Today The Chabauty topology

Let X be a locally compact metric space and let $\mathcal{C}(X)$ be the set of all closed subsets of X. The Chabauty topology on $\mathcal{C}(X)$ is gen'ed by

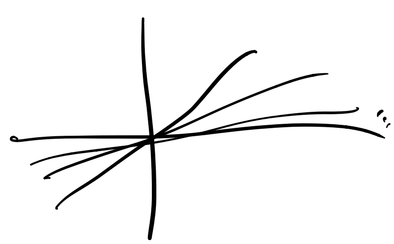
$$\underset{\substack{\uparrow \\ \text{compact} \\ \text{subset of } X}}{\mathcal{U}(K)} = \{ A \in \mathcal{C}(X) \mid A \cap K = \emptyset \}, \qquad \underset{\substack{\uparrow \\ \text{open in} \\ X}}{\mathcal{V}(U)} = \{ A \in \mathcal{C}(X) \mid A \cap U \neq \emptyset \}.$$

Fact $C_n \longrightarrow C \in \mathcal{C}(X)$ in the Chabauty top iff

1) any $x \in C$ is the limit of a sequence $x_n \in C_n$

2) any accum pt of a sequence $x_n \in C_n$ lie in C.

Ex If $X = \mathbb{R}^2$, lines through the origin at angle θ w/ the horizontal converge to the x-axis as $\theta \to 0$.



Facts 1) $\mathcal{C}(X)$ is compact

2) If X is compact, the Chabauty top is induced by the Hausdorff metric:

$$d_H(A,B) = \inf\{\varepsilon \mid A \subset N_\varepsilon(B),\ B \subset N_\varepsilon(A)\}$$

3) If X is proper (closed balls are compact), pick $p \in X$ and let $d_R(A,B)$ be the Hausdorff dist between $A \cap B(p,R)$, $B \cap B(p,R)$.

Then the Chabauty top on X is induced by

$$d(A,B) = \int_0^\infty e^{-R}\, d_R(A,B)\, dR$$

See B, 2017. There are lots of other metrics.

Rmk Curtis–Schori '74 show that when X is any compact, connected & locally connected metric space w/ ≥ 2 pts, then $\mathcal{C}(X)$ is homeo to the Hilbert cube $[0,1]^{\infty}$.

When G is a $\overline{\text{(l.c.s.c)}}$ topological group, the subspace $\mathrm{Sub}_G \subset \mathcal{C}(G)$ of closed subgroups of G is the Chabauty space of G.

Note: Sub_G is closed in $\mathcal{C}(G)$, hence compact.

Rmk Mahler's compactness thm says that for every $\varepsilon, V > 0$, the set of lattices L (discrete sgps $\cong \mathbb{Z}^n$) in $\mathbb{R}^n$ with

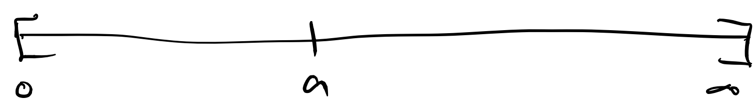
1) $|v| \geq \varepsilon \quad \forall\, v \in L \setminus 0$
2) $\mathrm{vol}(L \backslash \mathbb{R}^n) \leq V$

is compact, say in the Chabauty top.

Chabauty '50 introduced this top to generalize Mahler's thm to lattices in other top groups.

$\underline{Ex}$ $\mathrm{Sub}_{\mathbb{R}} = \{\mathbb{R}\} \cup \{a\mathbb{Z} \mid a>0\} \cup \{e\}$

$\cong$

$[0,\infty] = $ 

$\underline{\text{Thm}}$ (Hubbard - Pourezza '78)

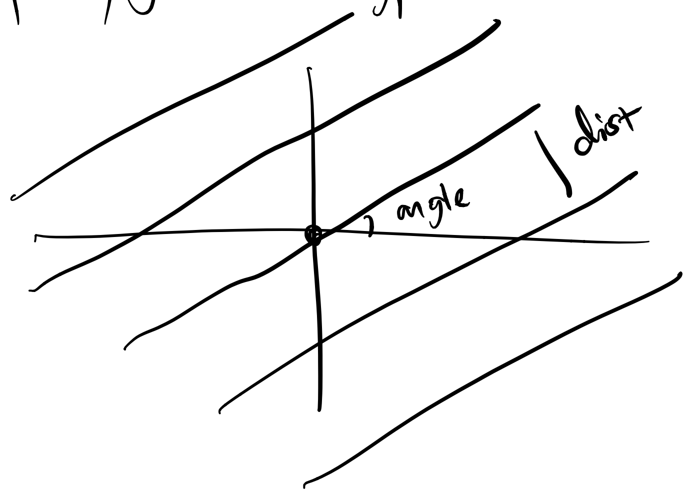
$\mathrm{Sub}_{\mathbb{R}^2}$ is homeo to S^4. Moreover, if

$$\mathcal{L} = \underbrace{\{\text{lattices}\}}_{\text{discrete, } \cong \mathbb{Z}^2} \subset \mathrm{Sub}_{\mathbb{R}^2}$$

then $(\mathrm{Sub}_{\mathbb{R}^2}, \mathcal{L}) \cong \text{Suspension}(S^3, S^3 \setminus \text{trefoil})$

$\underline{\text{Rmk}}$ 1) closed Sgps of $\mathbb{R}^2$ are isomorphic to either $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}\times\mathbb{R}$, $\mathbb{Z}^2$ or $\mathbb{R}^2$.

Easy to see that the space of lattices is 4-dim, and the complement is 2-dim, e.g. there are 2 degrees of freedom in specifying a sgp $\cong \mathbb{Z} \times \mathbb{R}$



2) The statement about the trefoil is related to the fact that

$$\left\{ \begin{array}{l} \text{unit area} \\ \text{lattices in } \mathbb{R}^2 \end{array} \right\} \cong SL_2\mathbb{R} / SL_2\mathbb{Z}$$

$$\cong T^1\left(\mathbb{H}^2 / SL_2\mathbb{Z} \right)$$

modular orbifold

$$\cong \text{trefoil complement}$$

Rmk In higher dims, $\mathrm{Sub}_{\mathbb{R}^n}$ isn't a mfld. But Kloeckner '08 shows it's a stratified space and is simply connected.

See also Bridson, de la Harpe, Kleptsyn '08 for lots of ex's.

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Groups vs framed X-mflds

S'pose G is a noncompact semisimple Lie gp, $K \subset G$ is a maximal compact sg'p and

$$X = G/K$$

is the assoc symmetric space, which we endow w/ a left G-inv't Riem metric.

Ex $G = \mathrm{Isom}\,\mathbb{H}^n$, $K = O(n)$, $X = \mathbb{H}^n$.

$G = SL_n\mathbb{R}$, $K = SO(n)$, $X =$ space of +-def quad forms on $\mathbb{R}^n$

Fact X has nonpositive sectional curvature, and is diffeo to $\mathbb{R}^m$ for some m.

Let $\mathrm{Sub}_G^{df} \subset \mathrm{Sub}_G$ be the set of discrete, torsion free sgps of G.

If $\Gamma \in \mathrm{Sub}_G^{df}$, the quotient $M = \Gamma \backslash X$ is a Riem manifold locally isometric to X. We call such M X-mnflds.

e.g., $\mathbb{H}^n$-manifolds are hyp n-mnflds.

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Def A framed Riem manifold is a

pair (M, f), where M is a Riem manifold and f is a base frame, an ONB for some tangent space. We say

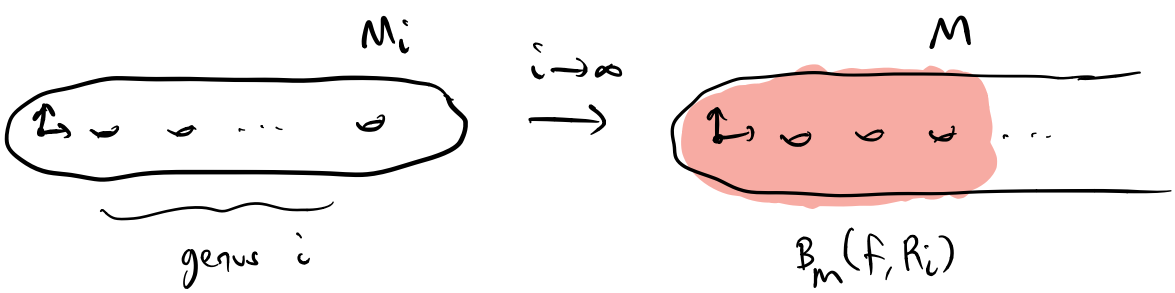
$$(M_i, f_i) \longrightarrow (M, f)$$

in the smooth topology if $\exists\, R_i \to \infty$ and embeddings

$$\varphi_i : \underset{M}{B}(f, R_i) \hookrightarrow M_i$$

$\uparrow$ ball around the basept of f

s.t. $(\varphi_i)_* f = f_i$ and if g_i, g are the Riem metrics, we have $\varphi_i^* g_i \xrightarrow{C^\infty} g$.



Thm Fix a base frame f_X for X. Then

$$\mathrm{Sub}_G^{df} \longrightarrow \left\{ \begin{array}{c} \text{framed } X\text{-mflds} \\ (M, f) \end{array} \right\} \Big/ \text{framed isometry.}$$

$$\Gamma \longmapsto \left(\Gamma \backslash X, \text{"}\Gamma f_X\text{"} \right)$$

$\uparrow$ projection of f_X to the quotient.

see ABBGNRS '17

AB Unimodular measures

on the space of Riem maflds.
is a homeo onto its image.

<u>Rmk</u> 1) When $G = \mathrm{Isom}(\mathbb{H}^n)$, $X = \mathbb{H}^n$, the map is surjective, so a <u>homeo</u>

In general, not surjective, since G may not act transitively on the frame bundle of X.

2) The Chabauty top on $\mathrm{Sub}_{\mathrm{Isom}\,\mathbb{H}^n}$, interpreted geometrically as above is really important when $n = 2, 3$; e.g. in Thurston's program to solve his geometrization conjecture.

A <u>case study</u>: $G = \mathrm{Isom}^+ \mathbb{H}^2 \cong PSL_2\mathbb{R}$,
joint w/ Lazarovich, Leitner (BLL)

Closed sgps of G are either discrete, <u>elementary</u> (they have a finite orbit in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$), or $= G$.

The ~~elementary groups~~ Example include

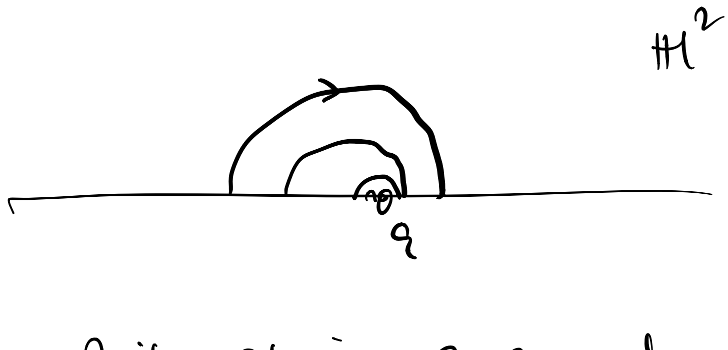
1) The group $A(\gamma)$ of translations along a geodesic $\gamma \subset \mathbb{H}^2$.

2) The group $U(q)$ of all parabolic isometries fixing $q \in \partial\mathbb{H}^2$.

3) The group $K(p)$ of rotations around $p \in \mathbb{H}^2$.

There are others: cyclic groups, ∞-dihedral, $\mathbb{R} \rtimes \mathbb{R}$, $\mathbb{R} \rtimes \mathbb{Z}$. The topology on the space of elem grps is completely understood, e.g. if $\gamma_n \longrightarrow q \in \partial\mathbb{H}^2$,

$$A(\gamma_n) \longrightarrow U(q).$$



See Baik–Clavier 2013 and BLL.

Discrete gps $\Gamma \leq G$ correspond to framed hyp 2-orbifolds. (Orbifold instead of mfld since gps aren't torsion free.)

For every topological 2-orbifold S w/ finite top type, let

$$\mathrm{Sub}_G^S = \{\Gamma \mid {}_{\Gamma}\backslash^{\mathbb{H}^2} \cong S \text{ and has finite vol}\}.$$

$$\subseteq \mathrm{Sub}_G.$$

Then Sub_G^S is a framed version of the moduli space $\mathcal{M}(S)$ of all fin vol hyperbolic structures on S, the map

$$\begin{aligned} \mathrm{Sub}_G^S &\longrightarrow \mathcal{M}(S) \\ \Gamma &\longmapsto [{}_{\Gamma}\backslash^{\mathbb{H}^2}] \end{aligned}$$

is a fiber orbibundle and Sub_G^S is a manifld of dimension $= \dim \mathcal{M}(S) + 3$.

So, Sub_G contains the elementary gps, all these ∞-many fin dim framed moduli spaces Sub_G^S, but also tons of $\Gamma \in \mathrm{Sub}_G$ whose quotients are ∞ type surfaces, or orbifolds.

<u>Thm</u> (BLL) The path components of Sub_G are

1) $\{G\}$

2) the space Sub_G^S, where S is a sphere w/ 3 cone pts

3) the component Sub_G' of id.

<u>Conjecture</u> Sub_G' is simply connected.