to 0 \quad \in \mathrm{Ban}_{G(\mathbb{Q}_p)}(E)$$

Main theorem (Em1, Thm. 0.5): $\underline{\text{Natural}}$ isomorphism:

Gal eq.
$G(\mathbb{Q}_p)$-eq.
Hecke eq.

$$H^1(K^p, W) \xrightarrow{\sim} \left(\widetilde{H}^1(K^p, W)\right)_{sm} = \left(\widetilde{H}^1(K^p, E)\otimes W\right)_{sm}$$

§2. Locally algebraic & Banach space representations of $G(\mathbb{Q}_p)$.

$\underline{\text{Lemma}}$: Let $W_{/E}$ irred. alg. rep of $G_{/\mathbb{Q}_p}$, $V_{/E}$ a $G(\mathbb{Q}_p)$-rep.

$\underline{\text{Then}}$:
$$\underbrace{\left(\varinjlim_{K_p} \mathrm{Hom}_{K_p}(W, V)\right)}_{H^0_{st}(G(\mathbb{Z}_p),\, V\otimes W^\vee)} \otimes_E W \xrightarrow{ev_W} V \quad \text{is injective.}$$

$$\left(E[K_p] \twoheadrightarrow \mathrm{End}_E(W)\right) \text{ by irred.}$$

Pf. ETS at finite level K_p.
$$W^\vee \otimes W \xrightarrow{\sim} \left(\mathrm{End}_E(W)\right)^\vee \hookrightarrow \left(E[K_p]\right)^\vee \xrightarrow[ev_1]{} E$$

$$\Rightarrow \quad V\otimes W^\vee \otimes W \hookrightarrow V\otimes\left(\mathrm{Fct}(K_p, E)\right) \longrightarrow V\otimes E$$

take invariant by $k \mapsto (k, k, 1)$ #

Corollary: $$\bigoplus_{\substack{W \text{ irr} \\ \text{alg.}}} H^0_{st}(V\otimes W^\vee)\otimes W \xhookrightarrow[\oplus ev_W]{} V$$

Def: $V^{l\,alg} :=$ image of $\bigoplus_W ev$

$\uparrow$

is a $G(\mathbb{Q}_p)$-rep.

Upshot: The main theorem is: $$\bigoplus_{\substack{W \text{ irr.} \\ \text{alg. rep. of } GL_{2/E}}} H^1(K^p, W)\otimes_E W^\vee \xrightarrow{\sim} \left(\tilde{H}^1(K^p, E)\right)^{l\,alg}$$

$G(\mathbb{Q}_p)$, Gal, Hecke <u>equivariant</u>

Let H be a compact p-adic analytic group

$\rightsquigarrow \mathcal{O}[[H]] = \varprojlim_{\substack{H' \trianglelefteq H \\ c.o.}} \mathcal{O}\left[\frac{H}{H'}\right]$ compact top. ring, p-adically complete

Thm (Lazard):

$\mathcal{O}[H] \hookrightarrow \mathcal{O}[[H]]$ with <u>dense</u> image

& $\mathcal{O}[[H]]$ is noetherian.

Basic example: $H = \mathbb{Z}_p \Rightarrow \mathcal{O}[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}[[X]]$ (X, ϖ)-adic topology.

$[0]-1 \longmapsto X$

Lemma [ST]: If $V \in \mathrm{Ban}_H(E)$ then $V^\vee := \mathrm{Hom}_E^{\mathrm{cont}}(V, E)$ is an $\mathcal{O}[[H]][\frac{1}{p}]$

Def: V is admissible if $V^\vee$ is fin. gen. / $\mathcal{O}[[H]][\frac{1}{p}]$

$$\varinjlim_{H'} \mathcal{C}(\tfrac{H}{H'}, E)$$

$/\!/$

Basic example: $V = \mathcal{C}^{\mathrm{cont}}(H, E)$ is admissible: $\mathcal{C}^{\mathrm{sm}}(H, E) \xhookrightarrow[\text{dense}]{} V$

$$\Rightarrow \quad \begin{tikzcd} V^\vee \arrow[r, hook] & \mathcal{O}[[H]] \otimes E \\ \mathcal{O}[[H]] \cdot ev_1 \arrow[u, phantom, sloped, "\subseteq"] & \end{tikzcd}$$

$$\Rightarrow V^\vee = \mathcal{O}[[H]][\tfrac{1}{p}].$$

Thm ([ST]): $\mathrm{Ban}_H^{\mathrm{adm}}(E) \xrightarrow{\sim} \mathrm{Mod}^{f.t}(\mathcal{O}[[H]][\frac{1}{p}])$

$V \longmapsto V^\vee$

in part., $\mathrm{Ban}_H^{\mathrm{adm}}(E)$ is abelian

Thm (Em 1, 2.1.5, 2.2.11):

1. $\tilde{H}^1(K^p, W) \in \mathrm{Ban}^{adm}_{G(\mathbb{Z}_p)}(E)$ (with unitary action if $E = W$)

2. $$\begin{tikzcd} H^1(K^p, W) \arrow[r, "dense"] \arrow[drr, "\sim"] & \hat{H}^1(K^p, W) \arrow[r, hook, "closed"] & \tilde{H}^1(K^p, W) \\ & & \tilde{H}^1(K^p, W)_{sm} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

edge of $H^i_{st}(G(\mathbb{Z}_p), \tilde{H}^j(K^p, W)) \Rightarrow H^{i+j}(K^p, W)$.

Completed cohomology for Shimura curves - II

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ICTS

Thm (Em1 2.1.5, 2.2.11) 1. $\tilde{H}^1(K^p, W),\ \hat{H}^1(K^p, W) \in \mathrm{Ban}^{adm}_{G(\mathbb{Z}_p)}(E)$

2.

$$\begin{tikzcd}
H^1(K^p, W) \arrow[r] \arrow[rrd] & \hat{H}^1(K^p, W) \arrow[r, leftrightarrow] & \tilde{H}^1(K^p, W) \arrow[d, phantom, "\cup"] \\
 & & (\tilde{H}^1(K^p, W))_{sm}
\end{tikzcd}$$

as an edge map: $H^i_{st}(G(\mathbb{Z}_p), \tilde{H}^j(K^p, W)) \Rightarrow H^{i+j}(K^p, W)$

<u>Rmk</u>: $H^i_{st}(G(\mathbb{Z}_p), \_)$: derived functors of $H^0_{st}(G(\mathbb{Z}_p), \_) : \mathrm{Ban}^{adm}_{G(\mathbb{Z}_p)}(E) \to$

$\to \mathrm{Rep}^{sm}_{G(\mathbb{Z}_p)}(E$

Lemma: $\mathrm{Ban}_H^{adm}(E)$ has enough injectives.

Pf: Example of yesterday: $\left(\mathcal{C}^{cont}(H,E)\right)^{\vee} = \mathcal{O}[[H]][\tfrac{1}{p}]$.

If $V \in \mathrm{Ban}^{adm}$ $\Rightarrow \left(\mathcal{C}^{cont}(H,E)\right)^{\oplus r} \twoheadrightarrow V^{\vee} \to 0$ $\exists\, r \in \mathbb{N}_{>0}$

$\Rightarrow V \hookrightarrow \mathcal{C}^{cont}(H,E)^{\oplus r}$ ([ST]) $\xleftrightarrow{\text{equiv.}}$

$\Rightarrow$ ETS: $\mathcal{C}^{cont}(H,E)$ is injective in Ban^{adm} $\Longleftrightarrow$ $\mathcal{O}[[H]][\tfrac{1}{p}]$ is projective in $\mathrm{Mod}^{ft.}(\mathcal{O}[[H]]$

#

Sketch of pf of the Thm.

Simplicity: $W=E$

Have $\mathcal{O}[[K_p]] \underset{\text{finite index}}{\hookrightarrow} \mathcal{O}[[G(\mathbb{Z}_p)]]$ $\Rightarrow$ $Y_{K^pK_p}(\mathbb{C})$ is a compact Riemann surface $\Rightarrow$ it has a finite triangulation $\downarrow$ T_K

Rmk: if $G = GL_n$ ⇝ Ash "Small dimensional class. spaces" 1984: $Y_{K^pK_p}(\mathbb{C}) \simeq$ compact def. retract

If: $K'_p \overset{\circ}{\trianglelefteq}_{c.o.} K_p$ $\quad \underset{\substack{\| \\ K^pK'_p}}{Y_{K'}}(\mathbb{C}) \xrightarrow{\iota} \!\!\!\!\to Y_K(\mathbb{C})$:

T_d : (finite set of) d-dim^el simplexes in T_K ③

$T_d(K'_p, \Delta) :=$ d-dim^el simplexes in $\iota^*(\Delta)$

Key:
- ·) principal homogeneous $\frac{K_p}{K'_p}$ - space
- ·) $\widehat{T}_d(\Delta) := \varprojlim_{K'_p} T_d(K'_p, \Delta) \simeq K_p$

Classical: $H^i(K'_pK^p, E)$ computed by:

$$\cdots\cdots \to \prod_{\Delta' \in T'_d} H^0(\Delta', E) \to \cdots\cdots \qquad \mathcal{C}^q(K^p)$$

$$\|$$

$$\prod_{\Delta \in T_d} \underbrace{\prod_{\Delta' \in T_d(K'_p, \Delta)} H^0(\Delta, E)}_{\mathcal{C}(\frac{K_p}{K'_p}, E) \otimes H^0(\Delta, E).}$$

pass to $\varinjlim_{K_p'}$

$H^i(K^p,E)$: is the coh. of $\quad \dots \longrightarrow \prod_{\Delta\in T_d} \mathcal{C}^\infty(\widehat{T}_d(\Delta),E)\otimes H^q(\Delta,E)^{\circlearrowleft K_p\text{-action}} \longrightarrow \dots$

$$(\widehat{\mathcal{C}}(K^p)^\bullet)$$

$$\wr\wr$$

$$\prod_{\Delta\in T_d}\left(\varinjlim_{K_p'}\left(\prod_{\Delta'\in T_d(K_p',\Delta)} H^q(\Delta,E)\right)^{\circlearrowleft \frac{K_p}{K_p'}}\right)$$

<u>Similar argument</u>: $\tilde{H}^i(K^p,E)$: $\quad \dots \longrightarrow \prod_{\Delta\in T_d} \mathcal{C}^{cont}(K_p,E)\otimes_E H^q(\Delta,E) \longrightarrow \dots$

Basic example 2: $\mathcal{C}^{cont}(K_p,E)\in \underbrace{\mathrm{Ban}^{adm}_{K_p}(E)}_{\text{Abelian by [ST]}}$ $\quad \Rightarrow \tilde{H}^i(K^p,E)\in \mathrm{Ban}^{adm}_{K_p}(E)$.

<u>Rmk</u>: same proof: replace $H^q(\Delta,E)$ by $H^q(\Delta,\mathcal{V}_W)$
to get $\tilde{H}^i(K^p,W)\in \mathrm{Ban}^{adm}(E)$.

Pf. (2a) Recall: if: $V^\bullet : V^0 \to V^1 \to \dots$ a complex in $\mathrm{Ban}^{\mathrm{adm}}_{K_p}(E)$; $H^0_{st}(K_p, \_)$ left exact

injective

$\Rightarrow$ spectral sequence in hypercohomology:

(see McCleary "User guide to spec. seq." CUP 58)

$$E_2^{i,j} = H^i_{st}(K_p, H^j(V^\bullet)) \Rightarrow H^{i+j}(H^0_{st}(K_p, V^\bullet))$$

(*)

Recall: $\tilde{H}^i(K^p, W)$: i-th coh. of $\dots \to \prod_{\Delta \in T_i} \mathcal{C}^{cont}(K_p, E) \otimes H^0(\Delta, \check{V}_W) \to \dots$

$H^i(K^p, W)$: ———— $\prod_{\Delta \in T_i} \mathcal{C}^{\infty}(K_p, E) \otimes H^0(\Delta, \check{V}_W) \to \dots$

Keys: 1 injective resol. of

2. $H^0_{st}(K_p, \mathcal{C}^{cont}(K_p, E)) = \mathcal{C}^{\infty}(K_p, E)$.

Conclusion: apply (*) to $V^\bullet = \hat{\mathcal{C}}(K^p)^\bullet$. ∗

Rem: **Here** $\tilde{H}^i(K^p,W) \underset{K_p\text{-eq.}}{\xrightarrow{\sim}} \tilde{H}^i(K^p,E)\otimes_E W$. In part, $\tilde{H}^i(K^p,W)\in \mathrm{Ban}^{adm}$.

Upshot of the theorem:

$$0\to H^1_{st}(\tilde{H}^0(K^p,W)) \longrightarrow H^1(K^p,W) \longrightarrow H^0_{st}(\tilde{H}^1(K^p,W)) \longrightarrow H^2_{st}(\tilde{H}^0(K^p,W)) \to$$

$\Rightarrow$ ETS that $H^i_{st}(\tilde{H}^0(K^p,W))=0$ for $i=1,2$.

§3 Locally analytic rep^{ns} & Lie algebra cohomology.

$G(\mathbb{Q}_p)$ locally $\simeq \mathbb{Z}_p^{4[F_p:\mathbb{Q}_p]}$ (locally)

Def: 1. $V\in \mathrm{Ban}(E)$, $f: G\to V$ is locally analytic if $\exists (\mathcal{U}_i)_i$ open cover of G s.t. $f|_{\mathcal{U}_i}$ is analytic.

2. $\mathrm{Ban}_G(E)\ni V\ni v$ is loc. analytic if $g\longmapsto g\cdot v$ is loc. an.

Hence: $V \in \mathrm{Ban}_{G(\mathbb{Q}_p)}(E) \rightsquigarrow$

$$\begin{tikzcd} V^{lan} = \{v \in V,\ v \text{ is loc. an}\} \circlearrowleft G(\mathbb{Q}_p) \\ V^{lal} \arrow[u, phantom, "\cup" description] \\ V_{sm} \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

Advantage: V^{lan} has an action of $\mathfrak{g} := \mathrm{Lie}(G(\mathbb{Q}_p)) \overset{\exp}{\dashrightarrow} G$ (defined in a nbh of 0)

$$x \longmapsto \sum_{m \geq 0} \frac{x^m}{m!}$$

$$\Rightarrow \quad x \cdot v = \left(\frac{d}{dt} (\exp(t \cdot x)) \cdot v \right) \Big|_{t=0}$$

Cohomology: $H^i(\mathfrak{g}, V^{an}) :=$ derived functor of $\underset{\mathrm{Mod}(U(\mathfrak{g}))}{M} \longmapsto M[\mathfrak{g}] = \{m \in M,\ \underset{\forall g \in \mathfrak{g}}{g \cdot m = 0}\}$

[Wie §7]

Thm (Em1, 1.1.13): Let $V \in \mathrm{Ban}^{adm}_{G(\mathbb{Z}_p)}(E)$ with compatible action of $G(\mathbb{Q}_p)$

Then: $H^i_{st}(V) \xrightarrow{\sim} H^i(\mathfrak{g}, V^{la})$.

Pf: Case $i=0$ $\quad x^m \cdot v := x\cdot(x\cdot \ldots \cdot(x\cdot v)\ldots)$ for $x\in\mathfrak{g}$, $m\geq 0$, $v\in V^{la}$

If $|t|<<0$, $\exp(t\cdot x)\cdot v = \sum_{m\geq 0} \frac{t^m}{m!}(x^m\cdot v)$

$\Rightarrow$ $x \underset{Lie}{\cdot} v = 0 \Leftrightarrow \exp(t\cdot x)\cdot v = v$ for all $|t|<<0$.

Since $\mathrm{Im}(\exp) \supseteq$ open subgroup of $G(\mathbb{Z}_p)$: v is fixed by K_p' $\exists K_p'$

$\Leftrightarrow x\cdot v = 0 \ \forall x\in\mathfrak{g}$.

Case $i>0$ **Recall:** $H^i_{st}(V)$ computed by applying $H^0_{st}(G(\mathbb{Z}_p),\_)$ to:

$$0\to V\to \bigoplus \mathcal{C}^{cont}(G(\mathbb{Z}_p),E)\to \bigoplus \mathcal{C}^{cont}(G(\mathbb{Z}_p),E)\dots$$

Note: $\left(\mathcal{C}^{cont}(G(\mathbb{Z}_p),E)\right)^{\ell an} = \mathcal{C}^{lan}(G(\mathbb{Z}_p),E)) = \left\{ f: G(\mathbb{Z}_p)\to E,\ f \text{ is loc. an.} \right\}$

Thm (Schneider-Teitelbaum "Algebras of p-adic distributions" 2003): the fct: $V\longmapsto V^{loc.an.}$ is exact.

↳

Upshot: have a resolution: $0\to V^{an}\to \bigoplus \mathcal{C}^{an}(G(\mathbb{Z}_p),E)\to \bigoplus\mathcal{C}^{an}(G(\mathbb{Z}_p),E))\to\dots$

Goal: prove that ↗ is $H^0(\mathfrak{g},\_)$ acyclic ($\Rightarrow$ the complex computes $H^i(\mathfrak{g},V^{an})$)

Next time: $H^i(\mathfrak{g},\mathcal{C}^{an}(G(\mathbb{Z}_p),E))=0\ \forall\, i>0.$

Course 3

①

Yesterday: Thm: $H^i_{st}(V) \simeq H^i(\mathfrak{g}, V^{la})$ $\left(V \in \mathrm{Ban}^{adm}_{G(\mathbb{Z}_p)}(E) + \mathcal{C}^0\ G(\mathbb{Q}_p)\text{-action} \right)$

Pf: We showed: $H^0_{st}(V) = H^0(\mathfrak{g}, V^{lan})$

Case $i>0$

Upshot: have a resolution:

$$\begin{tikzcd} & & \mathcal{U}(\mathfrak{g}) \arrow[d, phantom, "\circlearrowright"] & \\ 0 \arrow[r] & V^{an} \arrow[r] & \mathcal{C}^{an}(G(\mathbb{Z}_p), E)^{\oplus r_1} \arrow[r] & \cdots\cdots \end{tikzcd}$$

$(\mathcal{C}^{an}(V)^\bullet)$

[The complex $\mathcal{C}^{an}(V)^\bullet$ computes $H^i(\mathfrak{g}, V^{la})$ if $\mathcal{C}^{an}(V)^\bullet$ is $H^0(\mathfrak{g}, -)$ acyclic]

Goal: show $H^i(\mathfrak{g}, \mathcal{C}^{an}(G(\mathbb{Z}_p), E)) = 0 \quad \forall i>0$

The Chevalley-Eilenberg complex ([Wie, §7.7]): If $M \in \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$, $H^i(\mathfrak{g}, M)$ is computed by:

$$\cdots \to \mathrm{Hom}_{\mathbb{Q}_p}(\Lambda^i \mathfrak{g}, M) \longrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(\Lambda^{i+1} \mathfrak{g}, M) \longrightarrow \cdots \qquad (\mathcal{C}(M)^\bullet)$$

Specialize to $M = \mathcal{C}^{an}(\underset{\substack{\| \\ G(\mathbb{Z}_p)}}{H}, E) = \Gamma(H, \mathcal{O}_H)$ $\Rightarrow \mathrm{Hom}_{\mathbb{Q}_p}(\Lambda^i \mathfrak{g}, M) = (\Lambda^i \mathfrak{g})^\vee \otimes_{\mathbb{Q}_p} \Gamma(H, \mathcal{O}_H)$

structure sheaf of the l. an. variety H

$\cong \Gamma(H, \Omega^i_{H/\mathbb{Q}_p})$

since $\Lambda^i \mathfrak{g} \otimes_{\mathbb{Q}_p} \Gamma(H, \mathcal{O}) =$

$= \Lambda^i \Gamma(\underbrace{\text{tg bundle}}_{\substack{\| \\ \mathfrak{g} \times H}})$

as H is tot. disconnected

Upshot: $\mathcal{C}(H)^\bullet$ are the global sections of

$$\cdots \to \Omega^i_{H/\mathbb{Q}_p} \to \Omega^{i+1}_{H/\mathbb{Q}_p} \to \cdots \qquad (\mathcal{C}_{dR}(H))^\bullet$$

Key: $\underbrace{H \text{ locally} \simeq \mathbb{Z}_p^{\dim H}}_{\Rightarrow H \text{ is smooth !}}$ $\xRightarrow[\text{lemma}]{\text{Poincaré's}}$ $\mathcal{C}_{dR}(G(\mathbb{Z}_p))$ is a resolution of $\mathbb{Q}_p/H$

H is tot. discontinuous $\Rightarrow$ the global sections of $\mathcal{C}_{dR}(G(\mathbb{Z}_p))$ are a resolution of $\Gamma(H, \underline{\mathbb{Q}_p})$ $\Rightarrow$ $\underbrace{H^i_{dR}(G(\mathbb{Z}_p))=0}$

$= H^i(\mathfrak{g}, \mathcal{C}^{an}(G(\mathbb{Z}_p), E))$

#

Left to prove: $H^i(\mathfrak{gl}_2(\mathcal{O}_{F_p})|_{\mathbb{Z}_p}, \tilde{H}^0(K^p, W)^{la}) = 0$ for $i=1,2$

$$\tilde{H}^0(K^p, W)^{la} \cong \tilde{H}^0(K^p, E)^{la} \otimes_E W$$

as W is algebraic

idea:

1. $G(\mathbb{Q}_p) = G^{der}(\mathbb{Q}_p) \cdot \underbrace{Z(G)(\mathbb{Q}_p)}$

"easy" semisimple group

study $\pi_0(Y_K(\mathbb{C}))$

2. Künneth formula

1. $\mathfrak{gl}_2 = \mathfrak{sl}_2 \times \underset{\underset{\mathrm{Lie}(Z(G)(\mathbb{Q}_p))}{\shortparallel}}{\mathfrak{z}}$ $\qquad \chi$: central character of W

$$\tilde{H}^0(K^p,E)^{\mathrm{lan}} \otimes_E W \simeq \left(\tilde{H}^0(K^p,E)^{\mathrm{lan}} \otimes \chi\right) \otimes \boxed{(\chi^{-1} \otimes W)}$$

Advantage: $\quad \mathfrak{sl}_2$ acts trivially $\qquad \mathfrak{z}$ acts trivially

$\mathfrak{z}$ ___ non ___ $\qquad \mathfrak{sl}_2$ ___ non ___.

Whitehead Lemma [Wie §7.8]: $H^i(\mathfrak{sl}_2, \chi^{-1} \otimes W) = 0$, $i > 0$ since $\mathfrak{sl}_2$ is a semisimple Lie algebra (& $\dim \chi^{-1} \otimes W < \infty$)

__Lemma__: $\mathfrak{sl}_2 \circlearrowleft \tilde{H}^0(K^p,E)^{\mathrm{lan}} \otimes \chi$ & $H^i(\mathfrak{z}, \tilde{H}^0(K^p,E)) = 0$ for $i > 0$.

Pf:

$G(\mathbb{Z}_p)$-action $\circlearrowleft$ $\xrightarrow{\det}$ $\mathcal{O}_{F_p}^\times \circlearrowleft$

$$\begin{tikzcd} Y_{K^pGL_2(\mathbb{Z}_p)}(\mathbb{C}) \arrow[loop right] & & \pi_0(Y_{K^pK_p}(\mathbb{C})) \arrow[r, "\sim", "\text{via } \nu: D^\times \to F^\times"'] & T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / \nu(K^pK_p) \\ & \rightsquigarrow & & \\ Y_{K^pK_p(n)} \arrow[uu, "\text{Gal cover}"] & & \pi_0(Y_{K^pK_p(n)}(\mathbb{C})) \arrow[uu, "\text{torsor}"', "\left(\mathcal{O}_{F_p}^{\text{[illegible]}} / \varpi^{\text{[illegible]}}\right)^\times"] & \end{tikzcd}$$

$\downarrow$ cf [Ca, §1.2]

[De, Thm 2.5]

Hence: $H^0(K^pK_p(n), \mathcal{O}/\varpi^s) \xrightarrow{\sim} \mathcal{C}\left(\left(\mathcal{O}_{F_{\mathfrak{p}}}/\varpi^n\right)^\times, \mathcal{O}/\varpi^s\right)^{\oplus \# \pi_0(Y_{K^pK_p}(\mathbb{C}))}$

take $\varprojlim_{s} \varinjlim_{K_p(n)}$: $\tilde{H}^0(K^p, E) \cong \mathcal{C}^{cont}(\mathcal{O}_{F_{\mathfrak{p}}}^\times, E)^{\oplus \# \pi_0(Y_{K^pK_p}(\mathbb{C}))}$

As for comparison theorem: $H^i(\mathfrak{Z}, \tilde{H}^0(K^p, E)^{la}) = 0 \ \forall i > 0.$

Since $\mathcal{C}^{cont}(\mathcal{O}_{F_{\mathfrak{p}}}^\times, E) \xrightarrow{\sim} \mathcal{C}^{cont}(\mathcal{O}_{F_{\mathfrak{p}}}^\times, E) \otimes \chi$ as $\mathcal{O}_{F_{\mathfrak{p}}}^\times$-rep.

$\Rightarrow H^i(\mathfrak{Z}, \tilde{H}^0(K^p, E) \otimes \chi) = 0 \ \forall i > 0.$ $\#$

2. Künneth: $H^i(\mathfrak{g}_1 \times \mathfrak{g}_2, V_1 \otimes V_2) \xrightarrow{\sim} \bigoplus_{a+b=i} H^a(\mathfrak{g}_1, V_1) \otimes_E H^b(\mathfrak{g}_2, V_2)$

e.g. $H^1(\mathfrak{gl}_2, \tilde{H}^0(K^p, E)^{an} \otimes W) \longrightarrow \Big(\overbrace{H^1(\mathfrak{z}, \tilde{H}^0(K^p, E)^{an} \otimes \chi)}^{=0} \otimes H^0(\mathfrak{sl}_2, \chi^{-1} \otimes W) \Big) \oplus$

$\oplus \Big(H^0(\mathfrak{z}, \tilde{H}^0(K^p, E)^{an} \otimes \chi) \otimes \underbrace{H^1(\mathfrak{sl}_2, \chi^{-1} \otimes W)}_{=0} \Big)$

$\Rightarrow H^i(\mathfrak{gl}_2, \tilde{H}^0(K^p, W)^{an}) = 0.$ #

Conclusion:

$$\begin{tikzcd}
H^1(K^p, W) \arrow[r, "\text{dense}"] \arrow[rrd, bend right, "\simeq"'] & \hat{H}^1(K^p, W) \arrow[r, "\sim"] & \tilde{H}^1(K^p, W) \\
 & & \left(\tilde{H}^1(K^p, W)\right)_{sm}.
\end{tikzcd}$$

4. Hecke actions & relations with Galois rep^{ns}.

Recall:

Ker$(G \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m)$

c.c. of W

from the

$(\mathbb{R}^\times)^\circ \cong$

$$\varinjlim_{K} H^i(Y_K(\mathbb{C}), \mathcal{V}_W) \xrightarrow[\text{(France)}]{\sim} H^i\left(\mathrm{Lie}(G^d(\mathbb{R})), \mathrm{Lie}(K_\infty), \left(W \otimes \mathcal{A}\left(\frac{G(\mathbb{A})}{G(\mathbb{Q})}\right)[I_\chi^\infty]\right)\right)$$

If $K^{p'} \underset{c.o}{\triangleleft} K^p \Rightarrow Y_{K^{p'}K_p}(\mathbb{C}) \xrightarrow[\text{Galois}]{} Y_{K^pK_p}(\mathbb{C})$

As before:

$$\begin{tikzcd} 0 \arrow[r] & \hat{H}^i(K^p, W) \arrow[r] \arrow[d, hook] & \tilde{H}^i(K^p, W) \arrow[r] \arrow[d, hook] & V_p H^{i+1}(K^p, W) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \hat{H}^i(K^{p'}, W) \arrow[r] & \tilde{H}^i(K^{p'}, W) \arrow[r] & V_p H^{i+1}(K^{p'}, W) \arrow[r] & 0 \end{tikzcd}$$

closed embedding

$\frac{K^p}{K^{p'}} \circlearrowright$

& $\hat{H}^i(K^p, W) = (\hat{H}^i(K^{p'}, W))^{\frac{K^p}{K^{p'}}}$

Rmk on the pf: if H: fin. gp, $V \in \mathrm{Vect}(E)$ Hausdorff

$$\Rightarrow \pi_H : V \longrightarrow V^H \qquad \text{is a } \mathscr{C}^0 \text{ projection.}$$
$$v \longmapsto \frac{1}{|H|} \sum h.v$$

$$\begin{tikzcd}
\rightsquigarrow H^i(K^p, W) \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id.}"] & \tilde{H}^i(K^{p'}, W) \arrow[rr, "\pi_H^0"] & & H^i(K^p, W) \\
& H^i(K^{p'}, W^0) \arrow[u, hook, "\otimes E"] \arrow[r] & H^i(K^{p'}, \frac{1}{|H|}.W^0) \arrow[r, "\mathrm{tr}"] & H^i(K^p, \frac{1}{|H|}.W^0) \arrow[u]
\end{tikzcd}$$

Key: π_H extends to a $\mathscr{C}^0$ proj: $\tilde{H}^i(K^{p'}, W^0) \otimes E \longrightarrow \tilde{H}^i(K^p, \frac{1}{H}.W^0) \otimes E$

Upshot: Define $\tilde{H}^i(W) := \varinjlim_{K^p} \tilde{H}^i(K^p, W)$

Thm (Eml 2.2.16): **Have**: $G(\mathbb{A}_f) = \underbrace{G(\mathbb{Q}_p)}_{\substack{\text{admissible}\\ \text{Banach}}} \times \underbrace{G(\mathbb{A}_f^p)}_{\substack{\text{smooth}\\ \text{action}}} \curvearrowright \tilde{H}^i(W)$

1. $(\tilde{H}^i(W))^{K^p} \xleftarrow{\sim} \tilde{H}^i(K^p, W)$

2. $\tilde{H}^1(E)^{\mathrm{lalg}} \xleftarrow{\quad\sim\quad} \bigoplus_{\substack{W \text{ irr.}\\ \text{alg. rep}\\ \text{of } G/\mathbb{Q}}} H^1(W) \otimes W^\vee$

$$R^\square_{\bar{r}|G_{F_p}}$$

$$\circlearrowleft \text{ faithful}$$

Goal of tomorrow: $\tilde{H}^1(K^p, \mathcal{O})_{m_{\bar{r}}} \circlearrowleft \mathbb{T}$

Completed cohomology of Shimura curves - IV

- Hecke algebras
- Galois deformations
- flatness of $\tilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}_{\bar{r}}}$ over $R_{\bar{r}|G_{F_\wp}}$

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§5.1 : Hecke algebras

$g \in G(\mathbb{A}_f)$

$K \underset{c.o}{\leq} G(\mathbb{A}_f)$

$$\begin{tikzcd} Y_{K\cap(gKg^{-1})}(\mathbb{C}) \arrow[r, shift left, "\text{inclusion}"] \arrow[r, shift right, "\text{right mult by } g"'] & Y_K(\mathbb{C}) \arrow[r, Rightarrow] & H^1(K,W) \arrow[r, "T_g", "(g)^*\circ(\mathrm{tr})^*"'] & H^1(K,W) \end{tikzcd}$$

if $\ell \nmid p$ s.t. $D_\ell \simeq M_2(F_\ell)$

$(K)_\ell \simeq GL_2(\mathcal{O}_{F_\ell})$

$\rightsquigarrow$ $T_\ell := T_{\begin{pmatrix}1 & \\ & \varpi_\ell\end{pmatrix}}$

$S_\ell := T_{\begin{pmatrix}\varpi_\ell & \\ & \varpi_\ell\end{pmatrix}}$

$$\begin{tikzcd} \Big(\bigotimes'_{\ell \text{ as above}} \arrow[d, equal] & \mathcal{O}[T_\ell, S_\ell]\Big) \arrow[d, "\wr"] & \circlearrowright H^1(K, W) \\ \mathcal{H}(K^p) & \mathcal{O}[K_\ell \backslash G(F_\ell) / K_\ell] & \end{tikzcd}$$

$\mathbb{T}(K, W) :=$ faithful quotient acting on $H^1(K, W)$.

$$\mathbb{T}(K^p K_p', W^\circ/\varpi^s) \twoheadrightarrow \mathbb{T}(K^p K_p, W^\circ/\varpi^s) \quad \text{if} \quad K_p' \underset{\text{c.o.}}{\trianglelefteq} K_p$$

&

$$\mathcal{H}(K^p) \circlearrowright \tilde{H}^1(K^p, W^\circ)$$

$\Rightarrow$ $\mathbb{T}(K^p, W^\circ) :=$ faithful quotient acting on $\tilde{H}^1(K^p, W^\circ)$

$$\Big(H^1(K^p) \xhookrightarrow[\text{dense}]{} \tilde{H}^1(K^p)\Big) = \varprojlim_{K_p'} \mathbb{T}(K^p K_p', W^\circ)$$

Rmk: $\widetilde{H}^i(K^p, W^\circ) \simeq \widetilde{H}^i(K^p, \mathcal{O}) \otimes_{\mathcal{O}} W^\circ \Rightarrow \mathbb{T}(K^p, W^\circ) = \mathbb{T}(K^p, \mathcal{O})$.

if

$$\begin{tikzcd} \mathfrak{m} \arrow[d, mapsto] & \trianglelefteq & \mathbb{T}(K^p,\mathcal{O}) \arrow[d, Rightarrow] & \text{maximal} \\ \mathfrak{m}_{K_p} & \trianglelefteq & \mathbb{T}(K^pK_p,\mathcal{O}) & \end{tikzcd}$$

$$\widetilde{H}^i(K^p, W^\circ)_{\mathfrak{m}} = \varprojlim_{s} \varinjlim_{K_p} H^i(K^pK_p, W^\circ/\varpi^s)_{\mathfrak{m}_{K_p}}$$

Lemma: $\mathbb{T}(K^p,\mathcal{O})$ is ϖ-torsion free, reduced, semilocal.

Hence: $\widetilde{H}^1(K^p,W^\circ) = \bigoplus_{\substack{\mathfrak{m} \trianglelefteq \mathbb{T}(K^p,\mathcal{O}) \\ \text{max}^{d}}} \widetilde{H}^1(K^p,W^\circ)_{\mathfrak{m}}$ finite sum!

Key in the proof: 1. ETS $H^1(K^p, \mathbb{F})$ has finite support over $\mathbb{T}(K^p,\mathcal{O})/\varpi$

2. $H^1(K^p,\mathbb{F})_{\mathfrak{m}} \circlearrowleft G(\mathbb{Q}_p)$ $\underline{\underline{\text{smooth}}}$ $\Rightarrow$ ETS $\exists$ finite nb. of $\mathfrak{m}$ s.t. $H^1(K^p,\mathbb{F})_{\mathfrak{m}}^{K_p(1)} \neq 0$ #

is finite dim^{d} !!

5.2 Relation with Galois.

Fix: $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{F})$ $\mathcal{C}^0$, abs. irred.

$$\bar{\rho} = R_{\bar{r}|G_{F_\wp}}$$

$\rightsquigarrow$ $\mathfrak{m}_{\bar{r}} \trianglelefteq \mathbb{T}(K^p,\mathcal{O})$ max'l

goal: $\tilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}_{\bar{r}}} \circlearrowleft \mathbb{T}(K^p,\mathcal{O}) \leftarrow R_{\bar{r}|G_{F_\wp}}$

show: if $x : R_{\bar{\rho}} \to \mathcal{O}'$ is a loc. $\overset{\sigma}{\text{alg}}$. hom

$\Rightarrow$ $\tilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}_{\bar{r}}} \otimes_{R_{\bar{\rho}},x} \mathcal{O}'$ has $\mathcal{O}'$-rank ≥ 1

[Breuil-Herzig-Hu-M-Schraen
Hu-Wang

Construct: $M_\infty \big/ \underbrace{R^\square_{\bar{\rho}}[[x_i]]}$ with $\cdot)$ $\mathcal{C}^0$ $G(\mathbb{Q}_p)$-act. s.t. $\cdot)$ M_∞ is proj. as $S_\infty[[G(\mathbb{Z}_p)]]$-mod

$\mathcal{O}[[y_i,z_i]] = S_\infty \longrightarrow R_\infty$

$(y_i,z_i) \longmapsto \mathfrak{a}_\infty$

$\cdot)$ $S_\infty \to R_\infty \to \mathbb{T}(K^p,\mathcal{O})_{\mathfrak{m}_{\bar{r}}}$

$\cdot)$ $\left(M_\infty/\mathfrak{a}_\infty\right)^d \simeq \tilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}_{\bar{r}}}$

compatible with

Comparison $H^i_{ét}(Y_K \times_F \mathbb{C}, \mathcal{V}_{W,et}) \xrightarrow{\sim} H^i(Y_K(\mathbb{C}), \mathcal{V}_W)$

($Y_K/_F$ smooth!)

! replace $Y_K(\mathbb{C})$ by $Y_K(\mathbb{C}) / Z^\circ_\infty$

& $\mathrm{Ker}(Z(G(\mathbb{R})) \to \mathbb{R}^*)^{Z^\circ_\infty}$ $\curvearrowright$ W trivial

$\Rightarrow \tilde{H}^i(K^p, W^\circ/\varpi^n)$ have a $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$-action.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{G_F}$

Properties: 1. the G_F-action is $\mathcal{C}^\circ$

2. commutes with $G(\mathbb{Q}_p) \times \mathbb{T}(K^p, \mathcal{O})$

3. $H^1(K^p, W) \xrightarrow{\sim} \tilde{H}^1(K^p, W)_{sm}$ is G_F-eq.

⚠ $H^i_{ct}(K_p, \tilde{H}^j(K^p, W)) \Rightarrow H^{i+j}(K^p, W)$ ← given by simplicial cohomology.

→ alternative proof: → Hochschild-Serre
→ take the $\varinjlim_{K_p}$
→ take $\varprojlim$

Def: $r: G_F \to Gl_2(E)$ is *modular* if $\mathrm{Hom}_{G_F}(r, H^1_{et}(K^p, W)) \neq 0$.

$\downarrow$

$\bar{r}: G_F \to Gl_2(\mathbb{F}) \rightsquigarrow \mathfrak{m}_{\bar r} := \mathrm{Ker}\left(\begin{array}{rcl} \mathbb{T}(K^p, \mathcal{O}) & \longrightarrow & \mathbb{F} \\ T_\ell & \longmapsto & \mathrm{Tr}(\bar r(\mathrm{Frob}_\ell^{-1})) \\ S_\ell & \longmapsto & \det(\bar r(\mathrm{Frob}_\ell^{-1}))\cdot N(\ell) \end{array}\right)$

mod ϖ-red. of a G_F-stable lattice in r

assume: abs. irred.

r is *modular* if $H^1_{et}(K^p, \mathbb{F})_{\mathfrak{m}_{\bar r}} \neq 0$.

Properties: 1. $\tilde{H}^1(K^p, W^0)_{m_F}/\varpi \xrightarrow{\sim} \tilde{H}^1(K^p, W^0/\varpi)_{m_F}$

2. $\tilde{H}^1(K^p, W)_{m_F} \xrightarrow{\sim} \mathcal{C}^{cont}(H, E)^{\oplus n}$ $\exists\, n$ $\quad H \underset{c.o.}{\leq} G(\mathbb{Q}_p)$

$H^1(K^p, W^0/\varpi^s)_{m_F} \in \mathrm{Rep}^{sm}_{\mathcal{P}_H}(\mathcal{O}/\varpi^s)$ is injective

Main idea: $s' \geq s$

$$0 \to \mathcal{V}_{W^0/\varpi^{s'}} \xrightarrow{\varpi^s} \mathcal{V}_{W^0/\varpi^{s'}} \to \mathcal{V}_{W^0/\varpi^s} \to 0$$

⇝ long exact seq.

$$\dots \to H^0(K^pK_p, W^0/\varpi^s) \to H^1(\_) \to H^1(\ ) \to H^1(\ ) \to H^2(\_)$$

$$H^0(K^pK_p, W^0/\varpi^s) \cong \bigoplus \mathcal{C}^{cont}\left([illegible], W^0/\varpi^s\right)$$

abelian action of G_F $\quad \dfrac{\mathcal{O}^\times_{F_p}}{\varpi^n}$

$\to (H^0)_{m_F} = 0 \overset{\text{Poincaré}}{\Rightarrow} (H^2)_{m_F}$

Technical reasons: $\bar{r}|_{\mathrm{Gal}(\bar{\mathbb{Q}}/F(\zeta_p))}$ is abs. irred.

- fix v_1 s.t. $N(v_1) \not\equiv 1 \bmod p$, $\bar{r}|_{G_{F_{v_1}}}$ has $\neq$ Frob. eval.

Simplicity: $\bar{r}|_{G_{F_\ell}}$ is unramified $\forall \ell \neq p$

$(K^p)_\ell$ is a max'l order if $\ell \neq p, v_1$

is pro v_1-Iwahori at v_1

Global deformation problems:

$$R \longmapsto \left\{ \text{lifts } r : G_{F,\{p,v_1\}} \to GL_2(R) \text{ s.t. } \det r = \chi_{cyc}^{-1} \right\}$$

↙

representable by a $\mathrm{CLN}_{\mathcal{O}}$

$$R_\emptyset^{univ} \overset{?}{\dashrightarrow} \mathbb{T}(K^pK_p, \mathcal{O})_{\mathfrak{m}_{\bar{r}}}$$

$$H^1(K^pK_p, W) \xrightarrow{\sim} \bigoplus r_\pi \otimes (\pi_f)^{K^pK_p}$$

$$\pi = \pi_\infty \otimes \pi_f \subseteq \mathcal{A}\big(G(F)\backslash G(\mathbb{A}_F)\big)$$

$\pi_\infty \simeq W$ dim > 1

s.t. $WD(r_\pi|_{G_{F_\ell}}) = LL_\ell(\pi_\ell \otimes |\det|^{-\frac{1}{2}})$

(Carayol (AENS '86)
Saito ("HMF & p-adic HT")

$\Downarrow$

construct
$$\begin{tikzcd} r_{\mathfrak{m}} : G_F \arrow[r] \arrow[rrd, bend right, "(\text{Carayol})"'] & \prod_\pi GL_2(E) \arrow[r, "\sim"] & GL_2(\mathbb{T}(K^p)\otimes E) \\ & & GL_2(\mathbb{T}(K^p,\mathcal{O})) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

+ conditions
Satake (π_ℓ)

by LGC, $r_{\mathfrak{m}}$ is unramified outside p, v_1 & is a lift of $\bar{r}$

⇒ get
$$\begin{tikzcd} R_\emptyset^{univ} \arrow[r, two heads] & \mathbb{T}(K^p, \mathcal{O})_{\mathfrak{m}_{\bar{r}}} \\ R_p[[x_i]] \arrow[u] & \end{tikzcd}$$

Flatness:

M_∞ proj. f.t. / $S_\infty[[K_p(1)/Z_p(1)]]$ ⇒ M_∞ is Cohen-Macaulay
$R_\infty[[K_p(1)/Z_p(1)]]$.

local

• properties of M_∞: $\left(M_\infty/\mathfrak{m}_{\bar{r}}\right)^\vee = \underbrace{H^1(K^p, \mathbb{F})[\mathfrak{m}_{\bar{r}}]}$ ⇒ Noncommutative miracle flatness

$\downarrow$

M_∞ is flat / R_∞

$\boxed{j_{\mathbb{F}[[K_p(1)/Z_p(1)]]}(\ )} = 2[F_p:Q_p]$

$\downarrow$

Yongquan
Haoran's talks

$=$

• Formula: $\boxed{j_{R_\infty[[K_p(1)/Z_p(1)]]}(M_\infty)} = \dim(R_\infty) - \dim S_\infty$
$= 2[F_p:Q_p]$