Modulo p representations of GL_2

1) Generalities

2) $GL_2(\mathbb{Q}_p)$

3) $GL_2(L)$ $\quad L/\mathbb{Q}_p$ unr.

1. Generalities

p prime number $\quad \mathbb{F}$ finite field of char p.

G $\quad$ p-adic Lie group

A **smooth** rep of G over $\mathbb{F}$ is (π, V)

V $\quad$ $\mathbb{F}$-vector space $\quad \pi : G \to GL_{\mathbb{F}}(V)$

$\forall v \in V, \quad \mathrm{Stab}_G(v) \subset G$ is open.

Examples $\quad$ 1) $\chi : G \to \mathbb{F}^\times$ loc constant.

2) $H \subset G$ cocompact subgroup of G.

$$\mathrm{Ind}_H^G(W) = \left\{ f : G \to W \;\middle|\; \text{loc constant} + f(hg) = \sigma(h) f(g) \;\; \forall h \in H \; \forall g \in G \right\}.$$

(σ, W) smooth rep of H

$g \cdot f = f(\cdot\, g)$. $\quad$ smooth rep of G.

<u>Compact case</u> K compact p-adic Lie group.

$\Rightarrow$ profinite group $K = \varprojlim_{\substack{U \lhd K \\ \circ}} K/U$.

open normal

$\mathbb{F}[[K]] = \varprojlim_{\substack{U \lhd K \\ \circ}} \mathbb{F}[K/U]$ completed group algebra (Iwasawa algebra)

compact top $\mathbb{F}$-algebra.

(π, V) smooth rep of K. $v \in V$. $\mathbb{F}[[K]]v = \mathbb{F}[K/U]v$.

$U \subset \mathrm{Stab}_G(v)$

$\rightsquigarrow$ unique structure of $\mathbb{F}[[K]]$-module on V extending the $\mathbb{F}[K]$-module of V.

This $\mathbb{F}[[K]]$-module is smooth: each vector $v \in V$. killed by an open ideal in $\mathbb{F}[[K]]$

$$\left(\begin{array}{c}\text{smooth rep}\\ \text{of } K \text{ over } \mathbb{F}\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{smooth } \mathbb{F}[\![K]\!]\\ \text{-modules}\end{array}\right).$$

<u>Duality</u>: V $\mathbb{F}$-vs.

$V^\vee = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ top of pointwise convergence

$\rightarrow$ compact top $\mathbb{F}$-vs.

$$\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = \varprojlim_{\substack{W \subset V \\ \dim_{\mathbb{F}} W < \infty}} \mathrm{Hom}_{\mathbb{F}}(W, \mathbb{F})$$

If W is a compact top $\mathbb{F}$-vs,

$W^\vee = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(W, \mathbb{F})$ $\mathbb{F}$-vs.

$$\left(\mathbb{F}\text{ vs}\right) \underset{W^\vee \leftarrow\!\shortmid\, W}{\overset{V \mapsto V^\vee}{\rightleftarrows}} \left(\text{compact top } \mathbb{F}\text{-vs}\right)$$

anti-equivalence

$$\Rightarrow \begin{pmatrix} \text{smooth rep} \\ \text{of } K \text{ on} \\ \mathbb{F}\text{ vs} \end{pmatrix} \xleftrightarrow{\text{anti}} \begin{pmatrix} \text{compact} \\ \mathbb{F}[\![K]\!]\text{-modules} \end{pmatrix}$$

$$V \longmapsto V^\vee$$

<u>Remark</u> $K \xrightarrow{\sim} K$ $\mathbb{F}[\![K]\!] \simeq \mathbb{F}[\![K]\!]^{op}$

$g \longmapsto g^{-1}$ left modules $\longleftrightarrow$ right modules

<u>Structure results on $\mathbb{F}[\![K]\!]$</u> (Lazard)

<u>Rk</u> K is a pro-p-group. $\underset{\{0\}}{V \neq}$ smooth rep of K, $V^K \neq \{0\}$.

$\Rightarrow$ The only irreducible smooth rep of K is the trivial.

$\Rightarrow \mathbb{F}[\![K]\!]$ has a unique open maximal ideal.

Theorem (Lazard, 1965) G p-adic Lie group.

- G contains an open pro-p-group.
- G is moreover a pro-p-group,

$\mathbb{F}[\![G]\!]$ is complete local noetherian ring.

$$gr(\mathbb{F}[\![G]\!]) = \bigoplus_{i\geq 0} m_G^i / m_G^{i+1}$$ graded $\mathbb{F}$-algebra is noetherian.

($m_G \subset \mathbb{F}[\![G]\!]$ max ideal)

- In general, G always contains an open pro-p-group $H \subset G$ st $gr(\mathbb{F}[\![H]\!]) \simeq \mathbb{F}[x_1, \dots, x_d]$ with $d = \dim(H) = \dim(G)$.

Remark: P is an analytic pro-p-group.

$\mathfrak{m}_P \subset \mathbb{F}[\![P]\!]$ max ideal

$$\mathfrak{m}_P = \sum_{g \in P} ([g]-1)\mathbb{F}[\![P]\!]$$

V is smooth rep of P, $(V^P)^\vee \simeq V^\vee / \mathfrak{m}_P V^\vee \simeq V^\vee \otimes_{\mathbb{F}[\![P]\!]} \mathbb{F}[\![P]\!]/\mathfrak{m}_P$

Use topological Nakayama lemma.

Corollary (π, V) is a smooth rep of G over $\mathbb{F}$. TFAE

(i) $\exists P \subset G$ open pro-p-group in G st $\dim_{\mathbb{F}} V^P < +\infty$

(ii) $\forall K \subset G$ compact open $\dim_{\mathbb{F}} V^K < \infty$

(iii) $\forall K \subset G$ ———————— $V^\vee$ is f.g as a $\mathbb{F}[\![K]\!]$-module.

[Use moreover $K' \subset K$ open of K compact $\mathbb{F}[\![K]\!]$ is finite free as $\mathbb{F}[\![K']\!]$-module].

If these conditions are satisfied, (π, V) is said to be admissible.

$\mathbb{F}[\![K]\!]$ noetherian for K compact

$\rightsquigarrow$ category of adm smooth of G is abelian.

<u>Example</u> $K = \mathbb{Z}_p$. $\mathbb{F}[\![\mathbb{Z}_p]\!] \simeq \mathbb{F}[\![X]\!]$

$[1] \longmapsto (1+X)$.

Admissible rep of $\mathbb{Z}_p$: $\mathcal{C}(\mathbb{Z}_p, \mathbb{F})^{\oplus r} \oplus \underbrace{F}$

space of loc cst functions $\mathbb{Z}_p \to \mathbb{F}$.

fd rep of $\mathbb{Z}_p$ over $\mathbb{F}$. $\rho \curvearrowright \mathbb{Z}_p$

$$\mathcal{C}(\mathbb{Z}_p, \mathbb{F})^\vee = \mathrm{Ind}_{\{1\}}^{\mathbb{Z}_p}(\mathbb{F})^\vee \simeq \mathbb{F}[\![\mathbb{Z}_p]\!].$$

Dimension theory for admissible smooth rep of G.

K is an analytic pro-p-group, $\mathbb{F}[\![K]\!]$ "looks like" a regular ring.

Recall (Serre) A is noetherian commutative ring of "finite global dimension", then A is regular

$$\mathrm{Max}\{n \geq 0 : \mathrm{Ext}^n_A(M, N) \neq 0\} < +\infty.$$

Def (Björk) A ring A is Auslander regular if

1) A is noetherian (left and right) and A has a finite global dimension (left and right).
2) A satisfies the Auslander condition.

<u>Notation</u> if M is an A-module,

$$j_A(M) = \inf\{n \geq 0 : \mathrm{Ext}^n_A(M,A) \neq 0\}$$

<u>Auslander condition</u> Say that A has the Auslander property if

$\forall$ f.g. A-module M, $\forall N \subset \mathrm{Ext}^j_A(M,A)$ sub-A-module, $j_A(N) \geq j$.

[A commutative regular ring.

$$j_A(M) = \mathrm{codim}_{\mathrm{Spec}(A)}(\mathrm{Supp}(M))$$

$\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{Ext}^j_A(M,A)_{\mathfrak{p}} \simeq \mathrm{Ext}^j_{A_\mathfrak{p}}(M_\mathfrak{p}, A_\mathfrak{p}) = 0$ if $j > \dim(A_\mathfrak{p}) = \mathrm{codim}_{\mathrm{Spec} A}(V(\mathfrak{p}))$.

$A_\mathfrak{p}$ is regular ring]

<u>Properties</u>: A Auslander regular.

M f.g. A-module (left).

1) If $N \subset \mathrm{Ext}^j_A(M,A)$ $N \neq 0$.

$$j_A(N) = j \quad \text{(Purity condition)}.$$

2) $0 \to M' \to M \to M'' \to 0$

$$j_A(M) = \inf(j_A(M'), j_A(M'')).$$

$j_A(M)$ = "codimension of the support of M"

To prove 1), use <u>Auslander condition</u>

+ Biduality spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_A(\mathrm{Ext}^{-q}_A(M,A),A)) \Rightarrow M.$$

Theorem (Venjakob) If K is a compact p-adic Lie group, p-torsion free,
then $\mathbb{F}[\![K]\!]$ is Auslander regular of global dim is $\dim(G)$

Consequence (π, V) is a smooth admissible representation of G.

$K \subset G$ open compact subgroup, p-torsion free.

$$\operatorname{codim}_G(\pi, V) = j_{\mathbb{F}[\![K]\!]}(V^\vee)$$

f.g. $\mathbb{F}[\![K]\!]$-module

does not depend on choice of K.

$K' \subset K$ open $\qquad \mathbb{F}[\![K]\!]$ finite free $\mathbb{F}[\![K']\!]$.

$$\operatorname{Ext}^i_{\mathbb{F}[\![K]\!]}(M, \mathbb{F}[\![K]\!]) \simeq \operatorname{Ext}^i_{\mathbb{F}[\![K']\!]}(M, \mathbb{F}[\![K']\!]).$$

$$j_{\mathbb{F}[\![K]\!]}(V^\vee) = j_{\mathbb{F}[\![K']\!]}(V^\vee).$$

$$0 \leq \operatorname{codim}(\pi, V) \leq \dim(G).$$

$$\dim_G(\pi, V) = \dim(G) - \operatorname{codim}_G(\pi, V).$$

<u>Particular case</u> $H \subset G$ open st

$$gr(\mathcal{O}[\![H]\!]) \simeq \mathbb{F}[x_1, \dots, x_d, \varepsilon]$$

$$\varepsilon \simeq \mathfrak{p}$$

(π, V) adm rep of G.

$V^\vee$ $\quad \mathbb{F}[\![H]\!]$-module.

$\rightsquigarrow$ $\mathfrak{m}_H$-adic filtration

$$gr(V^\vee) = \bigoplus \mathfrak{m}_H^i V^\vee / \mathfrak{m}_H^{i+1} V^\vee \quad gr(\mathbb{F}[\![H]\!])\text{-module}$$

$$\dim_G(\pi, V) = \dim(\operatorname{Supp}_{\mathbb{A}^d_{\mathbb{F}}} gr(V^\vee)).$$