Ex (Limit of connected sgps that's not connected)

In $SL_2\mathbb{R}$, set $A_n = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ and

$$H_n = A_n SO(2) A_n^{-1} = \left\{ \begin{pmatrix} \cos\theta & -\frac{1}{n^2}\sin\theta \\ n^2\sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}$$

Then $H_n \underset{\text{homeo.}}{\cong} S^1$ and in $\mathrm{Sub}_{SL_2\mathbb{R}}$,

$$H_n \longrightarrow H = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$\cong \mathbb{R} \sqcup \mathbb{R}.$$

Let G be a Lie group. An invariant random subgroup (IRS) of G is a random closed sgp of G whose law is a Borel prob measure μ on Sub_G that's invt under the G-action by conjugation.

Ex 1) Normal sgps $N \trianglelefteq G$; the atomic measure $\mathbb{1}_N$ is G-invt.

2) A discrete sgp $\Gamma \leq G$ is a <u>lattice</u> if $\Gamma\backslash G$ admits a G-invt finite measure.

<u>Note</u>: All G in this talk are unimodular, i.e. they admit a Radon measure μ that's invt under both left/right mult. This measure is unique up to scale, and is called the Haar measure on G.

If $\Gamma \leq G$ is discrete, μ locally projects to a G-invt Radon measure $\mathrm{vol}_{\Gamma\backslash G}$ on $\Gamma\backslash G$, and Γ is a lattice $\iff$ this is a finite measure.

$\approx$

If $\Gamma \leq G$ is a lattice, we can construct an IRS that's a.s. a conjugate of Γ by

$$\begin{tikzcd} G \arrow[r, "g \mapsto g^{-1}\Gamma g"] \arrow[d] & \mathrm{Sub}_G \\ \Gamma\backslash G \arrow[ur, dashed] & \end{tikzcd}$$

$\mathrm{vol}_{\Gamma\backslash G} \xrightarrow{\text{push forward + scale}} \mu_\Gamma$, an IRS

G-invariance of $\mathrm{vol}_{\Gamma\backslash G}$ $\Rightarrow$ conjugation invariance of μ_Γ.

$\underline{\text{Ex}}$ S'pose $M = \Gamma\backslash\mathbb{H}^n$ is a finite vol hyp n-manfld, i.e. Γ is a lattice in $G = \mathrm{Isom}(\mathbb{H}^n)$.

G acts simply transitively on the frame bundle $F\mathbb{H}^n$, so after fixing a base frame $f_{\mathbb{H}^n}$, we get

$$\begin{tikzcd}
G \arrow[r, "g \mapsto g_* f_{\mathbb{H}^n}"] \arrow[d] & F\mathbb{H}^n \arrow[d] \\
\Gamma\backslash G \arrow[r, "\cong"] \arrow[dr] & \Gamma\backslash F\mathbb{H}^n \cong FM \arrow[d] \\
 & \mathrm{Sub}_G
\end{tikzcd}$$

$\mathrm{vol}_{\Gamma\backslash G} \mapsto \mu_\Gamma$

$f \mapsto$ the conjugate Δ of Γ w/ $(\Delta\backslash\mathbb{H}^n, Df_{\mathbb{H}^n}) \cong (M, f)$.

Here, the measure $\mathrm{vol}_\Gamma \circ$ corresponds (up to scale) to the measure $\mathrm{vol}_{\mathcal{F}M}$ obtained by integrating the unique $O(n)$-invt prob measure on each $\mathcal{F}M_p$ against the Riem measure on M, so we can regard our IRS as the sgp corr. to a $\mathrm{vol}_{\mathcal{F}M}$-random choice of baseframe for M.

3) S'pose $G \curvearrowright (X,\mu)$ is a pmp action. Varadarajan $\Rightarrow$ $\forall x \in X$, the stabilizer G_x is closed and $x \mapsto G_x$ is mble, $\Rightarrow$ the stabilizer of a μ-random $x \in X$ is an IRS of G.

<u>ABBGNRS '17</u> Every IRS of G arises this way.

(Abert - Bergeron - B - Gelander, Nikolov, Raimbault, Samet.)

Let G be a noncompact simple linear Lie group. Borel Density Thm $\Rightarrow$ any sgp $H \leq G$ s.t. G/H has a G-inv't finite measure is discrete and Zariski dense in G, i.e. it's not contained in a nontrivial Lie sgp of G w/ finitely many connected comps. ($\Rightarrow$ lattices are Zariski dense)

Thm (ABBGNRS ~'17) If H is an IRS then H is a.s. either

1) e
2) G
3) discrete + Zariski dense.

The rank $\mathrm{rank}_{\mathbb{R}}(G)$ is the maximal dim of an $\mathbb{R}$-diagonalizable sgp of G.

If $X = G/K$ is the assoc symmetric space, this is the same as the max'l dim of a totally geodesic $\mathbb{R}^d \hookrightarrow X$, which we denote $\mathrm{rank}_{\mathbb{R}}(X)$.

Ex 1) $\mathrm{rank}_{\mathbb{R}}(SL_n\mathbb{R}) = n-1$, the dim of the diagonal sgp.

2) $\mathrm{rank}_{\mathbb{R}}(\mathbb{H}^n) = 1 \quad \forall n$.

"higher rank"

Normal Sgp Thm (Margulis) If $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, and $\Gamma \leq G$ is a lattice, then any normal sgp $N \leq \Gamma$ is either finite index or trivial. (Show Γ/N amenable and has (T) $\Rightarrow$ finite.)

Thm (Nevo-Stuck-Zimmer)

If $\mathrm{rank}_{\mathbb{R}} G \geq 2$, the only ergodic IRSs of G are e, G and the random conjugate of a lattice Γ. (Laws 1_e, 1_G, μ_Γ.)

Actually stated in terms of pmp actions of G, but any IRS comes from such an action.

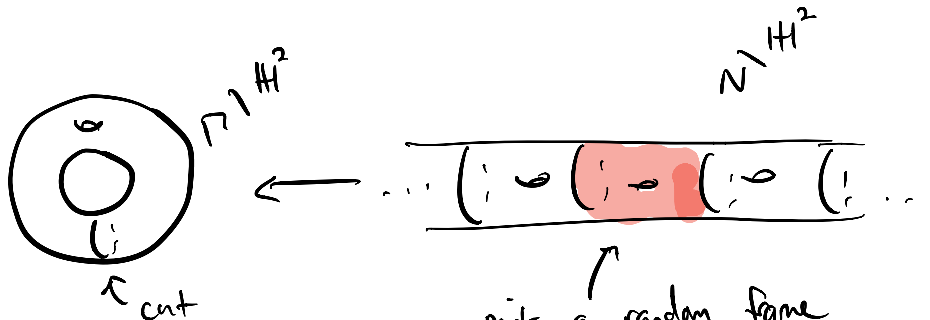
IRSs in rank 1

Ex (Normal sgps)

It's known that the normal sgp thm fails for rank 1 G, so $\exists$ lattice $\Gamma \leq G$ w/ ∞-index nontrivial $N \trianglelefteq \Gamma$. To get an IRS of G, push forward the prob measure on $\Gamma \backslash G$ under the map

$$\Gamma \backslash G \xrightarrow{g \mapsto g^{-1} N g} \mathrm{Sub}_G$$

Ex

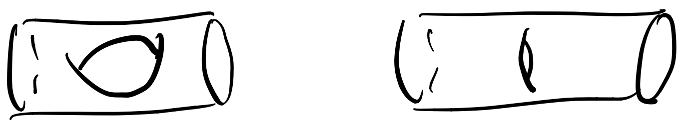


picks a random frame in some fundamental domain for the covering map, and look at the sgp corr. to this framed hyp mnfld.

This gives an IRS that's ∞-grained almost surely.

<u>Ex</u> (Shift spaces, ABBGNRS '17)

Pick two hyp n-mnflds N_0, N_1 w/ geodesic ∂, such that ∂N_i is the union of 2 copies of some fixed hyp $(n-1)$-mnfld Σ, and $\mathrm{vol}\, N_0 = \mathrm{vol}\, N_1$.

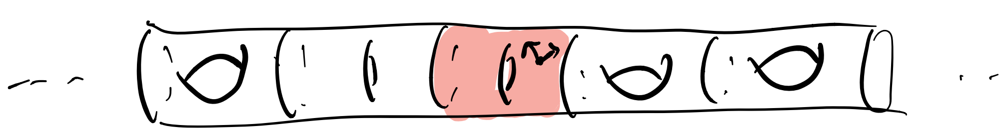


N_0 $\qquad$ N_1

Given a sequence $\alpha = (a_i) \in \{0,1\}^{\mathbb{Z}}$, construct N_α by gluing together copies of N_0, N_1 as indicated by α.

$\cdots \; 0\,1\,1\,0\,0 \; \cdots$ $\qquad$ α

$\downarrow$



N_α

If ν is a shift invt measure on $\{0,1\}^{\mathbb{Z}}$, construct a framed random hyp n-mnfld by selecting α ν-randomly, and then choosing a baseframe randomly from the copy of $N_{\alpha(0)}$ corr. to $0 \in \mathbb{Z}$.

<u>Thm</u> The random sgp of $\mathrm{Isom}(\mathbb{H}^n)$ corr. to the above framed hyp n-mnfld is an IRS. Moreover if ν is ergodic, so is our IRS, and if we choose N_i, ν carefully, this IRS isn't induced from any IRS of a lattice.

<u>Weak convergence</u>

If X is a compact metric space, $\mathcal{P}(X)$ the space of Borel prob measure on X. Then $\mu_n \to \mu \in \mathcal{P}(X)$ <u>weakly</u> if

$$\int f \, d\mu_n \longrightarrow \int f \, d\mu$$

$\forall$ bounded cnts $f : X \to \mathbb{R}$.

Fact $\mathcal{P}(X)$ is weakly compact.

If G is a Lie gp, then Sub_G is compact, so $\mathcal{P}(\mathrm{Sub}_G)$ is compact metrizable.

Since the G-action is cts,

$$\mathrm{IRS}(G) \subset \mathcal{P}(\mathrm{Sub}_G)$$

is closed, hence compact.

Idea Study sequences of lattices $\Gamma_i \leq G$ by analyzing weak limits of μ_{Γ_i}.

Application Let G be a noncompact simple Lie gp w/ $\mathrm{rank}_{\mathbb{R}} G \geq 2$. NSZ $\Rightarrow$ the only ergodic IRSs of G are 1_e, 1_G, μ_Γ for Γ a lattice.

Thm (ABBGNRS '17) If Γ_i is any sequence of pairwise nonconjugate lattices in G, then $\mu_{\Gamma_i} \longrightarrow 1_e$.

"Pf" Given Γ_i, pass to a subseq s.t. μ_{Γ_i} converges to some IRS μ, which ends up being ergodic.

So, $\mu = 1_e, 1_G$ or $\mu = \mu_\Gamma$ for some Γ. G is an isolated pt in Sub_G, so $\mu \neq 1_G$, so must rule out μ_Γ. $\square$

Def If $M = \Gamma \backslash X$ ($X = G/K$) is an X-mnfld, the *injectivity radius* of M at $p \in M$ is the largest r s.t. $B_M(p, r)$ is a topological ball.

Prop $\mu_{\Gamma_i} \longrightarrow 1_e \iff \forall R > 0$, we have

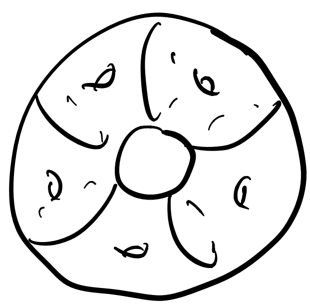
$$\frac{\mathrm{vol}(p \in M_i \mid \mathrm{inj}_{M_i}(p) > R)}{\mathrm{vol}(M_i)} \longrightarrow 1,$$

where here $M_i = \Gamma_i \backslash X$.

"If Γ_i is a sequence of lattices in a higher rank simple Lie group, the $M_i = \Gamma_i \backslash X$ have huge inj nearly everywhere for large i"

$\underline{\text{Ex}}$ (Not true in rank 1)

M_i



$i = 5$

i copies of

$N =$ 

As $i \to \infty$, these are noncom. lattices but inj stays uniformly bdd.