The Best Experienced Payoff Dynamic in the Ultimatum Minigame

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March 12, 2025

Ultimatum game

- Ultimatum game (Güth et al., 1982) one of the most well-known in game theory
- Proposer suggests division of value a
- Receiver can accept or reject
- Rejection generates zero payoff to both
- Subgame perfect Nash equilibrium (SPNE)
  - offer 0
  - Equilibrium payoff $(a, 0)$

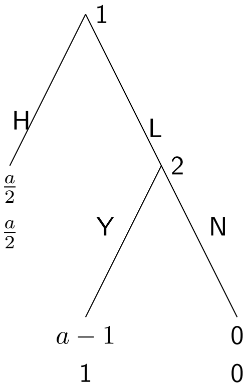
Literature

- Experimental literature shows offers of $40-50\%$ fairly common
- Offers below 20% rejected (Camerer, 2003; Güth and Kocher, 2014)
- Various factors have been suggested to explain the discrepancy between the theoretical prediction and observed behavior
  - altruism and fairness norms (Levine, 1998; Fehr and Schmidt, 1999)
  - cross-cultural differences (Henrich et al., 2005)
  - size of stakes (Andersen et al., 2011)
- We apply evolutionary game theory to the problem
  - Best experienced payoff (BEP) dynamic (Sethi, 2000; Sandholm et al., 2019)
  - Examine whether a state with significant positive offer can be stable

Ultimatum Minigame

- Original ultimatum game with a continuous range of offers $[0, a]$ too cumbersome for evolutionary analysis
- Consider a simplified version called ultimatum minigame (Gale et al., 1995)
- Proposer offers H (high) or L (low)
- H offer is $\frac{a}{2}$. Always accepted and game ends
  - motivated by experimental findings that 50% offers quite common
- L offer is 1. May be accepted (Y) or rejected (N)
  - Rejection leads to $(0, 0)$ payoff

Extensive form



- $a > 2$
- SPNE: (L, Y)

Population game

	Y	N	
H	$\frac{a}{2}, \quad \frac{a}{2}$	$\frac{a}{2}, \quad \frac{a}{2}$	x_1
L	$a-1, \;$ 1	0, 0	x_2
	y_1	y_2	

- Players from two populations randomly matched to play normal form
- x_1 is the proportion of player 1 playing H
- y_1 is the proportion of player 2 playing Y
- (L, Y) or $(x_1, y_1) = (0, 1)$ strict equilibrium
- In addition, $(1, y_1)$ with $y_1 \in \left[0, \frac{a}{2(a-1)}\right]$ a connected component of Nash equilibria

BEP dynamic

- Agents try their strategies k times. Choose the strategy that generates highest average payoff
  - procedural rationality (Osborne and Rubinstein, 1998)
- Play that strategy till the next random revision opportunity
- Let $w_1(y_1)$ and $w_2(x_1)$ be the probabilities of player 1 and 2 choosing H and Y respectively
- Under $k-$BEP dynamic

$$\dot{x}_1(t) = w_1(y_1(t)) - x_1(t)$$
$$\dot{y}_1(t) = w_2(x_1(t)) - y_1(t).$$

- In case of a tie in average payoffs, assume players play SPE strategies (L, Y)

Applications of BEP dynamic

- BEP dynamic applied to a variety of games
  - common pool resources (Cárdenas et al, 2015), public goods game (Mantilla et al., 2020), the centipede game (Sandholm et al., 2019), the prisoner's dilemma (Arigapudi et al., 2021), trust game (Arigapudi and Lahkar, 2024)
- Non-Nash stable rest points can arise. Explain observed non-Nash behavior
- Under BEP strategy revision process, agents lack information required to play Nash equilibrium
  - they do not best respond
  - but as k increases, the dynamic behaves like the best response dynamic

Replicator dynamic in Gale et al. (1995)

- Gale et al. (1995) apply the replicator dynamic to the ultimatum minigame
- Connected component of Nash equilibria plays major role
- Connected component has high offers
- Locally stable under the replicator dynamic
- Can explain high offers under this dynamic
- This component does not play any role in our analysis
- Instead, stable non-Nash states arise with significant high offer

1−BEP dynamic

- Players try each strategy only once
- Player 1: Average payoff from H is $\frac{a}{2}$
- Average payoff from L is

$$a - 1 \text{ with probability } y_1$$
$$0 \text{ with probability } 1 - y_1.$$

- As $a > 2$, $a - 1 > \frac{a}{2}$. Hence, $w_1(y_1) = 1 - y_1$. So,

$$\dot{x}_1 = 1 - y_1 - x_1. \tag{1}$$

1-BEP dynamic

- Player 2: Average payoff from Y is

$$\frac{a}{2} \text{ with probability } x_1$$
$$1 \text{ with probability } 1 - x_1.$$

- Average payoff from N is

$$\frac{a}{2} \text{ with probability } x_1$$
$$0 \text{ with probability } 1 - x_1$$

- As $a > 2$, $\frac{a}{2} > 1$. Hence,

$$w_2(x_1) = x_1 + (1 - x_1)^2 = 1 - x_1 + x_1^2$$

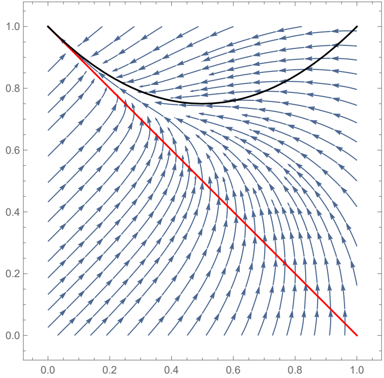
- Therefore,

$$\dot{y}_1 = 1 - x_1 + x_1^2 - y_1. \tag{2}$$

Convergence under $1-$BEP dynamic

Proposition

Consider the $1-$BEP dynamic (1) and (2) in the large population ultimatum minigame. This dynamic has a unique rest point, $(x_1^*, y_1^*) = (0, 1)$. This is the strict equilibrium. Moreover, this rest point is globally asymptotically stable.



Sample size $k \geq 2$

- If $k = 1$, global convergence to the strict equilibrium
- What about $k \geq 2$?
- (L, Y) unstable if (Theorem 3, Arigapudi et al., 2021)

$$ku_1(H, Y) > (k-1)u_1(L, Y) + u_1(L, N)$$
$$u_2(H, N) + (k-1)u_2(L, N) > ku_2(L, Y).$$

- Intuitively, a revising player prefers his non-equilibrium action when most of the k sampled opponents play their equilibrium strategy

Proposition

Let the sample size $k \geq 2$. Then, the strict equilibrium (L, Y) is (locally) stable under the $k-$BEP dynamic if

$$k \geq \min\left\{2\left(\frac{a-1}{a-2}\right), \frac{a}{2}\right\}. \tag{3}$$

Otherwise, it is unstable. Hence, if $k \geq 3$, (L, Y) is stable for all $a > 2$. If $k = 2$, (L, Y) is stable if $a \leq 4$ and unstable if $a > 4$.

Non-Nash stability

- Thus, when $k = 2$, the strict equilibrium may be unstable if $a > 4$
  - A non-Nash state may be (almost) globally stable
  - (almost) because strict equilibrium always a rest point
- If $k \geq 3$, the strict equilibrium is always stable
  - but it may only be locally stable and not globally
  - some other non-Nash state may also be locally stable
- We examine these possibilities for $k = 2$ and $k = 3$

$2-$BEP Dynamic: Player 1

- Player 1's average payoff from H is $\frac{a}{2}$
- Average payoff from L

$$(a-1) \text{ with probability } y_1^2$$
$$\frac{a-1}{2} \text{ with probability } 2y_1(1-y_1)$$
$$0 \text{ with probability } (1-y_1)^2$$

- $a > 2$ implies $a - 1 > \frac{a}{2} > \frac{a-1}{2}$. Therefore,

$$w_1(y_1) = 2y_1(1-y_1) + (1-y_1)^2$$

- Hence,

$$\begin{aligned} \dot{x}_1 &= 2y_1(1-y_1) + (1-y_1)^2 - x_1 \\ &= 1 - y_1^2 - x_1. \end{aligned} \tag{4}$$

$2-$BEP dynamic: Player 2's average payoff

- For player 2, average payoff from Y is

$$\frac{a}{2} \text{ with probability } x_1^2$$
$$\frac{a}{4} + \frac{1}{2} \text{ with probability } 2x_1(1-x_1)$$
$$1 \text{ with probability } (1-x_1)^2$$

- Average payoff from N is

$$\frac{a}{2} \text{ with probability } x_1^2$$
$$\frac{a}{4} \text{ with probability } 2x_1(1-x_1)$$
$$0 \text{ with probability } (1-x_1)^2$$

$2-$BEP dynamic for player 2: Two cases

- For player 2, $a > 4$ implies $\frac{a}{2} > \frac{a}{4} + \frac{1}{2} > \frac{a}{4} > 1$. Hence,

$$w_2(x_1) = x_1^2 + 2x_1(1-x_1)(2x_1(1-x_1) + (1-x_1)^2) + (1-x_1)^4$$

- Therefore,

$$\begin{aligned} \dot{y}_1 &= x_1^2 + 2x_1(1-x_1)(2x_1(1-x_1) + (1-x_1)^2) + (1-x_1)^4 - y_1 \\ &= x_1^2 + 4x_1^2(1-x_1)^2 + 2x_1(1-x_1)^3 + (1-x_1)^4 - y_1. \end{aligned} \quad (5)$$

- $a \in (2, 4]$ (tie-breaking) implies $\frac{a}{2} > \frac{a}{4} + \frac{1}{2} > 1 \geq \frac{a}{4}$. Hence,

$$w_2(x_1) = x_1^2 + (2x_1(1-x_1) + (1-x_1)^2)^2$$

- Hence, probability of Y is higher than when $a \in (2, 4]$. We obtain

$$\begin{aligned} \dot{y}_1 &= x_1^2 + (2x_1(1-x_1) + (1-x_1)^2)^2 - y_1 \\ &= 1 - x_1^2 + x_1^4 - y_1. \end{aligned} \quad (6)$$

Results for $2-$BEP dynamic

Proposition

Consider the $2-$BEP dynamic consisting of (4) and (5) in the large population ultimatum minigame when $a > 4$. This dynamic has two rest points, $(x_1, y_1) = (0, 1)$ and $(x_1, y_1) = (0.5268, 0.6879)$. The interior rest point $(0.5268, 0.6879)$, which is not a Nash equilibrium, is almost globally asymptotically stable under the $2 - BEP$ dynamic.

Proposition

Consider the $2-$BEP dynamic consisting of (4) and (6) in the large population ultimatum minigame F when $a \in (2, 4]$. The dynamic has a unique rest point $(x_1, y_1) = (0, 1)$, which is the strict equilibrium of F. This rest point is globally asymptotically stable under this dynamic.

Phase portraits for the $2-$BEP dynamic

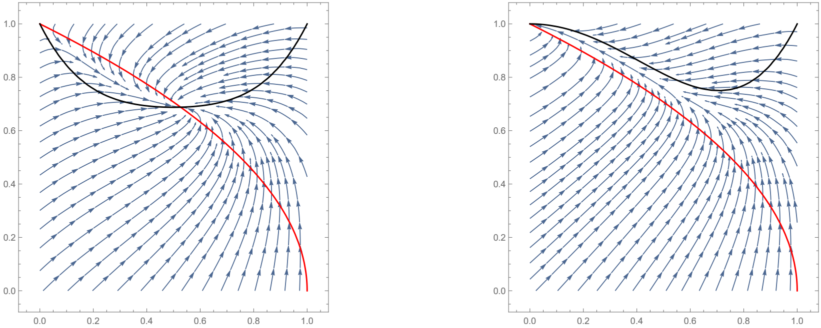


Figure: Phase portraits of the $2-$BEP dynamic. In the left panel, $a > 4$. In the right panel, $a \in (2, 4]$

- Intuitively, greater probability of Y (low offer is accepted) when $a \in (2, 4]$
- Hence, player 1 has lower incentive to make high offer
- Drives dynamics to the strict equilibrium (L, Y)

2-BEP dynamic: Summary

- When $a > 4$, a non-Nash state is (almost) globally asymptotically stable
- At that state, more than 50% of high offers
- But when $a \in (2, 4]$, the strict equilibrium is globally asymptotically stable
- All offers are low and accepted
- Thus, chances of high offer increase with higher stakes
- Stakes matter (Andersen et al., 2021)

$k = 3$: Player 1's payoffs

- Player 1's average payoff from H is $\frac{a}{2}$
- Average payoff from L is

$$
\begin{aligned}
&a - 1 \text{ with probability } y_1^3 \\
&\frac{2}{3}(a-1) \text{ with probability } 3y_1^2(1-y_1) \\
&\frac{1}{3}(a-1) \text{ with probability } 3y_1(1-y_1)^2 \\
&0 \text{ with probability } (1-y_1)^3.
\end{aligned}
$$

$k = 3$: Player 2's payoffs

- Average payoff from Y is

$$\frac{a}{2} \text{ with probability } x_1^3$$
$$\frac{a+1}{3} \text{ with probability } 3x_1^2(1-x_1)$$
$$\frac{a+4}{6} \text{ with probability } 3x_1(1-x_1)^2$$
$$1 \text{ with probability } (1-x_1)^3.$$

- Average payoff from N is

$$\frac{a}{2} \text{ with probability } x_1^3$$
$$\frac{a}{3} \text{ with probability } 3x_1^2(1-x_1)$$
$$\frac{a}{6} \text{ with probability } 3x_1(1-x_1)^2$$
$$0 \text{ with probability } (1-x_1)^3.$$

3−BEP dynamic: $a > 6$

- For player 1, $a - 1 > \frac{2}{3}(a-1) > \frac{a}{2} > \frac{1}{3}(a-1)$. Hence,

$$w_1(y_1) = 3y_1(1-y_1)^2 + (1-y_1)^3$$

- Therefore,

$$\begin{aligned}\dot{x}_1 &= 3y_1(1-y_1)^2 + (1-y_1)^3 - x_1 \\ &= 1 - 3y_1^2 + 2y_1^3 - x_1. \end{aligned} \tag{7}$$

- For player 2, $\frac{a}{2} > \frac{a+1}{3} > \frac{a}{3} > \frac{a+4}{6} > \frac{a}{6} > 1$. Hence,

$$\begin{aligned}w_2(x_1) = x_1^3 + 3x_1^2(1-x_1)(3x_1^2(1-x_1) + 3x_1(1-x_1)^2 + (1-x_1)^3) \\ + 3x_1(1-x_1)^2(3x_1(1-x_1)^2 + (1-x_1)^3) + (1-x_1)^6.\end{aligned}$$

- Therefore,

$$\begin{aligned}\dot{y}_1 &= x_1^3 + 3x_1^2(1-x_1)(3x_1^2(1-x_1) + 3x_1(1-x_1)^2 + (1-x_1)^3) \\ &\quad + 3x_1(1-x_1)^2(3x_1(1-x_1)^2 + (1-x_1)^3) + (1-x_1)^6 - y_1 \\ &= 1 - 3x_1 + 12x_1^2 - 28x_1^3 + 39x_1^4 - 30x_1^5 + 10x_1^6 - y_1. \end{aligned} \tag{8}$$

$3-$BEP dynamic: Results

Proposition

Consider the $3-$BEP dynamic consisting of (7) and (8) in the large population ultimatum minigame when $a > 6$. The dynamic has three rest points.

1. *The strict equilibrium $(0, 1)$, which is (locally) asymptotically stable.*
2. $(x_1, y_1) = (0.0741, 0.8333)$, *which is a saddle point and is, hence, unstable.*
3. $(x_1, y_1) = (0.2114, 0.7037)$, *which is (locally) asymptotically stable.*

- For $a \leq 6$, we have four subcases; $a \in (4, 6]$, $a = 4$, $a \in (3, 4)$, $a \in (2, 3]$
- In each subcase, the strict equilibrium $(0, 1)$ is globally asymptotically stable

Phase portraits for the $3-$BEP dynamic

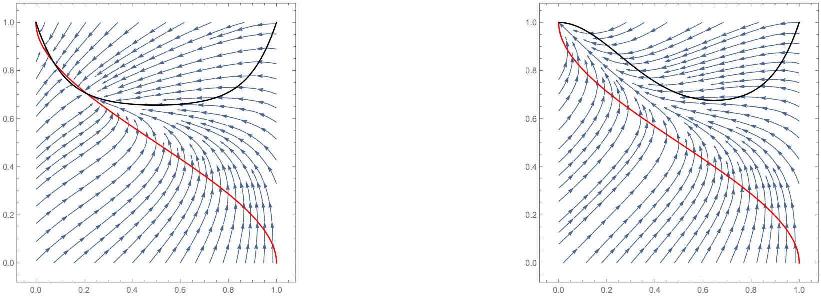


Figure: Phase portrait of the $3-$BEP dynamic. In the left panel, for $a > 6$. In the right panel, $a \in (4, 6]$.

- For other subcases where $a \leq 6$, phase portrait similar to the right one (same convergence)
- Similar intuition as in $k = 2$. For low a, probability of Y increases
- With greater chances of low offer being accepted, player 1 makes more low offers
- Convergence to (L, Y)

$3-$BEP dynamic: Summary

- When $a > 6$, locally stable non-Nash rest point
- Over 20% high offers
- Thus, if there is a significant proportion of high offers initially, it can persist
- But strict equilibrium also locally stable
- For $a \leq 6$, strict equilibrium globally stable

Conclusion

- We have considered the $2-$BEP and $3-$BEP dynamics
- A non-Nash rest point is almost globally asymptotically stable when $k = 2$ for $a > 4$
- A non-Nash rest point is locally asymptotically stable when $k = 3$ for $a > 6$
- General $k-$BEP dynamic intractable
- But consideration of $k = 4$ and $k = 5$ reveal a similar property
  - locally asymptotically stable non-Nash rest point for $a > 2k$ (stakes matter)
- Another interpretation
  - Given a, as k increases, the strict equilibrium becomes globally asymptotically stable
  - As k increases, the BEP dynamic behaves like the best response dynamic