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Analytic study of shell models of turbulence

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Abstract

In this paper we study analytically the viscous "sabra" shell model of the energy turbulent cascade. We prove the global regularity of solutions and show that the shell model has finitely many asymptotic degrees of freedom, specifically: a finite dimensional global attractor and globally invariant inertial manifolds. Moreover, we establish the existence of an exponentially decaying energy dissipation range for sufficiently smooth forcing.
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1. Introduction

Shell models of turbulence have attracted interest as useful phenomenological models that retain certain features of the Navier–Stokes equations (NSE). Their central computational advantage is the parameterization of the fluctuation of a turbulent field in each octave of wavenumbers $\lambda^n < |k_n| \leq \lambda^{n+1}$ by very few representative variables. The octaves of wavenumbers are called shells and the variables retained are called shell variables. Like in the Fourier representation of NSE, the time evolution of the shell variables is governed by an infinite system of coupled ordinary differential equations with quadratic nonlinearities, with forcing applied to the large scales and viscous dissipation affecting the smaller ones. Because of the very reduced number of interactions in each octave of wavenumbers, the shell models are drastic modifications of the original NSE in Fourier space.

The main object of this work is the "sabra" shell model of turbulence, which was introduced in [26]. For other shell models see [3,12,13,28]. A recent review of the subject emphasizing the applications of the shell models to the study of the energy cascade mechanism in turbulence can be found in [2]. It is worth noting that the results of this article apply equally well to the well-known Gledzer–Okhitani–Yamada (GOY) shell model, introduced in [28]. The main difference between shell models and the NSE written in Fourier space is the fact that the shell models contain only local interaction between the modes. We would like to point out that our analysis could be performed also in the presence of the interactions with long, but finite, range.

The sabra shell model of turbulence describes the evolution of complex Fourier-like components of a scalar velocity field denoted by u_n. The associated one-dimensional wavenumbers are denoted by k_n, where the discrete index n is referred to as the "shell index". The equations of motion of the sabra shell model of turbulence have the following form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \mathrm{i}(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \tag{1}$$

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for $n = 1, 2, 3, \ldots$, and the boundary conditions are $u_{-1} = u_0 = 0$. The wavenumbers k_n are taken to be

$$k_n = k_0 \lambda^n, \tag{2}$$

with $\lambda > 1$ being the shell spacing parameter, and $k_0 > 0$. Although the equation does not capture any geometry, we will consider $L = k_0^{-1}$ as a fixed typical length scale of the model. In analogy to the Navier–Stokes equations $\nu > 0$ represents a kinematic viscosity and f_n are the Fourier components of the forcing.

The three parameters of the model a, b and c are real. In order for the sabra shell model to be a system of the hydrodynamic type we require that in the inviscid ($\nu = 0$) and unforced ($f_n = 0, n = 1, 2, 3, \ldots$) case the model will have at least one quadratic invariant. Requiring conservation of the energy

$$E = \sum_{n=1}^{\infty} |u_n|^2 \tag{3}$$

leads to the following relation between the parameters of the model, which we will refer to as an energy conservation condition

$$a + b + c = 0. \tag{4}$$

Moreover, in the inviscid and unforced case the model possesses another quadratic invariant

$$W = \sum_{n=1}^{\infty} \left(\frac{a}{c}\right)^n |u_n|^2. \tag{5}$$

The physically relevant range of parameters is $|a/c| < 1$ (see [26] for details). For $-1 < \frac{a}{c} < 0$ the quantity W is not sign-definite and therefore it is common to associate it with the helicity — in analogy to the 3D turbulence. In that regime we can rewrite relation (5) in the form

$$W = \sum_{n=1}^{\infty} (-1)^n k_n^{\alpha} |u_n|^2, \tag{6}$$

for

$$\alpha = \log_{\lambda} \left|\frac{c}{a}\right|. \tag{7}$$

The 2D parameters regime corresponds to $0 < \frac{a}{c} < 1$. In that case the second conserved quadratic quantity W is identified with the enstrophy, and we can rewrite the expression (5) in the form

$$W = \sum_{n=1}^{\infty} k_n^{\alpha} |u_n|^2, \tag{8}$$

where α is also defined by Eq. (7).

The well-known question of global well-posedness of the 3D Navier–Stokes equations is a major open problem. In this work we study analytically the shell model (1). Specifically, we show global regularity of weak and strong solutions of (1) and smooth dependence on the initial data. This is a basic step for studying the long-time behavior of this system and for establishing the existence of a finite dimensional global attractor.

Our next result concerns the rate of the decay of the spectrum (i.e. $|u_n|$) as function of the wavenumber k_n. Having shown the existence of strong solutions, we prove that $|u_n|$ decays as a polynomial in k_n for n large enough, depending on the physical parameters of the model. However, if we assume that the forcing f_n decays exponentially with k_n, which is a reasonable assumption, because in most numerical simulations the forcing is concentrated in only finitely many shells, then we can show more. In fact, following the arguments presented in [21] and [20] we show that in this case the solution of (1) belongs to a certain Gevrey class of regularity, namely $|u_n|$ also decays exponentially in k_n — evidence of the existence of a dissipation range associated with this system, which is an observed feature of turbulent flows (see also [11] for similar results concerning 3D turbulence). Similar results on the existence of the exponentially decaying dissipation range in the GOY shell models of turbulence were obtained previously in [30] by the asymptotic matching between the nonlinear and viscous terms. Our results set a mathematically rigorous ground for that claim. Moreover we show that the exponentially decaying dissipation range exists for all values of the parameters of the model, in particular also for $c = 0$, improving the result of [30].

Classical theories of turbulence assert that the turbulent flows governed by the Navier–Stokes equations have a finite number of degrees of freedom (see, e.g., [13,25]). Arguing in the same vein one can state that the sabra shell model with non-zero viscosity has finitely many degrees of freedom. Such a physical statement can be interpreted mathematically in more than one way: as finite dimensionality of the global attractor, as existence of finitely many determining modes, and as existence of a finite dimensional inertial manifold.

Our first results concern the long-time behavior of the solutions of the system (1). We show that the global attractor of the sabra model has a finite fractal and Hausdorff dimensions. Moreover, we show that the number of determining modes of the sabra shell model equations is also finite. Remarkably, our upper bounds for the number of determining modes and the fractal and Hausdorff dimensions of the global attractor of the sabra shell model equation coincide (see Remark 1). This is to be contrasted with the present knowledge about Navier–Stokes equations, where the two bounds are qualitatively different. In the forthcoming paper we will investigate the lower bounds on the Hausdorff dimension of the global attractor.

Finally, and again in contrast with the present knowledge for the Navier–Stokes equations, we show that the sabra shell model equation has an inertial manifold $\mathcal{M}$. Inertial manifolds are finite dimensional Lipschitz globally invariant manifolds that attract all bounded sets in the phase space at an exponential rate and, in particular, contain the global attractor. We show that $\mathcal{M} = graph(\Phi)$, where Φ is a C^1 function which slaves the components u_n, for all $n \geq N$, as a function of $\{u_k\}_{k=1}^N$, for N large enough depending on the physical parameters of the problem, i.e. ν, f, λ, a, b, and c. The reduction of the system (1) to the manifold $\mathcal{M}$ yields a finite dimensional system of ordinary differential equations. This is the ultimate and best notion of system reduction that one could hope for. In other words, an inertial manifold is an exact rule for parameterizing the large modes (infinitely many of them) in terms of the low ones (finitely many of them). The concept of an inertial manifold for nonlinear evolution equations was first introduced in [17] (see also [8,9,18,31,33]).

2. Preliminaries and functional setting

Following the classical treatment of the NSE, and in order to simplify the notation we are going to write the system (1) in the following functional form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,u) = f \tag{9a}$$

$$u(0) = u^{in}, \tag{9b}$$

in a Hilbert space H. The linear operator A as well as the bilinear operator B will be defined below. In our case, the space H will be the sequences space ℓ^2 over the field of complex numbers $\mathbb{C}$. For every $u, v \in H$, the scalar product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$ are defined as

$$(u,v) = \sum_{n=1}^{\infty} u_n v_n^*, \quad |u| = \left(\sum_{n=1}^{\infty} |u_n|^2\right)^{1/2}.$$

We denote by $\{\phi_j\}_{j=1}^{\infty}$ the standard canonical orthonormal basis of H, i.e. all the entries of ϕ_j are zero except at the place j it is equal to 1.

The linear operator $A : D(A) \to H$ is a positive definite, diagonal operator defined through its action on the elements of the canonical basis of H by

$$A\phi_j = k_j^2 \phi_j,$$

where the eigenvalues k_j^2 satisfy Eq. (2). $D(A)$ — the domain of A is a dense subset of H. Moreover, it is a Hilbert space, when equipped with the graph norm

$$\|u\|_{D(A)} = |Au|, \quad \forall u \in D(A).$$

Using the fact that A is a positive definite operator, we can define the powers A^s of A for every $s \in \mathbb{R}$

$$\forall u = (u_1, u_2, u_3, \ldots), \quad A^s u = (k_1^{2s} u_1, k_2^{2s} u_2, k_3^{2s} u_3, \ldots).$$

Furthermore, we define the spaces

$$V_s := D(A^{s/2}) = \left\{ u = (u_1, u_2, u_3, \ldots) : \sum_{j=1}^{\infty} k_j^{2s} |u_j|^2 < \infty \right\},$$

which are Hilbert spaces equipped with the scalar product

$$(u,v)_s = (A^{s/2}u, A^{s/2}v), \quad \forall u, v \in D(A^{s/2}),$$

and the norm $|u|_s^2 = (u,u)_s$, for every $u \in D(A^{s/2})$. Clearly

$$V_s \subseteq V_0 = H \subseteq V_{-s}, \quad \forall s > 0.$$

The case of $s = 1$ is of a special interest for us. We denote by $V = D(A^{1/2})$ a Hilbert space equipped with a scalar product

$$((u, v)) = (A^{1/2}u, A^{1/2}v), \quad \forall u, v \in D(A^{1/2}).$$

We consider also $V' = D(A^{-1/2})$ — the dual space of V. We denote by $\langle \cdot, \cdot \rangle$ the action of the functionals from V' on elements of V. Note that the following inclusion holds

$$V \subset H \equiv H' \subset V',$$

and hence the H scalar product of $f \in H$ and $u \in V$ is the same as the action of f on u as a functional in the duality between V' and V

$$\langle f, u \rangle = (f, u), \quad \forall f \in H, \forall u \in V.$$

Observe also that for every $u \in D(A)$ and every $v \in V$ one has

$$((u, v)) = (Au, v) = \langle Au, v \rangle .$$

Since $D(A)$ is dense in V one can extend the definition of the operator $A : V \to V'$ in such a way that

$$\langle Au, v \rangle = ((u, v)), \quad \forall u, v \in V.$$

In particular, it follows that

$$\|Au\|_{V'} = \|u\|, \quad \forall u \in V. \tag{10}$$

Before proceeding and defining the bilinear term B, let us introduce the sequence analogue of Sobolev functional spaces.

Definition 1. For $1 \le p \le \infty$ and $m \in \mathbb{R}$ we define sequence spaces

$$w^{m,p} := \left\{ u = (u_1, u_2, \ldots) \in : \|A^{m/2}u\|_p = \left(\sum_{n=1}^{\infty} (k_n^m |u_n|)^p \right)^{1/p} < \infty \right\},$$

for $1 \le p < \infty$, and

$$w^{m,\infty} := \left\{ u = (u_1, u_2, \ldots) \in : \|A^{m/2}u\|_\infty = \sup_{1 \le n \le \infty} (k_n^m |u_n|) < \infty \right\}.$$

For $u \in w^{m,p}$ we define its norm

$$\|u\|_{w^{m,p}} = \|A^{m/2}u\|_p,$$

where $\| \cdot \|_p$ is the usual norm in the ℓ^p sequences space. The special case of $p = 2$ and $m \ge 0$ corresponds to the sequence analogue of the classical Sobolev space, which we denote by

$$h^m = w^{m,2}.$$

Those spaces are Hilbert with respect to the norm defined above and its corresponding inner product.

The above definition immediately implies that $h^1 = V$. It is also worth noting that $V \subset w^{1,\infty}$ and the inclusion map is continuous (but not compact) because

$$\|u\|_{w^{1,\infty}} = \|A^{1/2}u\|_\infty \le \|A^{1/2}u\|_2 = \|u\|.$$

The bilinear operator $B(u, v)$ will be defined formally in the following way. Let $u, v \in H$ be of the form $u = \sum_{n=1}^{\infty} u_n \phi_n$ and $v = \sum_{n=1}^{\infty} v_n \phi_n$. Then

$$B(u, v) = -\mathrm{i} \sum_{n=1}^{\infty} \left(a k_{n+1} v_{n+2} u_{n+1}^* + b k_n v_{n+1} u_{n-1}^* + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} v_{n-1} u_{n-2} \right) \phi_n, \tag{11}$$

where here again $u_0 = u_{-1} = v_0 = v_{-1} = 0$. It is easy to see that our definition of $B(u, v)$, together with the energy conservation condition

$$a + b + c = 0$$

implies that

$$B(u, u) = -\mathrm{i} \sum_{n=1}^{\infty} \left(a k_{n+1} u_{n+2} u_{n+1}^* + b k_n u_{n+1} u_{n-1}^* - c k_{n-1} u_{n-1} u_{n-2} \right) \phi_n,$$

which is consistent with (1). In the sequel we will show that our definition of $B(u, v)$ does indeed make sense as an element of H, whenever $u \in H$ and $v \in V$ or $u \in V$ and $v \in H$. For $u, v \in H$ we will also show that $B(u, v)$ makes sense as an element of V'.

Proposition 1. (i) *$B : H \times V \to H$ and $B : V \times H \to H$ are bounded, bilinear operators. Specifically,*

$$|B(u, v)| \le C_1 |u| \|v\|, \quad \forall u \in H, v \in V, \tag{12}$$

and

$$|B(u, v)| \le C_2 \|u\| |v|, \quad \forall u \in V, v \in H, \tag{13}$$

where

$$C_1 = (|a|(\lambda^{-1} + \lambda) + |b|(\lambda^{-1} + 1)),$$
$$C_2 = (2|a| + 2\lambda|b|).$$

(ii) *$B : H \times H \to V'$ is a bounded bilinear operator and*

$$\|B(u, v)\|_{V'} \le C_1 |u||v|, \quad \forall u, v \in H. \tag{14}$$

(iii) *$B : H \times D(A) \to V$ is a bounded bilinear operator and for every $u \in H$ and $v \in D(A)$*

$$\|B(u, v)\| \le C_3 |u||Av|, \tag{15}$$

where

$$C_3 = (|a|(\lambda^3 + \lambda^{-3}) + |b|(\lambda + \lambda^{-2})).$$

(iv) *For every $u \in H$, $v \in V$*

$$\mathrm{Re}(B(u, v), v) = 0, \tag{16}$$

Proof. To prove the first statement, let $u \in H$ and $v \in V$. H is a Hilbert space, therefore,

$$|B(u, v)| = \sup_{w \in H, |w| \le 1} |(B(u, v), w)|.$$

Using the form of the eigenvalues k_n given in (2) and applying the Cauchy–Schwartz inequality we get

$$\begin{aligned} |B(u, v)| &= \sup_{w \in H, |w| \le 1} \left| \sum_{n=1}^{\infty} a k_{n+1} v_{n+2} u_{n+1}^* w_n^* + b k_n v_{n+1} w_n^* u_{n-1}^* + a k_{n-1} w_n^* u_{n-1} v_{n-2} + b k_{n-1} w_n^* v_{n-1} u_{n-2} \right| \\ &\le (|a|(\lambda^{-1} + \lambda) + |b|(\lambda^{-1} + 1)) \sup_{w \in H, |w| \le 1} |w| \|v\|_{w^{1,\infty}} |u| \le C_1 |u| \|v\|, \end{aligned}$$

where the last inequality follows from the fact that $\|v\|_{w^{1,\infty}} \le \|v\|$. The case of $u \in V$ and $v \in H$ is proved in the similar way.

As for (ii), let $u, v \in H$ and $w \in V$. Then

$$\begin{aligned} |\langle B(u, v), w \rangle| &= \left| \sum_{n=1}^{\infty} B_n(u, v) w_n^* \right| \\ &\le \left| \sum_{n=1}^{\infty} a k_{n+1} v_{n+2} u_{n+1}^* w_n^* + b k_n v_{n+1} w_n^* u_{n-1}^* + a k_{n-1} w_n^* u_{n-1} v_{n-2} + b k_{n-1} w_n^* v_{n-1} u_{n-2} \right| \\ &\le (|a|(\lambda^{-1} + \lambda) + |b|(\lambda^{-1} + 1)) \|w\|_{w^{1,\infty}} |u||v| \le C_1 \|w\| |u||v|. \end{aligned} \tag{17}$$

Similar calculations can be applied to prove the statement (iii).

The statement (iv) follows directly from the energy conservation condition (4). □

Let us fix $m \in \mathbb{N}$ and consider $H_m = span\{\phi_1, \ldots, \phi_m\}$ and let P_m be an ℓ_2 orthogonal projection from ℓ^2 onto H_m. If $u \in H$, then

$$P_m u = u^m = (u_1^m, u_2^m, \ldots, u_m^m, 0, 0, \ldots),$$

where $u_n^m = (u, \phi_n)$. In order to simplify the notation we will write u_n instead of u_n^m, whenever it will not cause any confusion. We would like to obtain the same estimates as above in Proposition 1 for the truncated bilinear term $P_m B(u^m, v^m)$. Setting $u_j = u_j^m = 0$ for all $j = 0, -1$ and $j > m$ we get

$$P_m B(u^m, v^m) = -\mathrm{i} \sum_{n=2}^{m-1} \left(a k_{n+1} v_{n+2} u_{n+1}^* \phi_{n-1} + b k_n v_{n+1} u_{n-1}^* \phi_n + a k_{n-1} u_{n-1} v_{n-2} \phi_{n+1} + b k_{n-1} v_{n-1} u_{n-2} \phi_{n+1} \right).$$

We observe that estimates similar to (14), (18) and (16) hold for the truncated bilinear operator $P_m B(u^m, v^m)$. Namely, $\forall u^m, v^m \in H_m$

$$\left\| P_m B(u^m, v^m) \right\|_{V'} \le C_1 |u^m||v^m| \tag{18}$$

and $\forall u^m, v^m \in H$

$$\mathrm{Re}(P_m B(u^m, v^m), v^m) = 0. \tag{19}$$

3. Existence of weak solutions

In this section we prove the existence of weak solutions to the system (9). Our notion of weak solutions is similar to that of the Navier–Stokes equation (see, for example, [5,32]).

Theorem 2. *Let $f \in L^2([0, T), V')$ and $u^{in} \in H$. There exists*

$$u \in L^\infty([0, T], H) \cap L^2([0, T], V), \tag{20}$$

with

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2([0, T], V'), \tag{21}$$

satisfying the weak formulation of Eq. (9), namely

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v \right\rangle + \nu \langle Au, v\rangle + \langle B(u, u), v\rangle = \langle f, v\rangle \tag{22a}$$

$$u = u^{in}, \tag{22b}$$

for every $v \in V$.

The weak solution u satisfies

$$u \in C([0, T], H). \tag{23}$$

Proof. The proof follows standard techniques for equations of Navier–Stokes type. See, for example, [5,32] or [33]. We will use the Galerkin procedure to prove the global existence and to establish the necessary *a priori* estimates.

The Galerkin approximating system of order m for Eq. (9) is an m-dimensional system of ordinary differential equations

$$\frac{\mathrm{d}u^m}{\mathrm{d}t} + \nu A u^m + P_m B(u^m, u^m) = P_m f \tag{24a}$$

$$u^m(x, 0) = P_m u^{in}(x). \tag{24b}$$

We observe that the nonlinear term is quadratic in u^m, thus by the theory of ordinary differential equations, the system (24) possesses a unique solution for a short interval of time. Denote by $[0, T_m)$ the maximal interval of existence for positive time. We will show later that in fact, $T_m = T$.

Let us fix m and the time interval $[0, T]$. On $[0, T_m)$ we take the inner product $(\cdot, \cdot)_H$ of Eq. (24a) with u^m, considering the real part and using (19) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^m|^2 + \nu\|u^m\|^2 = \mathrm{Re}(P_m f, u^m).$$

The right hand side could be estimated using the Cauchy–Schwartz and Young inequalities

$$|(P_m f, u^m)| \le \|f\|_{V'}\|u^m\| \le \frac{1}{2\nu}\|f\|^2_{V'} + \frac{\nu}{2}\|u^m\|^2.$$

Plugging it into the equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|u^m|^2 + \nu\|u^m\|^2 \le \frac{1}{\nu}\|f\|^2_{V'}. \tag{25}$$

Integrating the last equation from 0 to s, where $0 < s \le T_m$, we conclude that for every such s,

$$|u^m(s)|^2 \le |P_m u^{in}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}\mathrm{d}t.$$

Thus, $\limsup_{s\to T_m^-} |u_m(s)|^2 < \infty$. Therefore, the maximal interval of existence of solution of the system (24a) for every m is $[0, T)$ — an interval where the terms of the equation make sense, and in particular, f. If $f \in L^2([0,\infty), V')$, then we can take $T > 0$ arbitrary large and show the existence of weak solutions on $[0, T)$. Now, we observe that

$$\|u^m\|^2_{L^\infty([0,T),H)} \le |P_m u^{in}|^2 + \frac{1}{\nu}\int_0^T \|f\|_{V'}\mathrm{d}t,$$

and hence

$$\text{The sequence } u^m \text{ lies in the bounded set of } L^\infty([0,T],H). \tag{26}$$

Next, integrating (25) over the interval $[0, T]$ we get

$$|u^m(T)|^2 + \nu\int_0^T \|u^m(t)\|^2\mathrm{d}t \le |u^m(0)|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}\mathrm{d}t,$$

and using the fact that $|u^m(0)| = |P_m u^{in}| \le |u^{in}|$ we conclude

$$\text{The sequence } u^m \text{ lies in the bounded set of } L^2([0,T],V). \tag{27}$$

Finally, in order to apply Aubin's Compactness Lemma (see, e.g., [1,5]) we need to estimate the norm $\left\|\frac{\mathrm{d}u_m}{\mathrm{d}t}\right\|_{V'}$. Consider the equation

$$\frac{\mathrm{d}u^m}{\mathrm{d}t} = -\nu A u^m - P_m B(u^m, u^m) + P_m f.$$

Then, (18) and (26) imply that $\|P_m B(u^m, u^m)\|_{V'}$ is uniformly bounded for every $0 \le t \le T$, and hence

$$P_m B(u^m, u^m) \in L^p([0,T], V'),$$

for every $p > 1$ and $T < \infty$. Moreover, from the fact that $\|P_m f\|_{V'} \le \|f\|_{V'}$ and the condition on f it follows that $P_m f \in L^2([0,T], V')$.

The relations (10) and (27) imply that the sequence $\{Au^m\}$ is uniformly bounded in the space $L^2([0,T],V')$. Therefore, using the fact that $f \in L^2([0,T],V')$ we conclude,

$$\text{The sequence } \frac{\mathrm{d}u^m}{\mathrm{d}t} \text{ lies in the bounded set of } L^2([0,T],V'). \tag{28}$$

At last, we are able to apply Aubin's Compactness Lemma. According to (26)–(28), there exists a subsequence $u^{m'}$, and $u \in H$ satisfying

$$u^{m'} \longrightarrow u \quad \text{weakly in } L^2([0,T],V) \tag{29a}$$

$$u^{m'} \longrightarrow u \quad \text{strongly in } L^2([0,T],H). \tag{29b}$$

We are left to show that the solution u that we found satisfies the weak formulation of Eq. (9) together with the initial condition. Let $v \in V$ be an arbitrary function. Take the scalar product of (24a) with v and integrate for some $0 \le t_0 < t \le T$ to get

$$(u^{m'}(t), v) - (u^{m'}(t_0), v) + \nu\int_{t_0}^t ((u^{m'}(\tau), v))\mathrm{d}\tau + \int_{t_0}^t (P_{m'}B(u^{m'}, u^{m'}), v)\mathrm{d}\tau = \int_{t_0}^t \langle P_{m'} f, v\rangle\,\mathrm{d}\tau. \tag{30}$$

The conclusion (29) implies that there exists a set $E \subset [0,T]$ of Lebesgue measure 0 and a subsequence $u^{m''}$ of $u^{m'}$ such that for every $t \in [0,T]\setminus E$, $u^{m''}(t)$ converges to $u(t)$ weakly in V and strongly in H. As for the third term

$$\lim_{m'\to\infty}\int_{t_0}^t ((u^{m'}(\tau), v))\mathrm{d}\tau \longrightarrow \int_{t_0}^t ((u(\tau), v))\mathrm{d}\tau,$$

because of (29a). The same holds for the forcing term. We are left to evaluate the nonlinear part. It is enough to check that

$$\int_{t_0}^t |(P_{m'}B(u^{m'}, u^{m'}), v) - (B(u,u), v)|\mathrm{d}\tau \longrightarrow 0, \quad \text{as } m' \to \infty. \tag{31}$$

Let us define $Q = Q_{m'} = I - P_{m'}$. Then, by definition of $B(u,u)$ and arguments similar to (14) we have

$$|(P_{m'}B(u^{m'}, u^{m'}), v) - B(u,u), v)| = \left|\sum_{n=m}^{\infty}\left(ak_{n+1}v_{n+2}u^*_{n+1}v^*_n + bk_n v_{n+1}v^*_n u^*_{n-1} + ak_{n-1}v^*_n u_{n-1}v_{n-2} + bk_{n-1}v^*_n v_{n-1}u_{n-2}\right)\right| \le C\|Qv\||Qu|^2 \le C\|v\||u - P_{m'}u|^2.$$

We see that (31) holds, because of (29b). Passing to the limit in (30) we finally conclude that u is weakly continuous in H and that it satisfies the weak form of the equation, namely

$$(u(t), v) - (u(t_0), v) + \nu \int_{t_0}^{t} ((u(\tau), v))\mathrm{d}\tau + \int_{t_0}^{t} (B(u, u), v)\mathrm{d}\tau = \int_{t_0}^{t} \langle f, v \rangle\, \mathrm{d}\tau.$$

In order to finish the proof we need to show that u satisfies (23). First, recall that $u \in L^2([0, T], V)$ and $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2([0, T], V')$. Therefore, it follows that $|u(t)|^2$ is continuous. Moreover, $u \in C_w([0, T], H)$ — the space of weakly continuous functions with values in H. Now, let us fix $t \in [0, T]$ and consider the sequence $\{t_n\} \subset [0, T]$ converging to t. We know that $|u(t_n)|$ converges to $|u(t)|$ and $u(t_n)$ converges weakly to $u(t)$. Then, it follows that $u(t_n)$ converges strongly to $u(t)$, and we finally conclude that

$$u \in C([0, T], H). \quad \square$$

4. Uniqueness of weak solutions

Next we show that the weak solutions depend continuously on the initial data and in particular the solutions are unique.

Theorem 3. *Let $u(t)$, $v(t)$ be two different solutions to Eq. (9) on the time interval $[0, T]$ with the corresponding initial conditions u^{in} and v^{in} in H. Then, for every $t \in [0, T]$ we have*

$$|u(t) - v(t)| \le \mathrm{e}^{K} |u^{in} - v^{in}|, \tag{32}$$

where

$$K = C_1 \int_0^t \|u(s)\|\mathrm{d}s. \tag{33}$$

Proof. Consider u, v — two solutions of Eq. (9a) satisfying the initial conditions $u(0) = u^{in}$ and $v(0) = v^{in}$. The difference $w = u - v$ satisfies the equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu A w + B(u, w) + B(w, u) - B(w, w) = 0,$$

with the initial condition $w(0) = w^{in} = u^{in} - v^{in}$.

According to Theorem 2, $\frac{\mathrm{d}w}{\mathrm{d}t} \in L^2([0, T), V')$ and $w \in L^2([0, T), V)$. Hence, the particular case of the general theorem of interpolation due to Lions and Magenes [27] (see also [32], Chapter III, Lemma 1.2) implies that $\langle \frac{\mathrm{d}w}{\mathrm{d}t}, w \rangle_{V'} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |w|^2$. Taking the inner product $\langle \cdot, w \rangle$ and using (14) and (16) we conclude that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |w|^2 \le \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |w|^2 + \nu \|w\|^2 = -\mathrm{Re}\langle B(w, u), w \rangle \le C_1 \|u\| |w|^2.$$

Gronwall's inequality now implies

$$|w(t)|^2 \le |w^{in}|^2 \exp\left(C_1 \int_0^t \|u(s)\|\mathrm{d}s \right). \tag{34}$$

Since $w^{in} = u^{in} - v^{in}$ and $u \in L^2([0, T), V)$ Eq. (32) follows. In particular, in the case that $u^{in} = v^{in}$, it follows that $u(t) = v(t)$ for all $t \in [0, T]$. $\square$

5. Strong solutions

In analogy to the Navier–Stokes equations we would like to show that the shell model Eq. (9) possesses even more regular solutions under an appropriate assumptions.

Let u^m be the solution of the Galerkin system (24). Taking the scalar product of Eq. (24a) with Au^m, and using Young and Cauchy–Schwartz inequalities we obtain

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u^m\|^2 + \nu |Au^m|^2 &\le |(P_m B(u^m, u^m), Au^m)| + |(f^m, Au^m)| \\ &\le C_1 \|u^m\| |u^m| |Au^m| + \frac{1}{\nu} |f|^2 + \frac{\nu}{4} |Au^m|^2 \\ &\le \frac{C_1^2}{\nu} \|u^m\|^2 |u^m|^2 + \frac{\nu}{4} |Au^m|^2 + \frac{1}{\nu} |f|^2 + \frac{\nu}{4} |Au^m|^2. \end{aligned}$$

Finally, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^m\|^2 + \nu|Au^m|^2 \leq \frac{2C_1^2}{\nu}\|u^m\|^2|u^m|^2 + \frac{2}{\nu}|f|^2.$$

By omitting for the moment the term $\nu|Au^m|^2$ from the left hand side of the equation, we can multiply it by the integrating factor $\exp(-\frac{2C_1^2}{\nu}\int_{t_0}^t |u^m(s)|^2\mathrm{d}s)$ for some $0 \leq t_0 \leq t$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u^m\|^2 \exp\left(-\frac{2C_1^2}{\nu}\int_{t_0}^t |u^m(s)|^2\mathrm{d}s\right)\right) \leq \frac{2}{\nu}|f|^2 \exp\left(-\frac{2C_1^2}{\nu}\int_{t_0}^t |u^m(s)|^2\mathrm{d}s\right) \leq \frac{2}{\nu}|f|^2. \tag{35}$$

Assuming that the forcing f is essentially bounded in the norm of H for all $0 \leq t \leq T$, namely that $f \in L^\infty([0,T],H)$ and denoting $|f|_\infty = \|f\|_{L^\infty([0,T],H)}$, we obtain by integrating equation (35)

$$\|u^m(t)\|^2 \leq \mathrm{e}^{\left(\frac{2C_1^2}{\nu}\int_{t_0}^t |u^m(s)|^2\mathrm{d}s\right)}\left(\|u^m(t_0)\|^2 + \frac{2}{\nu}|f|_\infty^2(t-t_0)\right). \tag{36}$$

In order to obtain a uniform bound on $\|u^m(t)\|^2$ for all $0 \leq t \leq T$ and for all m we need to estimate both the exponent and $\|u^m(t_0)\|^2$. The last quantity is of course bounded for $t_0 = 0$ if we assume that $P_m u^{in} \in V$ for all m, however we want to impose less strict assumptions. Let us first refine the estimates we made in the proof of the existence of weak solutions. Take the scalar product of Eq. (24a) with u^m to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^m|^2 + \nu\|u^m\|^2 = (f^m, u^m) \leq \frac{1}{2k_1^2\nu}|f|^2 + \frac{k_1^2\nu}{2}|u^m|^2,$$

and using the inequality $k_1^2|u^m|^2 \leq \|u^m\|^2$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|u^m|^2 + \nu\|u^m\|^2 \leq \frac{1}{k_1^2\nu}|f|^2. \tag{37}$$

Integrating from 0 to t, and once again using the fact that $f \in L^\infty([0,T],H)$, we get

$$\nu\int_0^t \|u^m(s)\|^2\mathrm{d}s \leq |u^m(0)|^2 + \frac{1}{k_1^2\nu}|f|_\infty^2 t. \tag{38}$$

On the other hand, Gronwall's inequality applied to (37) gives us

$$|u^m(t)|^2 \leq |u^m(0)|^2\mathrm{e}^{-\nu k_1^2 t} + \int_0^t \mathrm{e}^{-\nu k_1^2(t-s)}\frac{|f|^2}{k_1^2\nu}\mathrm{d}s,$$

and implies

$$|u^m(t)|^2 \leq |u^m(0)|^2\mathrm{e}^{-k_1^2\nu t} + \frac{|f|_\infty^2}{k_1^4\nu^2}(1-\mathrm{e}^{-k_1^2\nu t}), \tag{39}$$

where the last expression yields for every $t > 0$

$$|u^m(t)|^2 \leq |u^m(0)|^2 + \frac{|f|_\infty^2}{k_1^4\nu^2}. \tag{40}$$

Using Eq. (40) we can immediately evaluate the exponent in (36) by

$$\mathrm{e}^{\left(\frac{2C_1^2}{\nu}\int_{t_0}^t |u^m(s)|^2\mathrm{d}s\right)} \leq \mathrm{e}^{\frac{2C_1^2}{\nu}(t-t_0)\left(|u^m(0)|^2 + \frac{1}{k_1^4\nu^2}|f|_\infty^2\right)}. \tag{41}$$

The last step is to get a uniform bound for the $\|u^m(t_0)\|^2$. Let us integrate Eq. (37) from t to $t+\tau$. The result, after applying (40), is

$$\begin{aligned}\nu\int_t^{t+\tau}\|u^m\|^2\mathrm{d}s &\leq |u^m(t+\tau)|^2 - |u^m(t)|^2 + \frac{\tau}{k_1^2\nu}|f|_\infty^2\\ &\leq |u^m(0)|^2 + \frac{1}{k_1^2\nu}|f|_\infty^2\left(\frac{1}{k_1^2\nu}+\tau\right).\end{aligned}$$

The standard application of Markov's inequality implies that on every interval of length τ there exists a time t_0 satisfying

$$\|u^m(t_0)\|^2 \le \frac{2}{\tau}\left(\frac{1}{\nu}|u^m(0)|^2 + \frac{1}{k_1^2\nu^2}|f|_\infty^2\left(\frac{1}{k_1^2\nu}+\tau\right)\right). \tag{42}$$

Let us set $\tau \le \frac{1}{k_1^2\nu}$. Then, for every $t \ge \tau$ and an appropriate t_0 chosen in the interval $[t-\tau, t]$ we get a uniform bound for all $m \in \mathbb{N}$ and all $t \ge \tau$, by substituting (39) and (42) into (36)

$$\tau\|u^m(t)\|^2 \le 2\mathrm{e}^{\frac{2C_1^2}{k_1^2\nu^2}\left(|u^m(0)|^2+\frac{1}{k_1^4\nu^2}|f|_\infty^2\right)}\cdot\left(\frac{1}{\nu}|u^m(0)|^2+\frac{3}{k_1^4\nu^3}|f|_\infty^2\right). \tag{43}$$

To expand the estimate for all times $0 \le t \le \frac{1}{k_1^2\nu}$, we can apply the last inequality to the dyadic intervals of the form $[\frac{1}{k_1^2\nu}\frac{1}{2^{k+1}}, \frac{1}{k_1^2\nu}\frac{1}{2^k}]$ with $\tau = \frac{1}{k_1^2\nu}\frac{1}{2^{k+1}}$. Then, if $t \in [\frac{1}{k_1^2\nu}\frac{1}{2^{k+1}}, \frac{1}{k_1^2\nu}\frac{1}{2^k}]$, it satisfies $\tau \le t \le 2\tau$, and for such t inequality (43) implies

$$t\|u^m(t)\|^2 \le 4\mathrm{e}^{\frac{2C_1^2}{k_1^2\nu^2}\left(|u^m(0)|^2+\frac{1}{k_1^4\nu^2}|f|_\infty^2\right)}\cdot\left(\frac{1}{\nu}|u^m(0)|^2+\frac{3}{k_1^4\nu^3}|f|_\infty^2\right). \tag{44}$$

Since the last estimate does not depend on k we can conclude that it holds for all $0 \le t \le \frac{1}{k_1^2\nu}$.

Finally, passing to the limit in the Galerkin approximation we obtain

Theorem 4. *Let $T > 0$ and $f \in L^\infty([0,T], H)$. Then if $u^{in} \in H$, Eq. (9) possesses a solution $u(t)$ satisfying*

$$u \in L^\infty_{loc}((0,T], V)\bigcap L^2_{loc}((0,T], D(A))\bigcap L^\infty([0,T], H)\bigcap L^2([0,T], V).$$

Moreover, the following bound holds

$$\sup_{0<t\le\min\{\frac{1}{k_1^2\nu},T\}} \nu k_1^2 t\|u(t)\|^2 + \sup_{\frac{1}{k_1^2\nu}\le t\le T}\|u(t)\|^2 \le 6\mathrm{e}^{\frac{2C_1^2}{k_1^2\nu^2}\left(|u^{in}|^2+\frac{1}{k_1^4\nu^2}|f|_\infty^2\right)}\cdot\left(k_1^2|u^{in}|^2+\frac{3}{k_1^2\nu^2}|f|_\infty^2\right). \tag{45}$$

If the initial condition $u^{in} \in V$, then the solution $u(t)$ satisfies

$$u \in C([0,T], V)\bigcap L^2([0,T], D(A)). \tag{46}$$

6. Gevrey class regularity

6.1. Preliminaries and classical Navier–Stokes equation results

We start this section with a definition

Definition 2. Let us fix positive constants τ, $p > 0$, $q \ge 0$ and define the following class of sequences

$$G_\tau^{p,q} = \left\{u \in \ell_2 : |\mathrm{e}^{\tau A^{p/2}}A^{q/2}u| = \sum_{n=1}^\infty \mathrm{e}^{2\tau k_n^p}k_n^q|u_n|^2 < \infty\right\}.$$

We will say that the sequence $u \in \ell_2$ is Gevrey regular if it belongs to the class $G_\tau^{p,q}$ for some choice of τ, $p > 0$ and $q \ge 0$.

In order to justify this definition let $u = (u_1, u_2, u_3, \ldots)$ and suppose that u_n, $n = 1, 2, 3, \ldots$, are the Fourier coefficients of some scalar function $g(x)$, with corresponding wavenumbers k_n, defined by relation (2). Then, if $u \in G_\tau^{p,q}$, for some p, q and τ, the function $g(x)$ is analytic. The concept of the Gevrey class regularity for showing the analyticity of the solutions of the two-dimensional Navier–Stokes equations was first introduced in [20], simplifying earlier proofs. Later this technique was extended to the large class of analytic nonlinear parabolic equations in [21].

For $q = 0$ we will write $G_\tau^p = G_\tau^{p,0}$. Denote by $(\cdot,\cdot)_{p,\tau}$ the scalar product in G_τ^p and

$$|u|_{p,\tau} = \|u\|_{G_\tau^p}, \quad \forall u \in G_\tau^p.$$

In the case of $q = 1$ we will write

$$\|u\|_{p,\tau} = \|u\|_{G_\tau^{p,1}}, \quad \forall u \in G_\tau^{p,q},$$

and the scalar product in $G_\tau^{p,1}$ will be denoted by $((\cdot,\cdot))_{p,\tau}$.

Following the tools introduced in [20] and generalized in [21] we prove the following:

Theorem 5. *Let us assume that* $u(0) = u^{in} \in V$ *and the force* $f \in G^{p_1}_{\sigma_1}$ *for some* $\sigma_1, p_1 > 0$. *Then, there exists* T, *depending only on the initial data, such that for every* $0 < p \le \min\{p_1, \log_\lambda \frac{1+\sqrt{5}}{2}\}$ *the following holds*

(i) *For every* $t \in [0, T]$ *the solution* $u(t)$ *of Eq. (9) remains bounded in* $G^{p,1}_{\psi(t)}$, *for* $\psi(t) = \min\{t, \sigma_1\}$.

(ii) *Furthermore, there exists* $\sigma > 0$ *such that for every* $t > T$ *the solution* $u(t)$ *of Eq. (9) remains bounded in* $G^{p,1}_\sigma$.

Before proving the theorem, we prove the following estimates for the nonlinear term:

Lemma 6. *Let* $u, v, w \in G^{p,1}_\tau$ *for some* $\tau > 0$ *and* $p \le \log_\lambda \frac{1+\sqrt{5}}{2}$. *Then* $B(u, v)$ *belongs to* G^p_τ *and satisfies*

$$|(A^{1/2} e^{\tau A^{p/2}} B(u, v), A^{1/2} e^{\tau A^{p/2}} w)| \le C_4 \|u\|_{G^{p,1}_\tau} \|v\|_{G^{p,1}_\tau} \|w\|_{G^{p,1}_\tau}, \tag{47}$$

for an appropriate constant $C_4 > 0$.

Proof. Let us fix an arbitrary $p \le \log_\lambda \frac{1+\sqrt{5}}{2}$. Observe, that the operator $e^{\tau A^{p/2}}$ is diagonal in the standard canonical basis $\{\phi_n\}_{n=1}^\infty$ of H with corresponding eigenvalues $e^{\tau \widetilde{k}_n}$, $n = 1, 2, \ldots$, where $\widetilde{k}_n = \widetilde{\lambda}^n$ for some $\widetilde{\lambda} \le \frac{1+\sqrt{5}}{2}$. Therefore,

$$\begin{aligned}
|(A^{1/2} e^{\tau A^{p/2}} B(u, v), A^{1/2} e^{\tau A^{p/2}} w)| &\le \sum_{n=1}^\infty e^{2\tau \widetilde{k}_n} \Big| a k_{n+1} k_n^2 v_{n+2} u^*_{n+1} w^*_n + b k_n^3 v_{n+1} w^*_n u^*_{n-1} \\
&\quad + a k_n^2 k_{n-1} w^*_n u_{n-1} v_{n-2} + b k^*_n k_{n-1} w^*_n v_{n-1} u_{n-2} \Big| \\
&\le \sup_{n \ge 1} |e^{\tau \widetilde{k}_n} k_n w_n| \sum_{n=1}^\infty e^{\tau \widetilde{k}_n} \Big(|a k_{n+1} k_n v_{n+2} u^*_{n+1}| + \left| b k_n^2 v_{n+1} u^*_{n-1} \right| \\
&\quad + |a k_n k_{n-1} u_{n-1} v_{n-2}| + |b k_n k_{n-1} u_{n-2} v_{n-1}| \Big) \le C_4 \|w\|_{G^{p,1}_\tau} \|u\|_{G^{p,1}_\tau} \|v\|_{G^{p,1}_\tau},
\end{aligned} \tag{48}$$

where we can choose $C_4 = (|a|(\lambda^{-2} + \lambda^2) + |b|(1 + \lambda^2))$. The last inequality is the result of the Cauchy–Schwartz inequality and the fact that $e^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n+2} - \widetilde{\lambda}^{n+1})}$, $e^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n+1} - \widetilde{\lambda}^{n-1})}$ and $e^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n-1} - \widetilde{\lambda}^{n-2})}$ are less than or equal to 1 for all $\tau > 0$ and our specific choice of $\widetilde{\lambda}$. □

Now we are ready to prove the theorem:

Proof (*Of Theorem 5*). Let us take an arbitrary $0 < p \le \min\{p_1, \log_\lambda \frac{1+\sqrt{5}}{2}\}$ and define $\varphi(t) = \min(t, \sigma_1)$. For a fixed time t take the scalar product of Eq. (9a) with $Au(t)$ in $G^p_{\varphi(t)}$ to get

$$(u'(t), Au(t))_{p,\varphi(t)} + \nu |Au(t)|^2_{p,\varphi(t)} = (f, Au(t))_{p,\varphi(t)} - (B(u(t), u(t)), Au(t))_{p,\varphi(t)}. \tag{49}$$

The left hand side of the equation could be transformed in the following way:

$$\begin{aligned}
(u'(t), Au(t))_{p,\varphi(t)} &= (e^{\varphi(t) A^{p/2}} u'(t), e^{\varphi(t) A^{p/2}} Au(t)) \\
&= (A^{1/2} (e^{\varphi(t) A^{p/2}} u(t))' - \varphi'(t) A^{(p+1)/2} e^{\varphi(t) A^{p/2}} u(t), A^{1/2} e^{\varphi(t) A^{p/2}} u(t)) \\
&= \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{p,\varphi(t)} - \varphi'(t) (A^{(p+1)/2} u(t), A^{1/2} u(t))_{p,\varphi(t)} \\
&\ge \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{p,\varphi(t)} - |Au(t)|_{p,\varphi(t)} \|u(t)\|_{p,\varphi(t)} \\
&\ge \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{p,\varphi(t)} - \frac{\nu}{4} |Au(t)|^2_{p,\varphi(t)} - \frac{1}{\nu} \|u(t)\|^2_{p,\varphi(t)}.
\end{aligned} \tag{50}$$

The right hand side of Eq. (49) can be bounded from above in the usual way using Lemma 6, and the Cauchy–Schwartz and Young inequalities

$$\begin{aligned}
|(f, Au(t))_{p,\varphi(t)} - (B(u(t), u(t)), Au(t))_{p,\varphi(t)}| &\le |f|_{p,\varphi(t)} |Au(t)|_{p,\varphi(t)} + C_4 \|u\|^3_{p,\varphi(t)} \\
&\le \frac{1}{\nu} |f|^2_{p,\varphi(t)} + \frac{\nu}{4} |Au(t)|^2_{p,\varphi(t)} + C_4 \|u\|^3_{p,\varphi(t)}.
\end{aligned} \tag{51}$$

Combining (50) and (51), Eq. (49) takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{p,\varphi(t)} + \nu|Au(t)|^2_{p,\varphi(t)} &\le \frac{2}{\nu}|f|^2_{p,\varphi(t)} + \frac{2}{\nu}\|u(t)\|^2_{p,\varphi(t)} + C_4\|u\|^3_{p,\varphi(t)} \\ &\le \frac{2}{\nu}|f|^2_{p,\varphi(t)} + C_5 + C_4\|u\|^3_{p,\varphi(t)}.\end{aligned}$$

Denoting $y(t) = 1 + \|u(t)\|^2_{p,\varphi(t)}$ and $K = \frac{2}{\nu}|f|^2_{p,\varphi(t)} + C_5 + C_4^{2/3}$ we get an inequality

$$\dot{y} \le K y^{3/2}.$$

It implies

$$y(t) = 1 + \|u(t)\|^2_{p,\varphi(t)} \le 2y(0) = 2(1 + \|u^{in}\|^2),$$

for

$$t \le T_1(\sigma_1, \|u^{in}\|, |f|_{p,\varphi(t)}) = \frac{2-\sqrt{2}}{K} y^{-1/2}(0) = \frac{2-\sqrt{2}}{K}(1 + \|u^{in}\|^2)^{-1/2}.$$

According to Theorem 4, the solution $u(t)$ of Eq. (9) remains bounded in V for all time if we start from $u^{in} \in V$. Moreover, from the relation (46) we conclude that there exists a constant M, such that for all $t > 0$

$$\|u(t)\| \le M.$$

Hence we can repeat the above arguments starting from any time $t > 0$ and find that

$$|\mathrm{e}^{\varphi(T_2)A^{p/2}} A^{1/2} u(t)| \le 2 + 2M^2, \tag{52}$$

for all $t \ge T_2 = \frac{2-\sqrt{2}}{K}(1 + M^2)^{-1/2}$, and the theorem holds. □

6.2. A stronger result

We prove the following stronger version of Theorem 5; it is not known whether this holds for the Navier–Stokes equations.

Theorem 7. *Let us assume that $u(0) = u^{in} \in H$ and the force $f \in G^{p_1}_{\sigma_1}$ for some $\sigma_1, p_1 > 0$. Then, there exists T, depending only on the initial data, such that for every $0 < p \le \min\{p_1, \log_\lambda \frac{1+\sqrt{5}}{2}\}$ the following holds*

(i) *For every $t \in [0, T]$ the solution $u(t)$ of Eq. (9) remains bounded in $G^p_{\psi(t)}$, for $\psi(t) = \min\{t, \sigma_1\}$.*
(ii) *For every $t > T$ there exists $\sigma > 0$ such that the solution $u(t)$ of Eq. (9) remains bounded in G^p_σ.*

Before proving the theorem, we obtain the following estimates for the nonlinear term:

Lemma 8. *Let $u, v \in G^p_\tau$, $w \in G^{p,1}_\tau$ for some $\tau > 0$ and $p \le \log_\lambda \frac{1+\sqrt{5}}{2}$. Then $B(u, v)$ belongs to G^p_τ and satisfies*

$$|(\mathrm{e}^{\tau A^{p/2}} B(u, v), \mathrm{e}^{\tau A^{p/2}} w)| \le C_6\|u\|_{G^p_\tau}\|v\|_{G^p_\tau}\|w\|_{G^{p,1}_\tau}, \tag{53}$$

for an appropriate constant $C_6 > 0$.

Proof. Let us fix an arbitrary $p \le \log_\lambda \frac{1+\sqrt{5}}{2}$. Then we note that the operator $\mathrm{e}^{\tau A^{p/2}}$ is diagonal with eigenvalues $\mathrm{e}^{\tau\widetilde{k}_n}$, $n = 1, 2, \ldots$, where $\widetilde{k}_n = \widetilde{\lambda}^n$ for $\widetilde{\lambda} \le \frac{1+\sqrt{5}}{2}$. Hence,

$$\begin{aligned}|(\mathrm{e}^{\tau A^{p/2}} B(u, v), \mathrm{e}^{\tau A^{p/2}} w)| &\le \sum_{n=1}^{\infty} \mathrm{e}^{2\tau\widetilde{k}_n} \left|ak_{n+1}u_{n+2}v^*_{n+1}w^*_n + bk_n u_{n+1}w^*_n v^*_{n-1} - ck_{n-1}w^*_n v_{n-1}u_{n-2}\right| \\ &\le \sup_{n\ge1}|\mathrm{e}^{\tau\widetilde{k}_n}k_n w_n| \sum_{n=1}^{\infty}\mathrm{e}^{\tau\widetilde{k}_n}\left(\left|a\lambda u_{n+2}v^*_{n+1}\right| + \left|bu_{n+1}v^*_{n-1}\right| + \left|c\lambda^{-1}v_{n-1}u_{n-2}\right|\right) \\ &\le C_6\|w\|_{G^{p,1}_\tau}\|u\|_{G^p_\tau}\|v\|_{G^p_\tau},\end{aligned} \tag{54}$$

where we can choose $C_6 = (\lambda a + b + \lambda^{-1}c)$. The last inequality is the result of the Cauchy–Schwartz inequality and the fact that $\mathrm{e}^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n+2} - \widetilde{\lambda}^{n+1})}$, $\mathrm{e}^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n+1} - \widetilde{\lambda}^{n-1})}$ and $\mathrm{e}^{\tau(\widetilde{\lambda}^n - \widetilde{\lambda}^{n-1} - \widetilde{\lambda}^{n-2})}$ are less than or equal to 1 for all τ and the specific choice of $\widetilde{\lambda}$. □

Now we are ready to prove the theorem:

Proof (*Of Theorem 7*). Let us take an arbitrary $0 < p \le \min\{p_1, \log_\lambda \frac{1+\sqrt{5}}{2}\}$ and define $\varphi(t) = \min(t, \sigma_1)$. For a fixed time t take the scalar product of Eq. (9a) with $u(t)$ in $G^p_\varphi(t)$ to get

$$(u'(t), u(t))_{p,\varphi(t)} + \nu \|u(t)\|^2_{p,\varphi(t)} = (f, u(t))_{p,\varphi(t)} - (B(u(t), u(t)), u(t))_{p,\varphi(t)}. \tag{55}$$

The left hand side of the equation could be transformed in the following way:

$$\begin{aligned}(u'(t), Au(t))_{p,\varphi(t)} &= (\mathrm{e}^{\varphi(t)A^{p/2}} u'(t), \mathrm{e}^{\varphi(t)A^{p/2}} u(t)) \\ &= ((\mathrm{e}^{\varphi(t)A^{p/2}} u(t))' - \varphi'(t) A^{p/2} \mathrm{e}^{\varphi(t)A^{p/2}} u(t), \mathrm{e}^{\varphi(t)A^{p/2}} u(t)) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2_{p,\varphi(t)} - \varphi'(t)(A^{p/2} u(t), u(t))_{p,\varphi(t)} \\ &\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2_{p,\varphi(t)} - \|u(t)\|_{p,\varphi(t)} |u(t)|_{p,\varphi(t)} \\ &\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2_{p,\varphi(t)} - \frac{\nu}{4} \|u(t)\|^2_{p,\varphi(t)} - \frac{1}{\nu} |u(t)|^2_{p,\varphi(t)}.\end{aligned} \tag{56}$$

The right hand side of Eq. (55) could be bounded from above in the usual way using Lemma 8, and the Cauchy–Schwartz and Young inequalities

$$\begin{aligned}|(f, u(t))_{p,\varphi(t)} - (B(u(t), u(t)), u(t))_{p,\varphi(t)}| &\le |f|_{p,\varphi(t)} |u(t)|_{p,\varphi(t)} + C_6 \|u\|_{p,\varphi(t)} |u|^2_{p,\varphi(t)} \\ &\le \frac{3}{4} |f|^{4/3}_{p,\varphi(t)} + \frac{1}{4} |u(t)|^4_{p,\varphi(t)} + \frac{\nu}{4} \|u\|^2_{p,\varphi(t)} + \frac{C_6^2}{\nu} |u|^4_{p,\varphi(t)}.\end{aligned} \tag{57}$$

Combining (56) and (57), Eq. (55) takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2_{p,\varphi(t)} + \nu \|Au(t)\|^2_{p,\varphi(t)} &\le \frac{3}{2} |f|^{4/3}_{p,\varphi(t)} + \frac{2}{\nu} |u(t)|^2_{p,\varphi(t)} + \left(\frac{1}{2} + \frac{2C_6^2}{\nu}\right) |u|^4_{p,\varphi(t)} \\ &\le \frac{3}{2} |f|^{4/3}_{p,\varphi(t)} + C_7 |u|^4_{p,\varphi(t)},\end{aligned}$$

for some positive constant C_7.

Defining $y(t) = 1 + |u(t)|^2_{p,\varphi(t)}$ and $K = \frac{3}{2} |f|^{4/3}_{p,\varphi(t)} + C_7$ we get an inequality

$$\dot{y} \le K y^2.$$

It implies

$$y(t) = 1 + \|u(t)\|^2_{p,\varphi(t)} \le 2y(0) = 2(1 + |u^{in}|^2),$$

for

$$t \le T_1(\sigma_1, |u^{in}|, |f|_{p,\varphi(t)}) = \frac{1}{2K} y^{-1}(0) = \frac{1}{2K} (1 + |u^{in}|^2)^{-1}.$$

According to Theorem 2 the solution of Eq. (9) remains bounded in H for all time, if we start from $u^{in} \in H$. Moreover, from the estimate (23), it follows that there exists a constant M, such that for all $t > 0$

$$|u(t)| \le M.$$

Hence we can repeat the above arguments starting from any time $t > 0$ and find that

$$|\mathrm{e}^{\varphi(T_2)A^{p/2}} A^{1/2} u(t)| \le 2 + 2M^2, \tag{58}$$

for all $t \ge T_2 = \frac{1}{2K}(1 + M^2)^{-1}$, and the theorem holds. □

7. Global attractors and their dimensions

The first mathematical concept which we use to establish the finite dimensional long-term behavior of the viscous sabra shell model is the global attractor. The global attractor, $\mathcal{A}$, is the maximal bounded invariant subset of the space H. It encompasses all of the possible permanent regimes of the dynamics of the shell model. It is also a compact subset of the space H which attracts all the trajectories of the system. Establishment of finite Hausdorff and fractal dimensionality of the global attractor implies the possible parameterization of the permanent regimes of the dynamics in terms of a finite number of parameters. For the definition and further discussion of the concept of the global attractor see, e.g., [14,33].

In analogy with Kolmogorov's mean rate of dissipation of energy in turbulent flow we define

$$\epsilon = \nu \left\langle \|u\|^2 \right\rangle,$$

the mean rate of dissipation of energy in the shell model system. $\langle\cdot\rangle$ represents the ensemble average or a long-time average. Since we do not know whether such long-time averages converge for trajectories we will replace the above definition of ϵ by

$$\epsilon = \nu \sup_{u^{in}\in\mathcal{A}} \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|u(s)\|^2 \mathrm{d}s.$$

We will also define the viscous dissipation length scale l_d. According to Kolmogorov's theory, it should only depend on the viscosity ν and the mean rate of energy dissipation ϵ. Hence, pure dimensional arguments lead to the definition

$$l_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4},$$

which represents the largest spatial scale at which the viscosity term begins to dominate over the nonlinear inertial term of the shell model equation. In analogy with the conventional theory of turbulence this is also the smallest scale that one needs to resolve in order to get the full resolution for turbulent flow associated with the shell model system.

We would like to obtain an estimate of the fractal dimension of the global attractor for the system (9) in terms of another non-dimensional quantity — the generalized Grashoff number. Suppose the forcing term satisfies $f \in L^\infty([0,T],H)$ and define $\|f\|_{L^\infty([0,\infty),H)} = |f|_\infty$. Then we define the generalized Grashoff number for our system to be

$$G = \frac{|f|_\infty}{\nu^2 k_1^3}. \tag{59}$$

The generalized Grashoff number was first introduced in the context of the study of the finite dimensionality of long-term behavior of turbulent flow in [15]. To check that it is indeed non-dimensional we note that $\|f\|_{L^\infty([0,T],H)}$ has the dimension of $\frac{L}{T^2}$, where L is a length scale and T is a time scale. k_1 has the dimension of $\frac{1}{L}$ and the kinematic viscosity ν has the dimension of $\frac{L^2}{T}$. In order to obtain an estimate of the generalized Grashoff number, which will be used in the proof of the main result of this section, we can apply inequality (38) to get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|u(s)\|^2 \mathrm{d}s \le \frac{|f|_\infty^2}{\nu^2 k_1^2} = \nu^2 k_1^4 G^2. \tag{60}$$

Theorem 9. *The Hausdorff and fractal dimensions of the global attractor of the system of Eq. (9), $d_H(A)$ and $d_F(A)$ respectively, satisfy*

$$d_H(A) \le d_F(A) \le \log_\lambda\left(\frac{L}{l_d}\right) + \frac{1}{2}\log_\lambda(C_1(\lambda^2-1)). \tag{61}$$

In terms of the Grashoff number G the upper bound takes the form

$$d_H(A) \le d_F(A) \le \log_\lambda G^{1/2} + \frac{1}{2}\log_\lambda(C_1(\lambda^2-1)). \tag{62}$$

Proof. We follow [6] (see also [4,5,33]) and linearize the shell model system of Eq. (9) about the trajectory $u(t)$ in the global attractor. In Appendix B we show that the solution $u(t)$ is differentiable with respect to the initial data, hence the resulting linear equation takes the form

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \nu AU + B_0(t)U = 0 \tag{63a}$$

$$U(0) = U^{in}, \tag{63b}$$

where $B_0(t)U = B(u(t),U(t)) + B(U(t),u(t))$ is a linear operator. In order to simplify the notation, we will define

$$\Lambda(t) = -\nu A - B_0(t).$$

Let $U_j(t)$ be solutions of the above system satisfying $U_j(0) = U_j^{in}$, for $j = 1,2,\ldots,N$. Assume now that $U_1^{in}, U_2^{in},\ldots,U_N^{in}$ are linearly independent in H, and consider $Q_N(t)$ — an H-orthogonal projection onto the span $\{U_j(t)\}_{j=1}^N$. Let $\{\varphi_j(t)\}_{j=1}^N$ be the orthonormal basis of the span of $\{U_j(t)\}_{j=1}^N$. Notice that $\varphi_j \in D(A)$ since *span*$\{U_1(t),\ldots,U_N(t)\} \subset D(A)$.

Using the definition of $\Lambda(t)$ and the inequalities of the Proposition 1 we get

$$\begin{aligned}\mathrm{Re}(\mathrm{Trace}[\Lambda(t)\circ Q_N(t)]) &= \mathrm{Re}\sum_{j=1}^{N}(\Lambda(t)\varphi_j,\varphi_j)\\ &= \sum_{j=1}^{N} -\nu\|\varphi_j\|^2 - \mathrm{Re}(B(\varphi_j,u(t)),\varphi_j)\\ &\le -\nu\sum_{j=1}^{N}(k_j^2 + |(B(\varphi_j,u(t)),\varphi_j)|)\\ &\le -\nu k_0^2\lambda^2\frac{\lambda^{2N}-1}{\lambda^2-1} + C_1 N\|u(t)\|. \end{aligned} \tag{64}$$

According to the recent results of [4] (see also [5,6,33]), if N is large enough, such that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{Re}(\mathrm{Trace}[\Lambda(t)\circ Q_N(s)])\mathrm{d}s < 0,$$

then the fractal dimension of the global attractor is bounded by N. We need therefore to estimate N in terms of the energy dissipation rate. Using (64), the definition of ϵ, we find that it is sufficient to require N to be large enough such that it satisfies

$$\nu k_0^2\lambda^2\frac{\lambda^{2N}-1}{\lambda^2-1} > C_1 N\left(\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|u(t)\|^2\right)^{1/2} = C_1 N\left(\frac{\epsilon}{\nu}\right)^{1/2}. \tag{65}$$

Finally, $\epsilon\nu^{-3} = l_d^{-4}$ implies

$$\lambda^{2N} > \frac{\lambda^{2N}-1}{N} > C_1\frac{\lambda^2-1}{k_1^2 l_d^2} = C_1(\lambda^2-1)\left(\frac{L}{l_d}\right)^2,$$

which proves (61). Applying the estimate (60) to the inequality (64) we get the bound (62) in terms of the generalized Grashoff number. □

8. Determining modes

Let us consider two solutions of the shell model equations u, v corresponding to the forces $f, g \in L^2([0,\infty), H)$

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,u) = f, \tag{66}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v,v) = g. \tag{67}$$

We give a slightly more general definition of a notion of the determining modes than the one that is introduced previously in literature [16] (see also [14,15,23] and references therein).

Definition 3. We define a set of determining modes as a finite set of indices $\mathcal{M}\subset\mathbb{N}$, such that whenever the forces f, g satisfy

$$|f(t)-g(t)|\to 0, \quad \text{as } t\to\infty, \tag{68}$$

and

$$\sum_{n\in\mathcal{M}}|u_n(t)-v_n(t)|^2\to 0, \quad \text{as } t\to\infty \tag{69}$$

it follows that

$$|u(t)-v(t)|\to 0, \quad \text{as } t\to\infty. \tag{70}$$

The number of determining modes N of the equation is the size of the smallest such set $\mathcal{M}$.

We would like to recall the following generalization of the classical Gronwall's lemma which was proved in [23,24] (see also [14]).

Lemma 10. *Let $\alpha = \alpha(t)$ and $\beta = \beta(t)$ be locally integrable real-valued functions on $[0, \infty)$ that satisfy the following condition for some $T > 0$:*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\mathrm{d}\tau > 0, \tag{71}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)\mathrm{d}\tau < \infty, \tag{72}$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)\mathrm{d}\tau = 0, \tag{73}$$

where $\alpha^-(t) = \max\{-\alpha(t), 0\}$ and $\beta^+(t) = \max\{\beta(t), 0\}$. Suppose that $\xi = \xi(t)$ is an absolutely continuous non-negative function on $[0, \infty)$ that satisfies the following inequality almost everywhere on $[0, \infty)$:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} + \alpha\xi \le \beta. \tag{74}$$

Then $\xi(t) \to 0$ as $t \to \infty$.

A weaker version of the above statement was introduced in [15]. In fact this weaker version would be sufficient for our purposes.

Theorem 11. *Let G be a Grashoff number defined in (59). Then,*

(i) *The first N modes are determining modes for the shell model equation provided*

$$N > \frac{1}{2}\log_\lambda(C_1 G). \tag{75}$$

(ii) *Let u, v be two solutions of Eqs. (66) and (67) respectively. Let the forces f, g satisfy (68) and integer N be defined as in (75). If*

$$\lim_{t\to\infty} |u_N(t) - v_N(t)|, \quad \text{and} \quad \lim_{t\to\infty} |u_{N-1}(t) - v_{N-1}(t)| = 0,$$

then

$$\lim_{t\to\infty} |Q_N u(t) - Q_N v(t)| = 0.$$

Proof. Define $w = u - v$, which satisfies the equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu A w + B(u, w) + B(w, v) = f - g. \tag{76}$$

Let us fix an integer $m > 1$ and define $P_m : H \to H$ to be an orthogonal projection onto the first m coordinates, namely, onto the subspace of H spanned by m vectors of the standard basis $\{e_i\}_{i=1}^m$. Also define $Q_m = I - P_m$.

It is not hard to see that the second part of the theorem implies the first statement. Therefore it is enough for us to prove that if

$$\lim_{t\to\infty} |w_N(t)|, \quad \text{and} \quad \lim_{t\to\infty} |w_{N-1}(t)| = 0,$$

then for every $m > \frac{1}{2}\log_\lambda(C_1 G)$

$$\lim_{t\to\infty} |Q_m w| = 0.$$

Multiplying Eq. (76) by $Q_m w$ in the scalar product of H we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Q_m w|^2 + \nu\|Q_m w\|^2 = \mathrm{Re}(f - g, Q_m w) - \mathrm{Re}(B(u, w), Q_m w) - \mathrm{Re}(B(w, v), Q_m w). \tag{77}$$

For the left hand side of the equation we can use the inequality $k_{m+1}^2|Q_m w| \le \|Q_m w\|$. For the right hand side of the equation we observe that

$$\begin{aligned}\mathrm{Re}(B(u, w), Q_m w) &= \mathrm{Im}\sum_{n=m+1}^{\infty}\left(ak_{n+1}w_{n+2}u_{n+1}^* w_n^* + bk_n w_{n+1}w_n^* u_{n-1}^* + ak_{n-1}w_n^* u_{n-1}w_{n-2} + bk_{n-1}w_n^* w_{n-1}u_{n-2}\right)\\ &\le \left|ak_m w_{m+1}^* u_m w_{m-1} + ak_{m+1}w_{m+2}^* u_{m+1}w_m + bk_m w_{m+1}^* w_m u_{m-1}\right|.\end{aligned}$$

Moreover,

$$\mathrm{Re}(B(w,v),Q_m w) = \mathrm{Im}\sum_{n=m+1}^{\infty}\left(ak_{n+1}v_{n+2}w_{n+1}^*w_n^* + bk_n v_{n+1}w_n^*w_{n-1}^* + ak_{n-1}w_n^*w_{n-1}v_{n-2} + bk_{n-1}w_n^*v_{n-1}w_{n-2}\right)$$
$$\leq \left|ak_m w_{m+1}^* w_m v_{m-1} + ak_{m+1}w_{m+2}^* w_{m+1}v_m + bk_m w_{m+1}^* v_m w_{m-1}\right| + C_1\|v\||Q_m w|^2.$$

Finally defining $\xi = |Q_m w|^2$ we can rewrite the equation in the form

$$\frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t} + \alpha\xi \leq \beta,$$

where

$$\alpha(t) = \nu k_{m+1}^2 - C_1\|v(t)\|,$$

and

$$\beta(t) = (f-g, Q_m w) + \left|ak_m w_{m+1}^* u_m w_{m-1} + ak_{m+1}w_{m+2}^* u_{m+1}w_m + bk_m w_{m+1}^* w_m u_{m-1}\right|$$
$$+ \left|ak_m w_{m+1}^* w_m v_{m-1} + ak_{m+1}w_{m+2}^* w_{m+1}v_m + bk_m w_{m+1}^* v_m w_{m-1}\right|.$$

To see that $\beta(t)$ satisfies condition (73) of Lemma 10 it is enough to show that $\beta(t) \to 0$ as $t \to \infty$. It is clearly true, if we assume that the forces f, g satisfy (68) and $|w_m(t)|, |w_{m-1}(t)| \to 0$ as $t \to \infty$. Moreover, it is easy to see that $\alpha(t)$ satisfies the condition (72). Hence, in order to apply Lemma 10 we need to show that $\alpha(t)$ also satisfies the conditions (71), namely, that

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)\mathrm{d}\tau \geq \nu k_{m+1}^2 - C_1\left(\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|v(\tau)\|^2\mathrm{d}\tau\right)^{1/2} > 0. \tag{78}$$

To estimate the last quantity, we can use the bound (60) to conclude that if m satisfies

$$\lambda^{2m} > C_1 G,$$

then

$$|Q_m w| \to 0 \quad \text{as } t \to \infty. \quad \square$$

Remark 1. The existence of the determining modes for the Navier–Stokes equations both in two and three dimensions is known (see, e.g., [7,14,24] and reference therein). However, there exists a gap between the upper bounds for the lowest number of determining modes and the dimension of the global attractor for the two-dimensional Navier–Stokes equations for both the no-slip and periodic boundary conditions. Our upper bounds for the dimension of the global attractor and for the number of determining modes for the sabra shell model equation coincide. Recently it was shown in [22] that a similar result is also true for the damped–driven NSE and the Stommel–Charney barotropic model of ocean circulation.

9. Existence of inertial manifolds

In this section we prove the existence of a finite dimensional inertial manifold (IM). The concept of an inertial manifold for nonlinear evolution equations was first introduced in [17] (see also, e.g., [5,8,9,18,31,33]). An inertial manifold is a finite dimensional Lipschitz, globally invariant manifold which attracts all bounded sets in the phase space at an exponential rate and, consequently, contains the global attractor. In fact, one can show that the IM is smoother, in particular C^1 (see, e.g., [10,29,31]). The smoothness and invariance under the reduced dynamics of the IM implies that a finite system of ordinary differential equations is equivalent to the original infinite system. This is the ultimate and best notion of system reduction that one could hope for. In other words, an IM is an exact rule for parameterizing the large modes (infinitely many of them) in terms of the low ones (finitely many of them).

In this section we use Theorem 3.1 of [18] to show the existence of inertial manifolds for the system (9) and to estimate its dimension. Let us state Theorem 3.1 of [18] in the following way (see also [33], Chapter VIII, Theorem 3.1).

Theorem 12. *Let the nonlinear term of Eq. (81) $R(\cdot)$ be a differentiable map from $D(A)$ into $D(A^{1-\beta})$ satisfying*

$$|R'(u)v| \leq C_8|Au||A^{\beta}v| \tag{79a}$$

$$|A^{1-\beta}R'(u)v| \leq C_9|Au||Av|, \tag{79b}$$

for some $0 \leq \beta < 1$, for all $u, v \in D(A)$ and appropriate constants $C_8, C_9 > 0$, which depend on physical parameters a, b, c, ν and f.

Let $\tilde{b} > 0$ and $l \in [0, 1)$ be two fixed numbers. Assume that there exists an N large enough such that the eigenvalues of A satisfy

$$\frac{k_{N+1} - k_N}{k_{N+1}^{\beta} + k_N^{\beta}} \geq \max\left\{\frac{2}{l}(1+l)K_2, \left(\frac{1}{1-\beta}\right)\frac{\tilde{b}}{K_1}\right\} \tag{80}$$

where

$$K_1 = |A^{1-\beta} f| + 4C_9\rho^2,$$
$$K_2 = \frac{8}{\rho}|A^{1-\beta} f| + 26C_9\rho,$$

ρ is the radius of the absorbing ball in $D(A)$, whose existence is provided by Proposition 14.

Then Eq. (81) possesses an inertial manifold of dimension N which is a graph of a function $\Phi : P_N H \to Q_N D(A)$ with

$$\tilde{b} = \sup_{\{p \in P_N H\}} |A\Phi(p)|,$$

and

$$l = \sup_{\{p_1, p_2 \in P_N H, p_1 \neq p_2\}} \frac{|A(\Phi(p_1) - \Phi(p_2))|}{|A(p_1 - p_2)|},$$

is the Lipschitz constant of Φ.

In order to apply the theorem, we need to show that the sabra model equation satisfies the properties of the abstract framework of [18]. Let us rewrite our system (9) in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + R(u) = 0, \tag{81}$$

where the nonlinear term $R(u) = B(u, u) - f$.

First of all, we need to show that the nonlinear term $R(u)$ of Eq. (81) satisfies (79) for $\beta = 0$. Indeed, and based on (15), $R(u)$ is a differentiable map from $D(A)$ to $D(A^{1/2}) = V$, where

$$R'(u)w = B(u, w) + B(w, u), \quad \forall u, w \in D(A).$$

Moreover, the estimates (79) are satisfied according to Proposition 1, namely

$$|R'(u)v| = |B(u, v) + B(v, u)| \leq (C_1 + C_2)\|u\||v| \leq \frac{(C_1 + C_2)}{k_1}|Au||v|,$$

where the last inequality results from the fact that $k_1\|u\| \leq |Au|$ and the constant $C_8 = \frac{(C_1+C_2)}{k_1} = \frac{(|a|(\lambda^{-1}+\lambda)+|b|(\lambda^{-1}+1))+(2|a|+2\lambda|b|)}{k_1}$. In addition, one can prove as in Proposition 1 that

$$|AR'(u)v| \leq |AB(u, v)| + |AB(v, u)| \leq C_9'\|u\||Av| \leq C_9|Au||Av|,$$

where here again the last inequality results from the fact that $k_1\|u\| \leq |Au|$ and the constant $C_9 = C_9'/k_1$.

Finally, because of the form of the wavenumbers k_n (2), the spectral gap condition (80) is satisfied for $\beta = 0$. Therefore, we can apply Theorem 12 to Eq. (9), and conclude that the sabra shell model possesses an inertial manifold.

9.1. Dimension of inertial manifolds

Let us calculate the dimension of the inertial manifold of the sabra shell model equation for the specific choice of parameters

$$\beta = 0, \quad l = \frac{1}{2}, \quad \tilde{b} = \rho,$$

where ρ is the radius of the absorbing ball in the norm of the space $D(A)$ (see Proposition 14). In that case, the conditions (80) take the form

$$k_{N+1} - k_N \geq \max\left\{4K_2, \frac{2\rho}{K_1}\right\}, \tag{82}$$

and

Corollary 13. *Eq. (9) possesses an inertial manifold of dimension*

$$N \geq \max\left\{\log_\lambda \frac{4K_2}{\lambda - 1}, \log_\lambda \frac{2\rho}{(\lambda - 1)K_1}\right\}. \tag{83}$$

10. Conclusions

We have established the global regularity of the sabra shell model of turbulence. We shown analytically that the shell model enjoys some of the commonly observed features of real world turbulent flows. Specifically, we have established using the Gevrey regularity technique the existence of an exponentially decaying dissipation range, which is consistent with the observations established in [30] for the GOY model using asymptotic methods. Moreover, we have provided explicit upper bounds, in terms of the given parameters of the shell model, for the number of asymptotic degrees of freedom. Namely, we presented explicit estimates for the dimension of the global attractor and for the number of determining modes. In a forthcoming paper we will investigate the lower bounds on the Hausdorff dimension of the global attractor.

Finally, we have shown that the shell model possess a finite dimensional inertial manifold. That is, there exists an exact rule which parameterizes the small scales as a function of the large scales. The existence of inertial manifolds is not known for the Navier–Stokes equations. It is worth mentioning that the tools presented here can be equally applied to other shell models.

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Appendix A. Dissipativity in $D(A)$

In this section we prove that Eq. (9) is dissipative in different norms and that its solutions are real analytic with respect to the time variable with values in $D(A)$.

First, we start with the space H. Consider the relation (39), and by letting $t \to \infty$ we obtain

$$\limsup_{t\to\infty} |u(t)| \le \frac{|f|_\infty}{\nu k_1^2} = G\nu k_1, \tag{84}$$

where $|f|_\infty = \|f\|_{L^\infty([0,\infty),H)}$ and G is the generalized generalized Grashoff number, defined in (59).

The existence of the absorbing ball in the space V readily follows from Theorem 4. However, the exact formula for the radius of the absorbing ball is much more involved.

We already mentioned that if we assume that the forcing f belongs to some Gevrey class for all $t \ge 0$, then the existence of absorbing balls for solutions of Eq. (9) in the norms of V and $D(A)$ follows from the fact that in that case all the functions belonging to the global attractor are in some Gevrey class and hence are bounded in all weaker norms. In the current section we will show that the strong assumption on the force f to be in the Gevrey class could be dropped. In particular, we will assume that $f \in H$ is time independent. Our proof follows [19] (see also [5], chapter 12).

Fix $m \in \mathbb{N}$, let P_m be the projection of H onto the first m coordinates and denote

$$P_m u = u^m.$$

First, let us complexify our equation and all the relevant spaces and operators. Recall that the Galerkin approximating system (24) of order m for Eq. (9) is an m-dimensional system of ordinary differential equations, with analytic nonlinearity, and hence its solutions are locally complex analytic functions.

Let us consider the complex time variable $t = s\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $s \in \mathbb{R}_+$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\|u^m(s\mathrm{e}^{\mathrm{i}\theta})\|^2 = \frac{\mathrm{d}}{\mathrm{d}s}(u^m(s\mathrm{e}^{\mathrm{i}\theta}), Au^m(s\mathrm{e}^{\mathrm{i}\theta})) = 2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}}{\mathrm{d}t}u^m, Au^m\right).$$

Then, by taking the scalar product of (24a) with Au^m, multiplying by $\mathrm{e}^{\mathrm{i}\theta}$ and taking the real part we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|u^m(s\mathrm{e}^{\mathrm{i}\theta})\|^2 + \nu\cos\theta|Au^m(s\mathrm{e}^{\mathrm{i}\theta})|^2 &= \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(f, Au^m) - \mathrm{e}^{\mathrm{i}\theta}(P_m B(u^m, u^m), Au^m)\right)\\ &\le \frac{\nu\cos\theta}{2}|Au^m(s\mathrm{e}^{\mathrm{i}\theta})|^2 + \frac{|f|^2}{\nu\cos\theta} + \frac{C_1^2}{\nu\cos\theta}\|u^m(s\mathrm{e}^{\mathrm{i}\theta})\|^4,\end{aligned}$$

where the last line follows from (18) and Young's inequality. Finally we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\|u^m(s\mathrm{e}^{\mathrm{i}\theta})\|^2 + \nu\cos\theta|Au^m(s\mathrm{e}^{\mathrm{i}\theta})|^2 \le \frac{2|f|^2}{\nu\cos\theta} + \frac{2C_1^2}{\nu\cos\theta}\|u^m(s\mathrm{e}^{\mathrm{i}\theta})\|^4.$$

Now we are able to derive the bound

$$\|u^m(t)\|^2 \le 2(\|u^m(0)\|^2 + 1) \le 2(\|u^{in}\|^2 + 1), \tag{85}$$

provided $t = s\mathrm{e}^{\mathrm{i}\theta}$ satisfies

$$s \le \frac{1}{2K}(\|u^{in}\|^2 + 1)^{-1}, \tag{86}$$

where $K = \frac{2|f|^2}{\nu\cos\theta} + \frac{2C_1^2}{\nu\cos\theta}$.

The bounds (85) and (86) show that for every $m \in \mathbb{N}$, the functions $u^m(t) : \mathbb{C} \to \mathbb{C}^m$ are analytic in the domain

$$D = D(\nu, \|u^{in}\|, |f|) = \left\{ t = s\mathrm{e}^{\mathrm{i}\theta} : |\theta| < \frac{\pi}{2}, 0 < s \le \frac{1}{2K}(\|u^{in}\|^2 + 1)^{-1} \right\}.$$

Let γ be a small circle contained in the domain D. Then according to Cauchy formula

$$\frac{\mathrm{d}}{\mathrm{d}t}u^m(t) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{u^m(z)}{(z-t)^2}\mathrm{d}z.$$

Using the relation (85) we get

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}u^m \right\| \le \frac{\sqrt{2}}{r_\gamma}(\|u^m(0)\|^2 + 1)^{1/2},$$

where r_γ is the radius of γ. Let $M \subset D$ be a compact subset of D and denote $r_M = dist(M, \partial D)$, then

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}u^m \right\| \le \frac{2\sqrt{2}}{r_M}(\|u^m(0)\|^2 + 1)^{1/2}, \tag{87}$$

for all $t \in M$ and all $m \in \mathbb{N}$.

Finally, comparing the H norms of both sides of Eq. (24a), using the relations (12), (87) and the triangle inequality and we get

$$\begin{aligned}
\nu|Au^m(t)| &\le \left| \frac{\mathrm{d}}{\mathrm{d}t}u^m \right| + |f(t)| + |B(u^m, u^m)| \\
&\le k_1^{-1/2}\left\| \frac{\mathrm{d}}{\mathrm{d}t}u^m \right\| + |f(t)| + |B(u^m, u^m)| \\
&\le \frac{2\sqrt{2}}{r_M\sqrt{k_1}}(\|u^m(0)\|^2 + 1)^{1/2} + |f(t)| + C_1|u^m(t)|\|u^m(t)\|,
\end{aligned}$$

for all $t \in M$. Since all the summands at the right hand side are bounded (see the proofs of Theorems 2 and 4) and those bounds do not depend on m we can pass to the limit in m, using Vitali's convergence theorm for complex analytic functions, concluding that there exists a constant $E = E(\nu, |f|, \|u^{in}\|, \lambda, k_0, a, b, c)$, independent of m, such that

$$|Au(t)| \le E,$$

for all $t \in M$. Moreover, for all $t \in M$ the solution $u(t)$ is a complex analytic function with values in $D(A)$. Repeating this procedure starting instead of 0 from some $t_1 \in M$ we conclude that $|Au(t)|$ is uniformly bounded for all $t > 0$.

Proposition 14. *Let $f \in H$ and $u^{in} \in V$. Then the solution of Eq. (9) is an analytic function, with respect to the time variable with values in $D(A)$, and that it possesses absorbing balls in the spaces H, V and $D(A)$.*

Appendix B. Differentiability of the semigroup with respect to initial conditions

Theorems 2 and 3 show, in fact, that the initial-value problem (9) is well posed. This allows us to define the semigroup $S(t)$, i.e., the one-parameter semigroup family of operators

$$S(t) : u^{in} \in H \to u(t) \in H,$$

which are bounded for almost all $t \ge 0$. According to the Theorem 3, the mapping $S(t) : H \to H$ is Lipschitz continuous with respect to the initial data. The purpose of this section is to show that this mapping is also Fréchet differentiable with respect to the initial data (see, e.g., [33] (chapter VI, 8)).

First of all we wish to linearize the nonlinear term. Let us fix $T > 0$ and let $u, v \in L^\infty([0, T], H) \cap L^2([0, T], V)$ be solutions of Eq. (9) with the initial conditions $u(0) = u^{in}$ and $v(0) = v^{in}$. Then for almost every $t \in [0, T]$ we can write

$$B(u(t), u(t)) - B(v(t), v(t)) = B_0(t)(u(t) - v(t)) + B_1(t, u - v),$$

where $B_0(t) : H \to V'$ is a linear operator defined as

$$B_0(t)w = B(u(t), w(t)) + B(w(t), u(t)),$$

and

$$B_1(t, w) = -B(w(t), w(t)).$$

Let us consider a solution $U(t)$ of the linearized equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \nu AU + B_0(t)U = 0 \tag{88a}$$

$$U(0) = U^{in} = u^{in} - v^{in}, \tag{88b}$$

satisfying $U \in L^\infty([0, T], H) \cap L^2([0, T], V)$. Define

$$\varphi = u - v - U.$$

Using the fact that u, v satisfy Eq. (9) and U is the solution of (88) we can directly check that φ satisfies

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \nu A\varphi + B_0(t)\varphi = -B_1(t, u(t) - v(t)), \tag{89}$$

and $\varphi(0) = 0$. Multiplying by φ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi|^2 + \nu\|\varphi\|^2 + \langle B_0(t)\varphi, \varphi\rangle = -\langle B_1(t, u(t) - v(t)), \varphi\rangle. \tag{90}$$

According to Proposition 1 we can rewrite the last equation as the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi|^2 + \nu\|\varphi\|^2 \le C_8'|\varphi| \cdot |u| \cdot \|\varphi\| + C_9|u - v|^2 \cdot \|\varphi\|.$$

The fact that $u \in L^\infty([0, T], H)$ together with Young's inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}|\varphi|^2 \le \frac{\mathrm{d}}{\mathrm{d}t}|\varphi|^2 + \nu\|\varphi\|^2 \le \frac{C_8}{\nu}|\varphi|^2 + \frac{C_9}{\nu}|u - v|^4.$$

Our next step is to use Gronwall's inequality to get

$$|\varphi(t)|^2 \le \frac{C_9}{\nu}\int_0^T \mathrm{e}^{\frac{C_8}{\nu}(T-s)}|u(s) - v(s)|^4 \mathrm{d}s,$$

for every $t \in (0, T]$. Once again applying Theorem 3 we finally get

$$|\varphi(t)|^2 \le C_{10}|u^{in} - v^{in}|^4,$$

where $0 < C_{10} = \frac{C_9^2 \mathrm{e}^{4K}}{\nu}\int_0^T \mathrm{e}^{\frac{C_8}{\nu}(T-s)}\mathrm{d}s$ and K is the constant from the statement of the Theorem 3. It follows that

$$\frac{|u(t) - v(t) - U(t)|}{|u^{in} - v^{in}|} \le |u^{in} - v^{in}| \longrightarrow 0, \quad 0 < t \le T,$$

as $|u^{in} - v^{in}|$ tends to 0. That exactly means that $U(t)$ is a differential of $S(t)$ with respect to $u^{in} \in H$.

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