Reminder: URG $\leftrightarrow$ IRS

Leading principle:

- every URG behaves like a vertex transitive graph.
- every IRS behaves like a normal subgroup.

BUT ALSO: EVERY FINITE GRAPH IS A URG!

every finite index subgroup is an IRS!

<u>THIS allows a proving machine.</u>

Take your statement on finite graphs $F(G)$ / assume not, seq. of counterexamples, conv. subsequence pass to limit, $\not\!\!\!\!\;\;\;\;$ y

Example of "vertex transitive principle"

If $N \triangleleft F_2$ $\quad (F_2 : N) = \infty$

Claim: $N$ is not finitely generated

$Sch(F_d, N)$

core

$H < F_d$ IRS

of infinite index

$H \neq 1$

NO CORE!

We had:

Invariant Random Subgroup (IRS) of $\Gamma$

$\mathrm{Sub}_\Gamma$ : space of subgroups

$\Gamma \curvearrowright \mathrm{Sub}_\Gamma$ by conjugation

$\mu$ : $\Gamma$-invariant prob. measure

law of an IRS

Sources of IRS: $N \triangleleft \Gamma$ $\delta_N$

If $N_\Gamma(H)$ has finite index

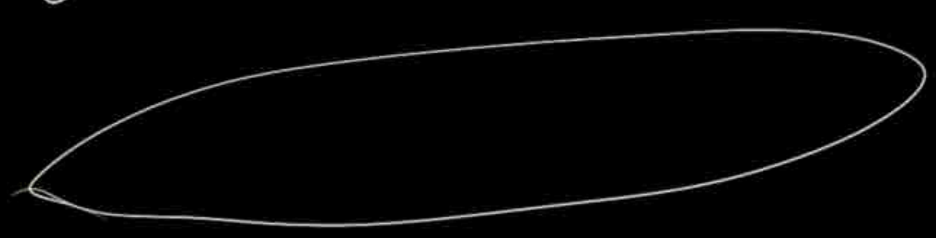
then a random conjugate of $H$ is an IRS.

Is there anything else?

- For some groups, NO: $SL_3(\mathbb{Z})$
- For others, a lot!: $F_d$

How to get an IRS?

$\Gamma \curvearrowright$



$(X,\mu)$ – Borel prob. space

$\Gamma \curvearrowright (X,\mu)$ by $\mu$-preserving maps

Let $x \in (X,\mu)$ be $\mu$-random

let $H = \mathrm{Stab}_\Gamma(x)$.

This is an IRS.

Take $g \in \Gamma$

Apply $g$

L: Every IRS arises as such. (!)

If you look at IRS as a field of stabilizers, then there is no natural way to take limits of IRS-es!

# KESTEN'S THEOREM:

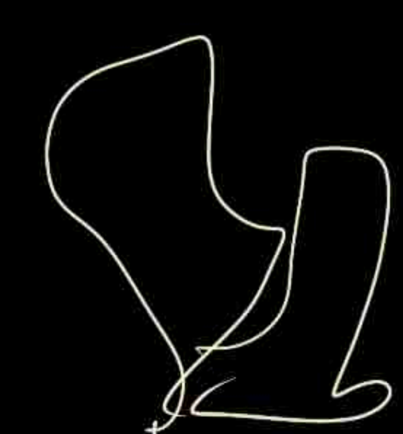
let $\Gamma = \langle S \rangle$ $\quad S = S^{-1}$ $\quad \mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$ RW

$G = Cay(\Gamma, S)$

$$\rho(\Gamma, S) = \rho(G) = \lim_{n \to \infty} \sqrt[2n]{p_{2n,e,e}}$$

where $p_{k,a,b}$ : prob RW of length k starting at a ends at b

$$p_{a,e,e} \cdot p_{b,e,e} \leq p_{a+b,e,e}$$



$\rho(G) = \| M_S \|$ $\quad M_S : \ell^2 \Gamma \to \ell^2 \Gamma$ averaging operator

T (Kesten) $\Gamma$ is amenable $\iff$ $\rho(G) = 1$

REAL Kesten!

$H < \Gamma$ subgroup $\quad \rho(\Gamma, S, H) = \lim \sqrt[2n]{p_{2n,e,H}}$

$p_{2n,e,H}$ in $\Gamma$ $=$ $p_{2n,e,e}$ in $Sch(\Gamma / H)$

$G = (\Gamma, S)$

N

Easy:

H is amenable $\Leftarrow$

$\rho(\Gamma, S) \le \rho(\Gamma, S, H)$

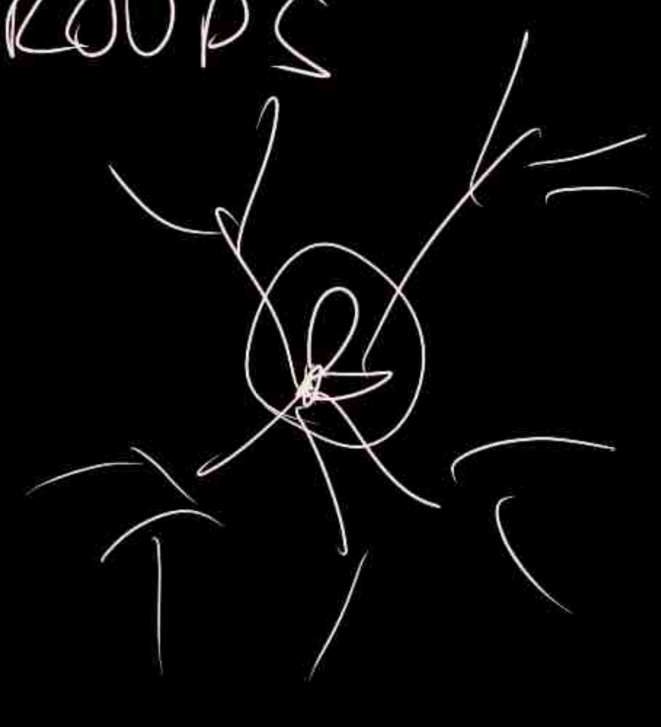
T [Is Kestens PhD thesis]

If $N \lhd \Gamma$, then

$\rho(\Gamma, S) = \rho(\Gamma, S, N) \iff$ N is amenable

NOT TRUE FOR JUST SUBGROUPS

$F_2 < F_{100000}$



T [Kesten for IRS] Let $H < \Gamma$ be an ergodic IRS.

Then

$$\rho(\Gamma, S, H) = \rho(\Gamma, S) \text{ a.s.}$$

$$\Updownarrow$$

$H$ is amenable a.s.

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Assume you know this!

Corollary: If $G_n$ are $d$-regular Ramanujan graphs, $|G_n| \to \infty \Rightarrow G_n \xrightarrow{BS} T_d$.

What is Ramanujan? $d$-regular

D: A finite $G$ is Ramanujan, if

$$\underbrace{\max_{i \neq 0} |\lambda_i|}_{\rho_0(G)} \leq \rho(T_d) = \frac{\sqrt{2d-1}}{d}$$

L [Alon-Boppana] $|G_n| \to \infty$ $d$-regular

$$\liminf \rho_0(G_n) \geq \rho(T_d).$$

"Proof" Assume the Corollary does not hold.

$\exists G_n$, Ramanujan, $G_n \not\xrightarrow[BS]{} T_d$.

Then $G_n$ are Schreier graphs for $\mathbb{F}_{d/2}$.

This gives a sequence of IRS $H_n$ "Ramanujan" but do not weakly converge to 1.

weak* convergence of IRS-es

$\Updownarrow$

BS convergence of $\mathrm{Sch}(\Gamma, S, H)$

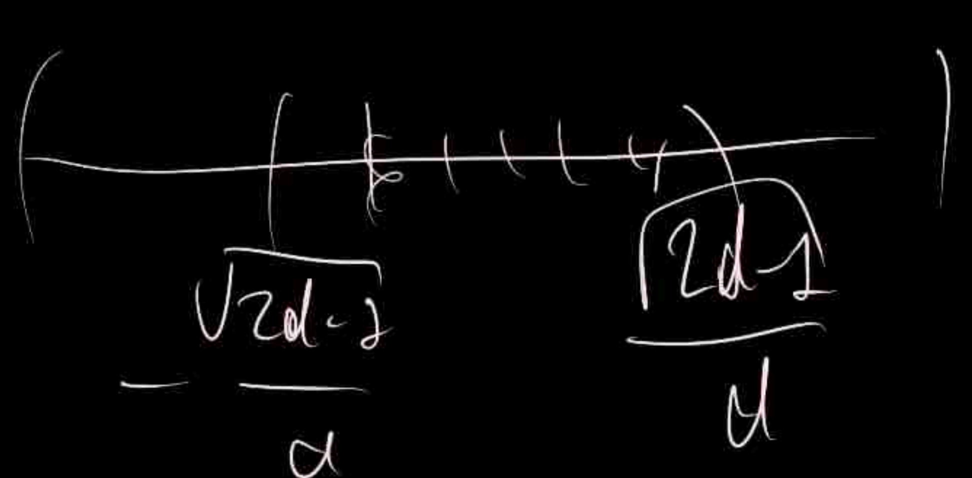
!

$H_n \not\xrightarrow{*} 1$, so some subsequence converges, to something else: IRS $H$.

$|F_d : H| = \infty$ a.s. ✓ $E$

$\rho(F_d, S, H) = \rho(F_d, S)$ Ramanujan.

WHY? Because the expected spectral measure is weak*-continuous.

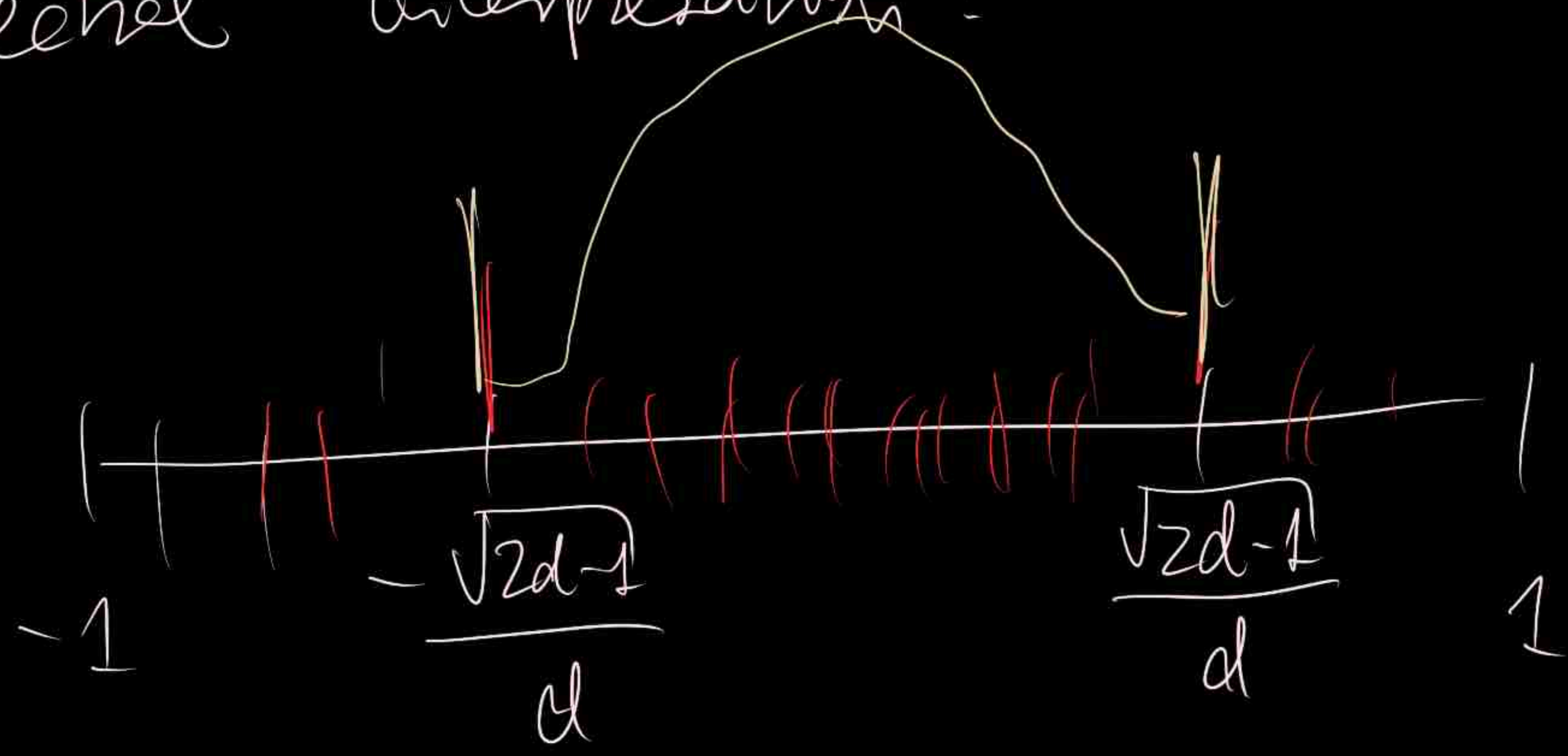



Apply T, $H$ is amenable a.s.

$H$ is cyclic a.s. f.g. $\Rightarrow H = 1$

↯

Spectral interpretation:




finite graph $G$

If I squeeze most of the $\lambda_i$ inside Ramanujan bound $\Rightarrow$ I know its shape!