QUANTUM ERGODICITY:

Let M be a compact [neg. curved] manifold, normalized volume μ.

The Laplacian has a discrete spectrum:

$$0 \le \lambda_0 \le \lambda_1 \le \dots, \quad b_n \in L^2(M)$$

$$\Delta b_n = \lambda_n b_n \qquad \|b_n\|_2 = 1$$

Let $\mu_n \in P(M)$ be defined by $\mu_n(A) = \int_A b_n^2 \, d\mu$

"square measure"

Conjecture [Rudnick-Sarnak, QUE] $\mu_n \xrightarrow{*} \mu$. STILL OPEN

Meaning of QUE: high energy eigenfunctions equidistribute: $\forall$ test function $a : M \to \mathbb{R}$

$$\int a \, d\mu_n \to \int a \, d\mu$$

T [Shnirelman, Zelditch, de Verdière]

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} \left| \int a \, d\mu_n - \int a \, d\mu \right|^2 = 0$$

Corollary: $\mu_n \xrightarrow{*} \mu$ for a density 1 subsequence

T [Lindenstrauss] $\mu_n \to \mu$ if M is arithmetic and b_n are joint for Hecke $+ \Delta$.

T [Anantharaman] lower bound on the entropy for any $\lim^* \mu_{n_i}$

T [Dyatlov-Jin] $\lim^* \mu_{n_i}$ is fully supported

PROBLEM: b_n are "frozen". No room to move.

We know b_n exists and thats it.

"Physics" would give a mixture of eigenfunctions

Berry's conjecture: "b_n behaves like a Gaussian eigenwave"

NO precise formulation!

This is what prompted the QUE question [Sarnak]

Best guess: look at the value distribution of b_n.

Conj [Hejhal - Rackner] $\nu_{b_n} \overset{*}{\to}$ normal [Gaussian]

WHAT IS NORMAL?
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

mean μ, variance σ^2

"Unique" attractor of sums of independent things.

PROBLEM: k-th moment $\int x^k \, d\nu_{b_n}$ MAY NOT CONVERGE!

If μ is compact

$$\mu_n \overset{*}{\to} \mu \iff \forall k \ \int x^k d\mu_n \to \int x^k d\mu$$

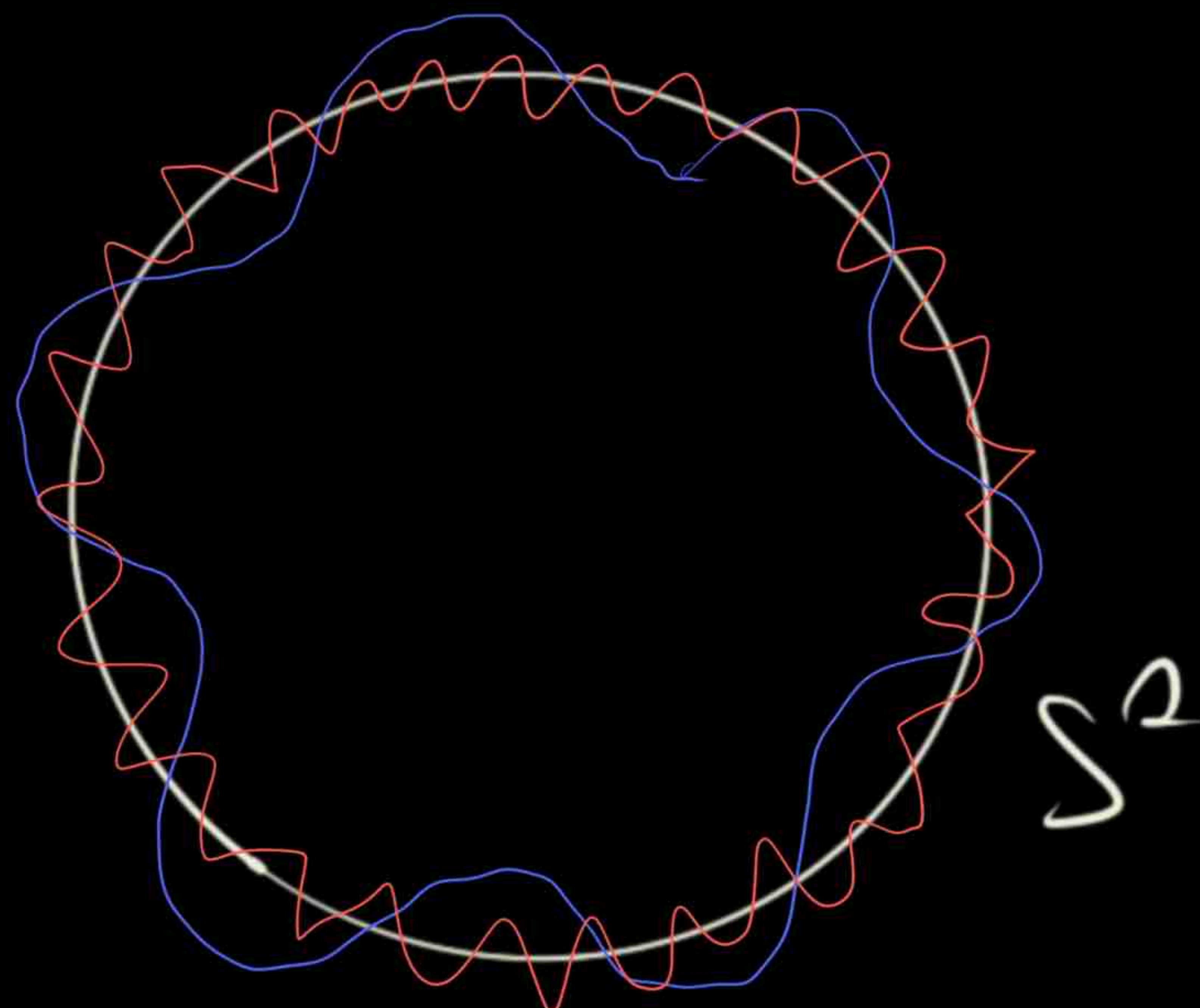
For 6-th moment, this conjecture fails!

IDEA: Take the limit of b_n themselves!

How? $M = S^1$

b_n are cosine waves

They dont converge to a meaningful object!



Hence the square measure idea of Sarnak-Rudnick

SOLUTION: rescale with the wavelength
then take a local sampling

for S^1 this gives the SIN wave.

$\boxed{\text{FRAMEWORK:}}$ Benjamini-Schramm convergence of manifolds [Abert - Biringer]

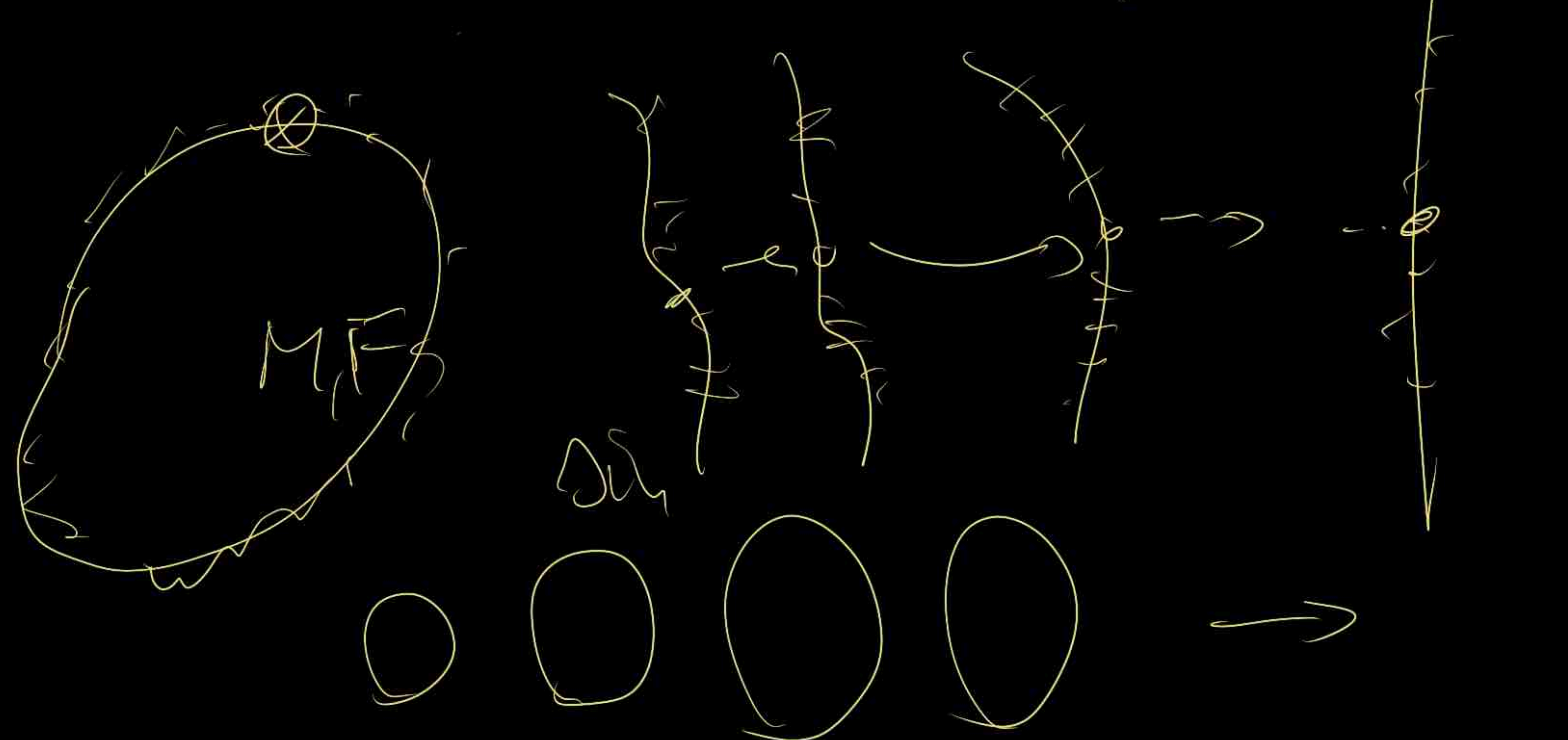
Space of rooted manifolds $\mathcal{M}$: rooted GH metric
rooted limit can do bad things

M finite volume manifold:
pick the root at random
+ weak* convergence

SAME for (M_n, F_n) where $F_n : M_n \to \mathbb{R}$

$(M_n, F_n) \quad (\cdots) \longrightarrow$

Sometimes the M_n are boring



What is the limit of manifolds?

Its a URM (unimodular) means, T^1 is invariant under geodesic flow.

What if $M_n \to X$ X fixed?

seems boring.

Example: M fixed, $M_n = n \cdot M \xrightarrow{BS} \mathbb{R}^d$!

BUT: $(M_n, F_n) \to (X, F)$ then F is NOT fixed but invariant random! (under the isometry group of X)

So (M_n, F_n) becomes a stationary object!

Fürstenberg for Szemerédi:

When $M_n \to (X, o)$ $\mathbb{R}^d$, $\mathbb{H}^d$

$\lim (M_n, F_n)$, (X, F)

F is distributionally invariant under $\mathrm{Iso}(X)$

CHANGE OF PERSPECTIVE

We had: (M, b_n) b_n is λ_n-eigenfunction

Let $M_n = \lambda_n M$ rescale the Riemannian metric.

Let $F_n = b_n$ same, but of eigenvalue 1!

$\lim (M, b_n)$ makes no sense

$\lim^{BS} (M_n, F_n)$ is an invariant random eigenwave on $\mathbb{R}^d$!

$\overset{\shortparallel}{(\mathbb{R}^d, F)}$

If exists, as lots of things may go wrong.

BERRY'S CONJECTURE IN MATH. FORM

[A - Bergeron - le Masson]

CONJECTURE: Let M be compact, neg. curved, λ_n, b_n as before. Then (M_n, b_n) BS converges to the Gaussian random eigenwave.

Now one can prove or disprove this.

T: Berry implies QUE.

Honest question: QUE is already f. hard. WHY make an even harder conjecture?

I.	II. MAKE ROOM	III. DIVIDE AND CONQUER!
JUST CAUSE ITS FUN	makes a "frozen" object into a rich dynamical one	opens much simpler problems!

T [Backhausz-Szegedy] Berry works for random d-regular graphs!

If G_n is random d-reg graph on n points b_n is an almost eigenfunction + QUE

$\Rightarrow$ Gaussian random eigenwave on T_d

Torus [Bourgain] solves for certain "nice" eigenfunctions on the Torus.

UNIFIED LEVEL AND EIGENVALUE APPROACH

$\Gamma \leq G$ lattice, $M = \Gamma \backslash X$.

Eigenvalue aspect: $\lambda_i \to \infty$, $b_i : M \to \mathbb{R}$

level aspect: $\lambda_i \to \lambda$, $\Gamma_n < \Gamma$, $M_n \xrightarrow{BS} X$

Same.

LEVEL BERRY: Let $\Gamma < SL_2(\mathbb{R})$ lattice

$M = \Gamma \backslash \mathbb{H}$ $\qquad$ $\Gamma_1 > \Gamma_2 > \dots \Gamma_n > \dots$

$M_n = \Gamma_n \backslash \mathbb{H}$ $\qquad$ b_n are λ_n-eigenfunctions with $\lambda_i \to \lambda$.

$$(M_n, b_n) \xrightarrow{*} (\mathbb{H}, F)$$

WIGNER WAVES

III. Let M be a compact neg. curved surface.
A Wigner wave is a BS limit of eigenfunctions
The farthest eigenwave from Gaussian is
SIN : $\sin(x)$ translated and rotated

Problem: Prove that SIN is <u>not</u> a Wigner wave!

Meaning: you can't locally copy and patch sin on M to turn it to an eigenfunction.

STILL HARD!

WHAT IS <u>JOINT GAUSSIAN</u>?

<u>F random function</u>, $x_1, \dots, x_d \in X$

$c_1, \dots, c_n \in \mathbb{R}$

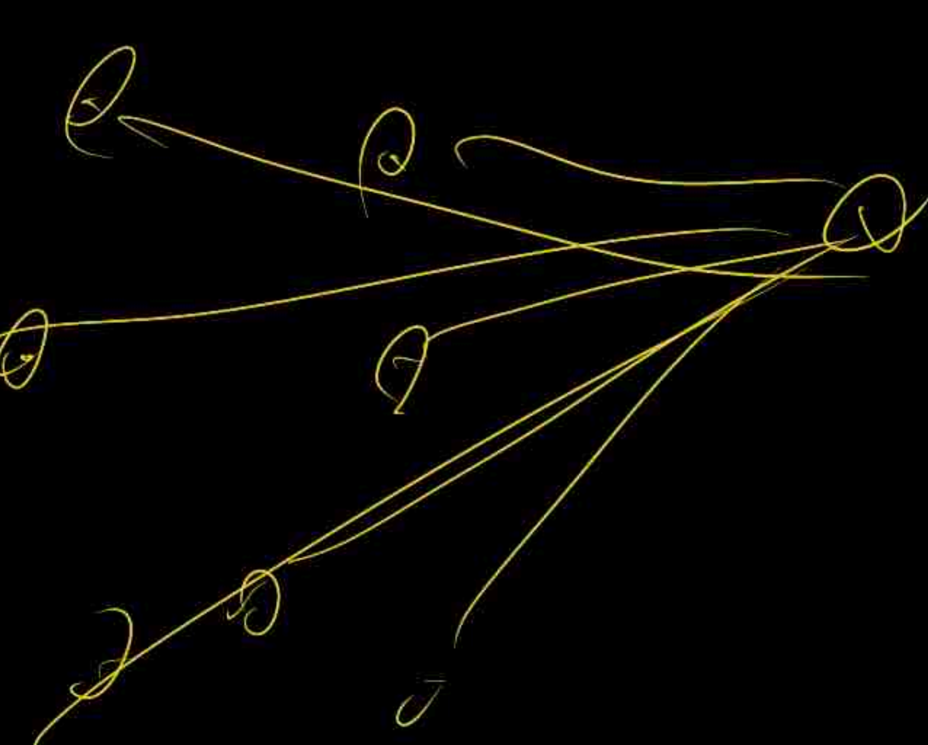
$\boxed{\sum c_i F(x_i)}$ has normal distribution

WHY CARE IF YOU DO DISCRETE GROUPS?

ℓ^2 eigenfunctions don't exist

Gaussians always do!!!

X discrete



– e is Gaussian

– iid Gaussians

– ia Gaussian

$\boxed{\cancel{X} \cdot M}$

X sym. space?

$\boxed{\text{Gaussian white noise}}$

Theorem $\forall \lambda$ $\exists$ unique

joint Gaussian λ-eigenfunction on $\mathbb{R}^d$, X

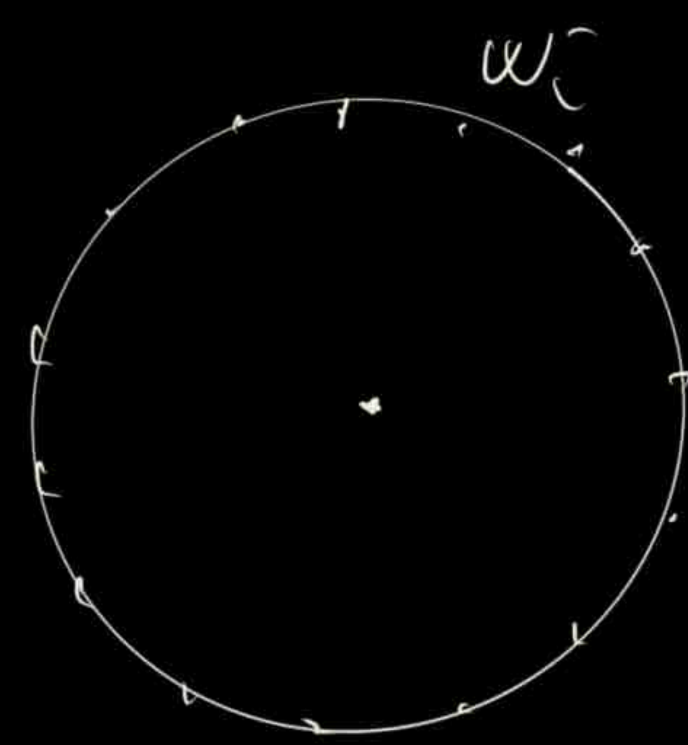
TAKE A MOMENT:

- already 6-th moment can go to hell

  reason: one bad point, maximum norm is too big.

- can already be seen on standard tori

  $\mathbb{R}^d/\mathbb{Z}^d$:



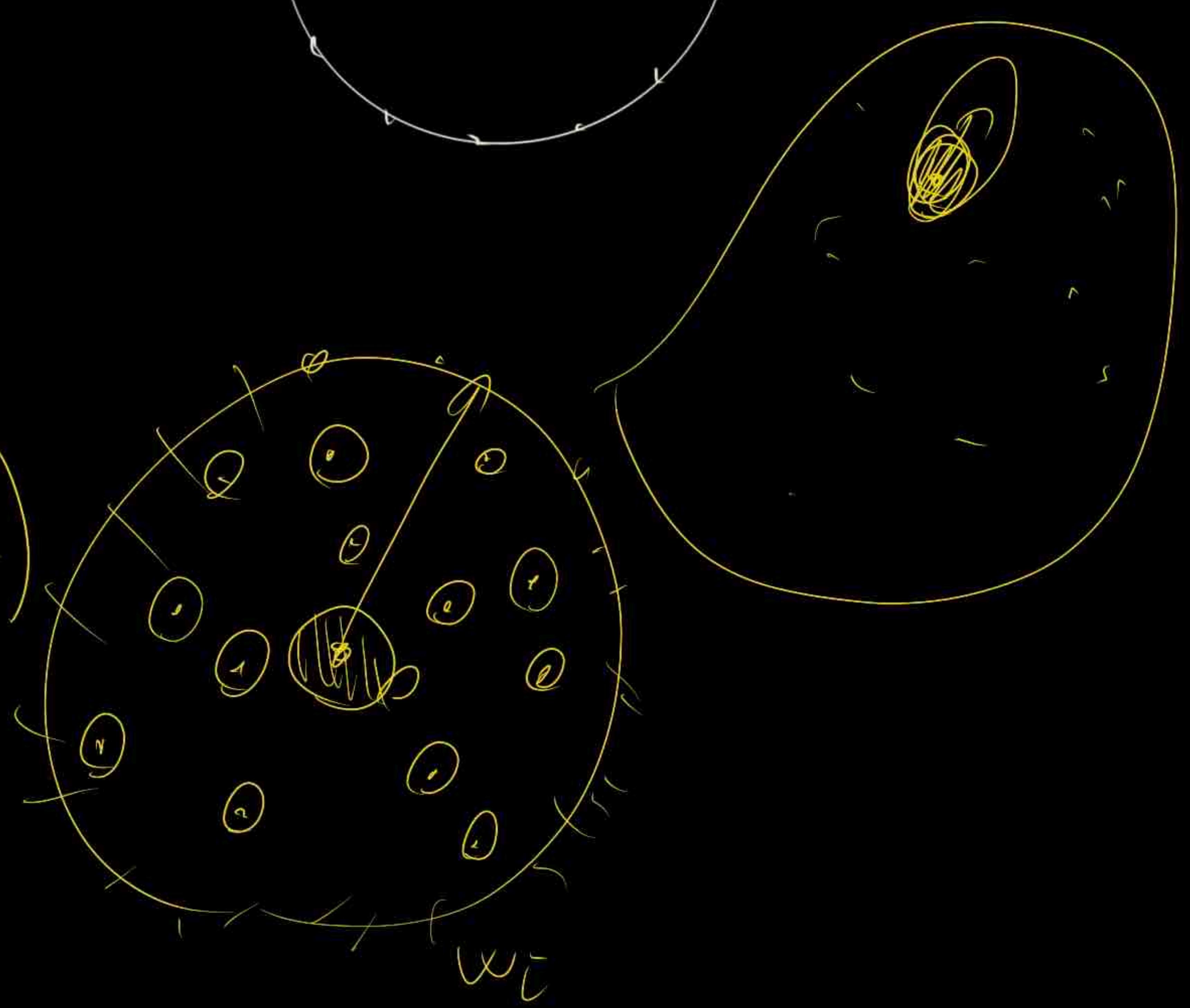
$f_w(z) = \cos\langle z, w\rangle$

all 1 at 0

torus:

$F_{w_j}(x) = \cos(w \cdot x)$

$x = 0$ 1



THANK THE ORGANIZERS!!!

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