J Nonlinear Sci (2017) 27:1065–1087
DOI 10.1007/s00332-017-9360-y

Journal of Nonlinear Science

CrossMark

Continuous Data Assimilation for a 2D Bénard Convection System Through Horizontal Velocity Measurements Alone

Aseel Farhat[1] · Evelyn Lunasin[2] · Edriss S. Titi[3,4]

Received: 26 March 2016 / Accepted: 6 January 2017 / Published online: 13 January 2017

Abstract In this paper we propose a continuous data assimilation (downscaling) algorithm for a two-dimensional Bénard convection problem. Specifically we consider the two-dimensional Boussinesq system of a layer of incompressible fluid between two solid horizontal walls, with no-normal flow and stress-free boundary conditions on the walls, and the fluid is heated from the bottom and cooled from the top. In this algorithm, we incorporate the observables as a feedback (nudging) term in the evolution equation of the *horizontal* velocity. We show that under an appropriate choice of the nudging parameter and the size of the spatial coarse mesh observables, and under the assumption that the observed data are error free, the solution of the proposed algorithm converges at an exponential rate, asymptotically in time, to the unique exact unknown reference solution of the original system, associated with the observed data on the horizontal component of the velocity.

Communicated by Paul Newton.

✉ Aseel Farhat
af7py@virginia.edu

Evelyn Lunasin
lunasin@usna.edu

Edriss S. Titi
titi@math.tamu.edu; edriss.titi@weizmann.ac.il

1 Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

2 Department of Mathematics, United States Naval Academy, Annapolis, MD 21402, USA

3 Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA

4 Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, 76100 Rehovot, Israel

Keywords Bénard convection · Boussinesq system · Continuous data assimilation · Signal synchronization · Nudging · Downscaling

Mathematics Subject Classification 35Q30 · 93C20 · 37C50 · 76B75 · 34D06

1 Introduction

The Bénard convection problem is a model of the Boussinesq convection system of an incompressible fluid layer, confined between two solid walls, which is heated from below in such a way that the lower wall maintains a temperature T_0, while the upper one maintains a temperature $T_1 < T_0$. In this case, after some change of variables and proper scaling (by normalizing the distance between the walls and the temperature difference), the two-dimensional Boussinesq equations that govern the perturbation of the velocity, pressure and temperature about the pure conduction steady state are

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p' = \theta \mathbf{e}_2, \tag{1.1a}$$

$$\frac{\partial \theta}{\partial t} - \kappa \Delta \theta + (u \cdot \nabla)\theta - u \cdot \mathbf{e}_2 = 0, \tag{1.1b}$$

$$\nabla \cdot u = 0, \tag{1.1c}$$

$$u(0; x_1, x_2) = u_0(x_1, x_2), \quad \theta(0; x_1, x_2) = \theta_0(x_1, x_2). \tag{1.1d}$$

In this paper, we will consider the above system with the following boundary conditions at walls:

- No-normal flow, and zero temperature at the solid walls, respectively,

$$u_2, \text{ and } \theta = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1. \tag{1.1e}$$

- Stress free at the solid walls,

$$\frac{\partial u_1}{\partial x_2} = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1. \tag{1.1f}$$

- For simplicity, we supplement the system with periodic boundary conditions,

$$u, \theta, p \text{ are periodic, of period } L, \text{ in the } x_1\text{-variable.} \tag{1.1g}$$

Here, $u(t; x_1, x_2) = (u_1(t; x_1, x_2), u_2(t; x_1, x_2))$ is the fluid velocity, and $p' = p'(t; x_1, x_2)$ is the modified pressure given by $p' = p - \left(x_2 - \frac{x_2^2}{2}\right)$ where $p = p(t; x_1, x_2)$ is the scaled pressure of the fluid in the domain $\tilde{\Omega} = (0, L) \times (0, 1)$, $\theta = \theta(t; x_1, x_2)$ is the scaled fluctuation of the temperature around the pure scaled conduction steady-state background temperature profile $1 - x_2$, and it is given by $\theta = T - (1 - x_2)$, where $T = T(t; x_1, x_2)$ is the scaled temperature of the fluid inside the domain $\tilde{\Omega}$, and κ and ν are the thermal diffusivity and kinematic viscosity, respectively.

The mathematical analysis of the Bénard convection system (1.1) has been studied in Foias et al. (1987) (see also Temam 1997), where the existence and uniqueness of weak solutions in dimension two and three were proved, along with the existence of a finite-dimensional global attractor in space dimension two. We remark that the analysis in Foias et al. (1987) was done considering the boundary conditions (1.1g) and Dirichlet boundary conditions for u and θ at the top and the bottom boundaries. The authors in Foias et al. (1987) remarked that one can handle other natural boundary conditions following similar analysis by simply modifying the definition of the function spaces. Different cases for boundary conditions were discussed in Foias et al. (1987). The special boundary conditions case we are considering above, in particular (1.1f), can be handled similarly.

1.1 Equivalent Formulation of the Bénard Problem with Periodic Boundary Conditions

Next, we will show that the solution of the initial-boundary value problem (1.1) is equivalent to the solution of system (1.1a)–(1.1b) subject to periodic boundary conditions with specific spatial symmetry.

Consider any smooth solution $u = (u_1, u_2)$ of (1.1) and perform an even extension of the horizontal component u_1, and the modified pressure $p^{'}$, across the boundary $x_2 = 0$:

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \quad \text{for } (x_1, x_2) \in (0, L) \times (-1, 0), \tag{1.2}$$

$$p^{'}(x_1, x_2) = p^{'}(x_1, -x_2), \quad \text{for} (x_1, x_2) \in (0, L) \times (-1, 0). \tag{1.3}$$

This, with the divergence-free condition, (1.1c), imposes an odd extension of the vertical component u_2 across the boundary $x_2 = 0$:

$$u_2(x_1, x_2) = -u_2(x_1, -x_2), \quad \text{for } (x_1, x_2) \in (0, L) \times (-1, 0). \tag{1.4}$$

The above extensions yield an odd extension on θ across the boundary $x_2 = 0$:

$$\theta(x_1, x_2) = -\theta(x_1, -x_2), \quad \text{for } (x_1, x_2) \in (0, L) \times (-1, 0). \tag{1.5}$$

In view of the above extensions and the original boundary conditions (1.1e)–(1.1g), we have

$$u_2 = \theta = 0, \quad \text{at } x_2 = -1, 0, 1, \tag{1.6a}$$

and

$$\frac{\partial u_1}{\partial x_2} = 0, \quad \text{at } x_2 = -1, 0, 1. \tag{1.6b}$$

It is important to observe that this space of periodic functions with the specific spatial symmetries (1.2)–(1.6) is equivariant (invariant) under the solution operator of the Bénard equations (1.1a)–(1.1c), subject to periodic boundary conditions. It is also clear that such symmetric solutions satisfy the physical boundary conditions (1.1e)–(1.1g). Hence all the results we obtain for the periodic case with the specific spatial symmetry are equally valid for the physical problem.

Based on this remark, in the rest of this paper, we will consider the Bénard problem on the extended fully periodic domain $\Omega = (0, L) \times (-1, 1)$ with the symmetries (1.2)–(1.6).

1.2 A Continuous Data Assimilation Algorithm Using Horizontal Velocity Measurements Only

Designing feedback control algorithms for dynamical systems has been the focus of many authors in the past decades, see, e.g., Cao et al. (2001), Leunberger (1971), Thau (1973), Nijmeijer (2001) and references therein. In the context of meteorology and atmospheric physics, feedback control algorithms with a data assimilation prospective have been studied, e.g., in Ghil et al. (1977, 1978). A finite-dimensional feedback control algorithm for stabilizing solutions of infinite-dimensional dissipative evolution equations, such as reaction-diffusion systems, the Navier–Stokes equations and the Kuramoto–Sivashinsky equation, has been proposed and studied in Azouani and Titi (2014), Cao et al. (2001) (see also Lunasin and Titi 2015). Based on the feedback control algorithm in Azouani and Titi (2014), a continuous data assimilation algorithm, where the spatial coarse mesh observational measurements of the full state variables are incorporated into the equations in the form of a linear feedback control term, was developed in Azouani et al. (2014). The algorithm was designed to work for general *linear and nonlinear* dissipative dynamical systems, and it can be outlined as follows: consider a general dissipative evolutionary equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u), \tag{1.7}$$

where the initial data $u(0) = u_0$ are missing. The algorithm is of the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(v) - \mu(I_h(v) - I_h(u)), \tag{1.8a}$$

$$v(0) = v_0, \tag{1.8b}$$

where $\mu > 0$ is a relaxation (nudging) parameter and v^0 is taken to be arbitrary initial data. $I_h(\cdot)$ represents an interpolant operator based on the spatially discrete observational measurements of a system at a coarse spatial resolution of size h, for $t \in [0, T]$. Notice that if system (1.8) is globally well posed and $I_h(v)$ converge to $I_h(u)$ in time, then we recover the reference $u(t, x)$ from the approximate solution $v(t, x)$. The main task is to find explicit estimates on $\mu > 0$ and $h > 0$, in terms of the physical parameters of the underlying problem, such that the approximate solution

$v(t)$ is converging to the reference solution $u(t)$ in time. Notice that this algorithm, in (1.8a), requires measurement of *all* state variables of the dynamical system (1.7).

The continuous data assimilation in the context of the incompressible 2D Navier–Stokes equations (NSE) was studied in Azouani et al. (2014) under the assumption that the data are noise free. A computational study of this algorithm was later presented in Altaf et al. (2015) and Gesho et al. (2016). The case when the observational data contain stochastic noise is treated in Bessaih et al. (2015). Most recently an extension of this algorithm for the case of discrete spatio-temporal measurements with error is studied in Foias et al. (2000); in addition, it is also shown there how to implement this algorithm in order to extract statistical properties of the relevant reference solution.

In Markowich et al. (2016), the authors analyzed an algorithm for continuous data assimilation for 3D Brinkman-Forchheimer-extended Darcy (3D BFeD) model of a porous medium, a model equation when the velocity is too large for classical Darcy's law to be valid. Furthermore, in Albanez et al. (2016), the proposed data assimilation algorithm was also applied to several three-dimensional subgrid scale turbulence models.

Analyzing the validity and success of a data assimilation algorithm when some state variable observations are not available is an important problem meteorology and engineering (Charney et al. 1969; Ghil et al. 1977, 1978) (see also Altaf et al. 2015). In a recent work Farhat et al. (2015a), a continuous data assimilation scheme for the two-dimensional incompressible Bénard convection problem was introduced. The data assimilation algorithm in Farhat et al. (2015a) constructs the approximate solutions for the velocity u and temperature fluctuations θ using only the observational data, $I_h(u)$, of the velocity field and *without any measurements for the temperature fluctuations*. Inspired by the recent algorithm proposed in Farhat et al. (2015a), we introduced an abridged dynamic continuous data assimilation for the 2D NSE in Farhat et al. (2015b). The proposed algorithm in Azouani et al. (2014) for the 2D NSE requires measurements for the two components of the velocity vector field. On the other hand, in Farhat et al. (2015b), we establish convergence results for an improved algorithm where the observational data needed to be measured and inserted into the model equation are reduced or subsampled. Our algorithm there requires observational measurements of *only one component* of the velocity vector field. An abridged data assimilation algorithm for the 3D Leray-α model, using observations in *only any two components and without any measurements on the third component of the velocity field*, was later analyzed in Farhat et al. (2015c). In a more recent paper Farhat et al. (2016), we proposed and analyzed a data assimilation algorithm for the 2D and 3D Bénard convection problem in porous medium that employs measurements of the *temperature only*.

Inspired by the previous works Farhat et al. (2015a, b), in this paper we propose and analyze a continuous data assimilation algorithm for a Bénard convection problem in the periodic box $\Omega = (0, L) \times (-1, 1)$ with the symmetries (1.2)–(1.6). Our algorithm requires measurements of *only the horizontal component of the velocity* to recover the full (velocity and temperature) reference solution of the 2D Bénard convection problem. This algorithm is given by

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla \tilde{p} = \eta \mathbf{e}_2 - \mu (I_h(v_1) - I_h(u_1)) \mathbf{e}_1, \tag{1.9a}$$

$$\frac{\partial \eta}{\partial t} - \kappa \Delta \eta - (v \cdot \nabla) \eta - v \cdot \mathbf{e}_2 = 0, \tag{1.9b}$$

$$\nabla \cdot v = 0, \tag{1.9c}$$

$$v(0; x) = v_0(x), \quad \eta(0; x) = \eta_0(x), \tag{1.9d}$$

subject to the spatial symmetries (1.2)–(1.4) on the solution (v, η), with fully periodic boundary condition in the box $\Omega = (0, L) \times (-1, 1)$. Here, (v_0, η_0) can be taken arbitrary and I_h is a linear interpolant operator constructed from the spatial discrete measurements of the horizontal component of the velocity u_1. Two types of interpolants can be considered. One is to be given by a linear interpolant operator $I_h : H^1 \to L^2$ satisfying the approximation property

$$\|\varphi - I_h(\varphi)\|_{L^2}^2 \leq c_0 h^2 \|\varphi\|_{H^1}^2, \tag{1.10}$$

for every $\varphi \in H^1$, where $c_0 > 0$ is a dimensionless constant. The other type is given by $I_h : H^2 \to L^2$, together with approximation property

$$\|\varphi - I_h(\varphi)\|_{L^2}^2 \leq c_0 h^2 \|\varphi\|_{H^1}^2 + c_0^2 h^4 \|\varphi\|_{H^2}^2, \tag{1.11}$$

for every $\varphi \in H^2$, where $c_0 > 0$ is a dimensionless constant. Notice that the interpolant operator $I_h(\varphi)$ [satisfying (1.10) or (1.11)], with φ satisfying the spatial symmetry (1.2), will also satisfy the spatial symmetry (1.2).

One example of an interpolant observable that satisfies (1.10) and (1.11), with φ satisfying the symmetry (1.2), is the orthogonal projection onto the low Fourier modes with wave numbers k such that $|k| \leq 1/h$. A physical example of an interpolant observable that satisfies (1.10) is based on local volume elements (local volume averages) that were studied in Azouani et al. (2014) and Jones and Titi (1992). A physical example of an interpolant observable that satisfies (1.11) is based on measurements at a discrete set of nodal points in Ω (see Appendix A in Azouani et al. 2014). Observe that one has to slightly modify the presentation given in Azouani et al. (2014) and Jones and Titi (1992) to fulfill the spatial symmetry condition (1.2).

In this paper, we give a rigorous justification that the solution of the data assimilation algorithm (1.9), (v, η) is converging to the exact reference solution of the Bénard convection problem (u, θ), subject to (1.2)–(1.6). We provide explicit estimates on the relaxation (nudging) parameter μ and the spatial resolution h of the observational measurements, in terms of physical parameters, that are needed in order for the proposed downscaling (data assimilation) algorithm to recover the reference solution under the assumption that the supplied data are error free. An extension of algorithm (1.9) for the case of measurements with stochastic noise and the case of discrete spatio-temporal measurements with error can be established by combining the ideas and tools reported in Bessaih et al. (2015) and Foias et al. (2000), respectively, with those presented here. We do not present the results in this paper in order to fix ideas by focusing on the main goal, i.e., observing only the horizontal component of the velocity.

2 Preliminaries

For the sake of completeness, this section presents some preliminary material and notation commonly used in the mathematical study of fluids, in particular in the study of the Navier–Stokes equations (NSE) and the Euler equations. For more detailed discussion on these topics, we refer the reader to Constantin and Foias (1988), Robinson (2001) and Temam (1995, 2001).

We begin by defining function spaces corresponding to the relevant physical boundary conditions. We define $\mathcal{F}_1$ to be the set of trigonometric polynomials in (x_1, x_2), with period L in the x_1-variable, that are even, with period 2, in the x_2-variable. We define $\mathcal{F}_2$ to be the set of trigonometric polynomials in (x_1, x_2) with period L in the x_1-variable, that are odd, with period 2, in the x_2-variable. We denote the space of smooth vector-valued functions which incorporates the divergence-free condition by

$$\mathcal{V} := \{\phi \in \mathcal{F}_1 \times \mathcal{F}_2 : \nabla \cdot \phi = 0\}.$$

Remark 2.1 We will use the same notation indiscriminately for both scalar and vector Lebesgue and Sobolev spaces, which should not be a source of confusion.

The closures of $\mathcal{V}$ and $\mathcal{F}_2$ in $L^2(\Omega)$ will be denoted by H_0 and H_1, respectively. H_0 and H_1 will be endowed with the usual scalar product

$$(u, v)_{H_0} = \sum_{i=1}^{2} \int_\Omega u_i(x) v_i(x)\,\mathrm{d}x \quad \text{and} \quad (\psi, \phi)_{H_1} = \int_\Omega \psi(x)\phi(x)\,\mathrm{d}x,$$

and the associated norms $\|u\|_{H_0} = (u, u)_{H_0}^{1/2}$ and $\|\phi\|_{H_1} = (\phi, \phi)_{H_1}^{1/2}$, respectively. We denote the closures of $\mathcal{V}$ and $\mathcal{F}_2$ in $H^1(\Omega)$ by V_0 and V_1, respectively. V_0 and V_1 are Hilbert spaces endowed by the scalar product

$$((u, v))_{V_0} = (u, v)_{H_0} + \sum_{i,j=1}^{2} \int_\Omega \partial_j u_i(x) \partial_j v_i(x)\,\mathrm{d}x$$

and

$$((\psi, \phi))_{V_1} = \sum_{j=1}^{2} \int_\Omega \partial_j \psi(x) \partial_j \phi(x)\,\mathrm{d}x,$$

and the associated norms $\|u\|_{V_0} = ((u, u))_{V_0}^{1/2}$ and $\|\phi\|_{V_1} = ((\phi, \phi))_{V_1}^{1/2}$, respectively.

Remark 2.2 Since $\theta, u_2 \in V_1$ are zero at $x_2 = -1, 0, 1$, because they are odd functions in the x_2-variable and are periodic in the x_1-variable (hence they have zero mean in Ω), then by the Poincaré inequality (2.6), below, $\|.\|_{V_1}$ defines a norm on V_1.

Let $D(A_0) = V_0 \cap H^2(\Omega)$ and $D(A_1) = V_1 \cap H^2(\Omega)$ and let $A_i : D(A_i) \to H_i$ be the unbounded linear operator defined by

$$(A_i\phi, \psi)_{H_i} = ((\phi, \psi))_{V_i}, \qquad i = 0, 1,$$

for all $\phi, \psi \in D(A_i)$.

Remark 2.3 Notice that for each $i = 0, 1$, there exists a complete orthonormal set of eigenfunctions $w_{i,j}$ of H_i such that $A_i w_{i,j} = \lambda_{i,j} w_{i,j}$ where $0 \le \lambda_{i,j} \le \lambda_{i,j+1}$ for $j \in \mathbb{N}$ and each $i = 0, 1$. The operator A_1 is a positive definite, while the operator A_0 is a nonnegative operator with finitely many eigenfunctions corresponding to the eigenvalue $\lambda = 0$. Using the Cauchy–Schwarz inequality and the elliptic regularity of the operator $A_0 + I$, one can show that $\|u\|_{H^2} \equiv \|u\|_{L^2} + \|A_0 u\|_{L^2}$. Moreover, in periodic boundary conditions, the operator $A_0 = -\Delta$.

We denote the Helmholtz–Leray projector from $L^2(\Omega)$ onto H_0 by $\mathcal{P}_\sigma$ and the dual space of V_i by $V_i^{'}$, for $i = 0, 1$. We define the bilinear map $B : V_0 \times V_0 \to V_0^{'}$ by the continuous extension of

$$\langle B(u, v), w\rangle_{V_0'} = \int_\Omega ((u \cdot \nabla)v \cdot w)\, \mathrm{d}x,$$

for $u, v, w \in \mathcal{V}$; we define its scalar analogue $\mathcal{B} : V_0 \times V_1 \to V_1^{'}$ by the continuous extension of

$$\langle \mathcal{B}(u, \theta), \phi\rangle_{V_1'} = \int_\Omega ((u \cdot \nabla)\theta \cdot \phi)\, \mathrm{d}x,$$

for $u \in \mathcal{V}$ and $\theta, \phi \in \mathcal{F}_2$.

These bilinear operators enjoy the algebraic property

$$\langle B(u, v), w\rangle_{V_0'} = -\langle B(u, w), v\rangle_{V_0'}, \tag{2.1a}$$

and

$$\langle \mathcal{B}(u, \theta), \phi\rangle_{V_1'} = -\langle \mathcal{B}(u, \phi), \theta\rangle_{V_1'}, \tag{2.1b}$$

for each $u, v, w \in V_0$ and $\theta, \phi \in V_1$. This property can be established easily through integration by parts, for smooth functions, with the use of (1.1c), and then extended by continuity of the relevant function spaces. Consequently, the above bilinear maps also enjoy the orthogonality property

$$\langle B(u, v), v\rangle_{V_0'} = 0, \qquad \text{and} \qquad \langle \mathcal{B}(u, \theta), \theta\rangle_{V_1'} = 0, \tag{2.2}$$

for each $u, v \in V_0$ and $\theta \in V_1$. Also, in two dimensions (under periodic boundary conditions), the bilinear operator $B(.,.)$ satisfies the following identities (see, e.g., Temam 2001):

$$(B(u, u), A_0 u)_{H_0} = 0, \tag{2.3a}$$

for each $u \in \mathcal{D}(A_0)$, and consequently

$$(B(u, w), A_0 w)_{H_0} + (B(w, u), A_0 w)_{H_0} + (B(w, w), A_0 u)_{H_0} = 0, \tag{2.3b}$$

for each u and $w \in \mathcal{D}(A_0)$.

Employing the above notation, we write the incompressible two-dimensional Bénard convection problem (1.1), with the relevant boundary conditions (1.2)–(1.6), in the functional form

$$\frac{du}{dt} + \nu A_0 u + B(u, u) = \mathcal{P}_\sigma(\theta \mathbf{e}_2), \tag{2.4a}$$

$$\frac{d\theta}{dt} + \kappa A_1 \theta + \mathcal{B}(u, \theta) - u \cdot \mathbf{e}_2 = 0, \tag{2.4b}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0. \tag{2.4c}$$

Next, we recall the two-dimensional Ladyzhenskaya inequality (Ref. Constantin and Foias 1988),

$$\|\varphi\|_{L^4}^2 \leq c\|\varphi\|_{L^2} \|\varphi\|_{H^1}, \quad \text{for every } \varphi \in H^1, \tag{2.5}$$

where c is a dimensionless, positive constant. Hereafter, c denotes a generic constant which may change from line to line. We also observe that we have the Poincaré inequality:

$$\|\varphi\|_{L^2}^2 \leq \lambda_1^{-1} \|\nabla \varphi\|_{L^2}^2, \quad \text{for all } \varphi \in V_1, \tag{2.6a}$$

$$\|\varphi\|_{V_1}^2 \leq \lambda_1^{-1} \|A_1 \varphi\|_{L^2}^2, \quad \text{for all } \varphi \in \mathcal{D}(A_1), \tag{2.6b}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the operator A_1.

Remark 2.4 Notice that for $w = (w_1, w_2) \in V_0$ one has $w_2 \in V_1$ and thus the Poincaré inequality is only valid for w_2, i.e.,

$$\|w_2\|_{L^2}^2 \leq \lambda_1^{-1} \|\nabla w_2\|_{L^2}^2, \quad \text{and} \quad \|\nabla w_2\|_{L^2}^2 \leq \lambda_1^{-1} \|A_1 w_2\|_{L^2}^2 = \lambda_1^{-1} \|\Delta w_2\|_{L^2}^2.$$

This is not the case for the horizontal component w_1. On the other hand, thanks to the boundary conditions, (1.2)–(1.5) and the divergence-free condition, (1.1c), we have

$$\begin{aligned} \|w_i\|_{H^2}^2 &= \|w_i\|_{L^2}^2 + \|\nabla w_i\|_{L^2}^2 + \sum_{j,k=1}^{2} \left\|\partial_j \partial_k w_i\right\|_{L^2}^2 \\ &\equiv \|w_i\|_{L^2}^2 + \|\nabla w_i\|_{L^2}^2 + \|\Delta w_i\|_{L^2}^2, \end{aligned}$$

for $i = 1, 2$. More precisely, $\|w_i\|_{H^2}^2 = \|w_i\|_{L^2}^2 + 2\,\|\Delta w_i\|_{L^2}^2$, for $i = 1, 2$.

Next, we will prove a lemma that we will use later in our analysis.

Lemma 2.5 *Let* $u = (u_1, u_2) \in V_0$*, then*

$$\|u_2\|_{L^2}^2 \le \|\nabla u_1\|_{L^2}^2 . \tag{2.7}$$

Proof Since $u \in V_0$, then $u = (u_1, u_2) \in H^1$, and it satisfies the spatial symmetries (1.2) and (1.4), in particular, $u_2(x_1, 0) = 0$. Moreover, u satisfies the divergence-free condition $\nabla \cdot u = 0$ in L^2. Therefore, one has

$$u_2(x_1, x_2) = u_2(x_1, 0) + \int_0^{x_2} \frac{\partial u_2}{\partial x_2}(x_1, s)\, \mathrm{d}s = -\int_0^{x_2} \frac{\partial u_1}{\partial x_1}(x_1, s)\, \mathrm{d}s.$$

By Cauchy–Schwarz inequality, we get

$$|u_2(x_1, x_2)| = \left| \int_0^{x_2} \frac{\partial u_1}{\partial x_1}(x_1, s)\, \mathrm{d}s \right| \le \left(\int_0^1 \left| \frac{\partial u_1}{\partial x_1}(x_1, s) \right|^2 \mathrm{d}s \right)^{1/2} .$$

Thus,

$$\int_0^L |u_2(x_1, x_2)|^2 \, \mathrm{d}x_1 \le \int_0^L \int_0^1 \left| \frac{\partial u_1}{\partial x_1}(x_1, s) \right|^2 \mathrm{d}x_1 \mathrm{d}s.$$

This implies that

$$\begin{aligned} \|u_2\|_{L^2}^2 = \int_{-1}^1 \int_0^L |u_2(x_1, x_2)|^2 \, \mathrm{d}x_1 \mathrm{d}s &= 2 \int_0^1 \int_0^L |u_2(x_1, x_2)|^2 \, \mathrm{d}x_1 \mathrm{d}s \\ &\le 2 \int_0^L \int_0^1 \left| \frac{\partial u_1}{\partial x_1}(x_1, s) \right|^2 \mathrm{d}x_1 \mathrm{d}s \\ &= \int_0^L \int_{-1}^1 \left| \frac{\partial u_1}{\partial x_1}(x_1, s) \right|^2 \mathrm{d}x_1 \mathrm{d}s \\ &\le \|\nabla u_1\|_{L^2}^2 . \end{aligned}$$

□

We will apply the following inequality which is a particular case of a more general inequality proved in Jones and Titi (1992).

Lemma 2.6 (Jones and Titi 1992) *Let* $\tau > 0$ *be fixed. Suppose that* $Y(t)$ *is an absolutely continuous function which is locally integrable and that it satisfies the following:*

$$\frac{\mathrm{d}Y}{\mathrm{d}t} + \alpha(t) Y \le \beta(t), \qquad \text{a.e. on } (0, \infty),$$

such that

$$\liminf_{t\to\infty} \int_t^{t+\tau} \alpha(s)\, ds \ge \gamma, \qquad \limsup_{t\to\infty} \int_t^{t+\tau} \alpha^-(s)\, ds < \infty, \tag{2.8}$$

and

$$\lim_{t\to\infty}\int_t^{t+\tau}\beta^+(s)\,ds = 0, \tag{2.9}$$

for some $\gamma > 0$*, where* $\alpha^- = \max\{-\alpha, 0\}$ *and* $\beta^+ = \max\{\beta, 0\}$. *Then,* $Y(t) \to 0$*, at an exponential rate, as* $t \to \infty$.

We also recall the following results from Foias et al. (1987) and Temam (1997) for the Bénard convection problem (2.4). These results were proved for a special case of boundary conditions: periodic in the x_1-variable and Dirichlet in the x_2-variable. The authors remarked that the analysis will follow similar steps for other natural boundary conditions. The same results hold for the boundary conditions we are considering in this paper: fully periodic boundary conditions with the spatial symmetries (1.2)–(1.6).

Theorem 2.7 (Existence and uniqueness of weak solutions) *Let* $T > 0$ *be fixed. Let* $\nu > 0$ *and* $\kappa > 0$*. If* $u_0 \in H_0$ *and* $\theta_0 \in H_1$*, then system* (2.4) *has a unique weak solution* (u, θ) *such that* $u \in C([0, T]; H_0) \cap L^2([0, T]; V_0)$ *and* $\theta \in C([0, T]; H_1) \cap L^2([0, T], V_1)$.

It was also shown in Foias et al. (1987) and Temam (1997) that the 2D Bénard convection system has a finite-dimensional global attractor.

Theorem 2.8 (Foias et al. 1987; Temam 1997, Existence of a global attractor) *Let* $T > 0$ *be fixed. If the initial data* $u_0 \in V_0$ *and* $\theta_0 \in V_1$*, then system* (2.4) *has a unique strong solution* (u, θ) *that satisfies* $u \in C([0, T]; V_0) \cap L^2([0, T]; \mathcal{D}(A_0))$ *and* $\theta \in C([0, T]; V_1) \cap L^2([0, T]; \mathcal{D}(A_1))$. *Moreover, system* (2.4) *is globally well posed and possesses a finite-dimensional global attractor* $\mathcal{A}$ *which is maximal among all the bounded invariant sets, and is compact in* $H_0 \times H_1$.

We will use the following bounds on (u, θ) later in our analysis.

Proposition 2.9 (Foias et al. 1987; Temam 1997, A variant of the maximum principle) *Let* (u, θ) *be a strong solution of* (2.4)*, then*

$$\theta(t; \cdot) = \tilde{\theta}(t; \cdot) + \bar{\theta}(t; \cdot),$$

where $-1 \le \tilde{\theta}(t; x) \le 1$ *and*

$$\left\|\bar{\theta}(t)\right\|_{H_1} \le \left(\|(\theta_0 - 1)_+\|_{H_1} + \|(\theta_0 + 1)_-\|_{H_1}\right) e^{-\kappa t},$$

for all $x \in \Omega$ *and* $t > 0$. *Here* $M_+ = \max\{M, 0\}$ *and* $M_- = \max\{-M, 0\}$ *for any real number* M.

Proposition 2.10 (Foias et al. 1987; Temam 1997, Uniform bounds on the solutions) *Let* (u, θ) *be a strong solution of* (2.4). *There exists* $t_0 > 0$*, which depends on norms of the initial data, such that for all* $t \ge t_0$,

$$\|\theta(t)\|_{H_1} \le a_0, \quad and \quad \|u(t)\|_{H_0} \le b_0, \tag{2.10}$$

$$\int_t^{t+1} \|u(s)\|_{V_0}^2 \, ds \le a_3, \quad \int_t^{t+1} \|\theta(s)\|_{V_1}^2 \, ds \le b_3, \tag{2.11}$$

$$\|u(t)\|_{V_0}^2 \le (a_2 + a_3)\, e^{a_1} =: J_0, \tag{2.12}$$

$$\|\theta(t)\|_{V_1}^2 \le (b_2 + b_3) e^{b_1} =: J_1, \tag{2.13}$$

where $a_0, a_1, a_2, a_3, , b_0, b_1, b_2, b_3, J_0$ and J_1 are positive constants that depend on L, ν, λ_1 and κ.

Moreover, the solution (u, θ) is analytic in time with values in $D(A)$ and all the H^m norms of $u(t)$ and $\theta(t)$ remain uniformly bounded in time for all $t \ge \delta$, for some $\delta > 0$.

3 Convergence Results

In this section, we derive conditions under which the approximate solution (v, η), of the data assimilation algorithm system (3.1), converges to the corresponding unique reference solution (u, θ) of the Bénard convection problem (2.4), as $t \to \infty$.

In functional form the data assimilation algorithm, system (1.9), reads as

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A_0 v + B(v, v) = \mathcal{P}_\sigma(\eta \mathbf{e}_2) - \mu \mathcal{P}_\sigma(I_h(v_1) - I_h(u_1))\mathbf{e}_1, \tag{3.1a}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + \kappa A_1 \eta + \mathcal{B}(v, \eta) - v \cdot \mathbf{e}_2 = 0, \tag{3.1b}$$

$$v(0) = v_0 \quad \eta(0) = \eta_0. \tag{3.1c}$$

Here, (u, θ) is a strong solution of the 2D Bénard convection problem (2.4), in the global attractor $\mathcal{A}$ corresponding to the observable measurements $I_h(u)$, that the above algorithm is designed to recover in a unique fashion, asymptotically in time, at an exponential rate.

Theorem 3.1 *Suppose that I_h satisfies the approximation property (1.10) and the spatial symmetry property (1.2). Let $(u(t), \theta(t))$, for $t \ge 0$, be a strong solution in the global attractor of (2.4).*

(1) Let $T > 0$, $v_0 \in V_0$ and $\eta_0 \in H_1$. Suppose that $\mu > 0$ is large enough such that

$$\mu \ge 2(K_1 + \nu), \tag{3.2}$$

where $K_1 = K_1(\nu, \kappa, L)$ is a constant defined in (3.19) below, and $h > 0$ is small enough such that $4\mu c_0^2 h^2 \le \nu$. Then, (3.1) has a unique solution (v, η) that satisfies

$$v \in C([0, T]; V_0) \cap L^2([0, T]; \mathcal{D}(A_0)), \tag{3.3a}$$

$$\eta \in C([0, T]; H_1) \cap L^2([0, T]; V_1), \tag{3.3b}$$

and also

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^2([0,T]; H_0), \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} \in L^2([0,T]; V_1^{'}). \tag{3.3c}$$

Moreover, the solution (v, η) *depends continuously on the initial data in the* $V_0 \times H_1$ *norm, and it satisfies*

$$\|u(t) - v(t)\|_{V_0}^2 + \|\theta(t) - \eta(t)\|_{H_1}^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

(2) Let $T > 0$, $v_0 \in V_0$ *and* $\eta_0 \in V_1$. *Suppose that* $\mu > 0$ *is large enough such that*

$$\mu \geq 2(K_1 + \nu) + 2K_2, \tag{3.4}$$

where $K_i = K_i(\nu, \kappa, L)$, $i = 1, 2$, *are constants defined below in* (3.19) *and* (3.28), *respectively, and suppose that* $h > 0$ *is small enough such that* $4\mu c_0^2 h^2 \leq \nu$. *Then,* (3.1) *has a unique strong solution* (v, η) *that satisfies*

$$v \in C([0,T]; V_0) \cap L^2([0,T]; \mathcal{D}(A_0)), \tag{3.5a}$$

$$\eta \in C([0,T]; V_1) \cap L^2([0,T]; \mathcal{D}(A_1)), \tag{3.5b}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^2([0,T]; H_0), \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} \in L^2([0,T]; H_1). \tag{3.5c}$$

Moreover, the strong solution (v, η) *depends continuously on the initial data, in the* $V_0 \times V_1$ *norm, and it satisfies*

$$\|u(t) - v(t)\|_{V_0}^2 + \|\theta(t) - \eta(t)\|_{V_1}^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

Proof Since we assume that (u, θ) is a reference solution of system (1.1), then it is enough to show the existence and uniqueness of the difference $(w, \xi) = (u - v, \theta - \eta)$. In the proof below, we will derive formal *a-priori* bounds on the difference (w, ξ), under the conditions that μ is large enough and h is small enough such that $4\mu c_0^2 h \leq \nu$. These *a-priori* estimates, together with the global existence and uniqueness of the solution (u, θ), form the key elements for showing the global existence of the solution (v, η) of system (1.9). The convergence of the approximate solution (v, η) to the exact reference solution (u, θ) will also be established under the tighter condition on the nudging parameter μ as stated in (3.2). Uniqueness can then be obtained using similar energy estimates.

As we stated above, the estimates we provide in this proof are formal, but can be justified by the Galerkin approximation procedure and then passing to the limit while using the relevant compactness theorems. We will omit the rigorous details of this standard procedure (see, e.g., Constantin and Foias 1988; Robinson 2001; Temam 2001) and provide only the formal *a-priori* estimates.

As above we define $w = u - v$, $\xi = \theta - \eta$. Then w and ξ satisfy the system

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu A_0 w + B(v, w) + B(w, u) = \mathcal{P}_\sigma(\xi \mathbf{e}_2) - \mu \mathcal{P}_\sigma(I_h(w_1)\mathbf{e}_1), \tag{3.6a}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} - \kappa A_1 \xi + \mathcal{B}(v, \xi) + \mathcal{B}(w, \theta) - w \cdot \mathbf{e}_2 = 0, \tag{3.6b}$$

$$w(0) = w_0 := u_0 - v_0, \tag{3.6c}$$

$$\xi(0) = \xi_0 := \theta_0 - \eta_0. \tag{3.6d}$$

Taking the L^2 inner product of (3.6a) and (3.6b) with w and ξ, respectively, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \nu\|\nabla w\|_{L^2}^2 + (B(w, u), w) = \int_\Omega \xi(w \cdot \mathbf{e}_2)\,\mathrm{d}x - \mu(I_h(w_1), w_1), \tag{3.7a}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{L^2}^2 + \kappa\|\nabla \xi\|_{L^2}^2 + (\mathcal{B}(w, \theta), \xi) = \int_\Omega \xi(w \cdot \mathbf{e}_2)\,\mathrm{d}x. \tag{3.7b}$$

By the Hölder and Young inequalities, Lemma 2.5 and Poincaré inequality (2.6), we have

$$\begin{aligned}\left|\int_\Omega \xi(w \cdot \mathbf{e}_2)\,\mathrm{d}x\right| &\le \|w_2\|_{L^2}\|\xi\|_{L^2} \\ &\le \frac{\kappa\lambda_1}{20}\|\xi\|_{L^2}^2 + \frac{c}{\kappa\lambda_1}\|w_2\|_{L^2}^2 \\ &\le \frac{\kappa}{20}\|\nabla\xi\|_{L^2}^2 + \frac{c}{\kappa\lambda_1}\|\nabla w_1\|_{L^2}^2.\end{aligned} \tag{3.8}$$

Young's inequality and Lemma 2.5 yield

$$\begin{aligned}|(\mathcal{B}(w, \theta), \xi)| &= |(\mathcal{B}(w, \xi), \theta)| \\ &\le \|\theta\|_{L^\infty}\|w\|_{L^2}\|\nabla\xi\|_{L^2} \\ &\le \frac{\kappa}{20}\|\nabla\xi\|_{L^2}^2 + \frac{c}{\kappa}\|\theta\|_{L^\infty}^2\|w\|_{L^2}^2 \\ &\le \frac{\kappa}{20}\|\nabla\xi\|_{L^2}^2 + \frac{c}{\kappa}\|\theta\|_{L^\infty}^2\left(\|w_1\|_{L^2}^2 + \|\nabla w_1\|_{L^2}^2\right).\end{aligned} \tag{3.9}$$

Also [(thanks to Ladyzhenskaya's inequality (2.5)]

$$\begin{aligned}|(B(w, u), w)| &\le \|\nabla u\|_{L^2}\|w\|_{L^4}^2 \\ &\le c\|\nabla u\|_{L^2}\|w\|_{L^2}\|w\|_{V_0} \\ &\le c\|\nabla u\|_{L^2}\|w\|_{L^2}\left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right)^{1/2}\end{aligned}$$

$$\begin{aligned}
&\le c\left\|\nabla u\right\|_{L^2}\left\|w\right\|_{L^2}^2 + c\left\|\nabla u\right\|_{L^2}\left\|w\right\|_{L^2}\left\|\nabla w\right\|_{L^2}\\
&\le c\left\|\nabla u\right\|_{L^2}\left\|w\right\|_{L^2}^2 + \frac{\nu}{20}\left\|\nabla w\right\|_{L^2}^2 + \frac{c}{\nu}\left\|\nabla u\right\|_{L^2}^2\left\|w\right\|_{L^2}^2\\
&\le \frac{\nu}{20}\left\|\nabla w\right\|_{L^2}^2 + c\left\|\nabla u\right\|_{L^2}\left(1+\frac{\left\|\nabla u\right\|_{L^2}}{\nu}\right)\left(\left\|w_1\right\|_{L^2}^2 + \left\|\nabla w_1\right\|_{L^2}^2\right).
\end{aligned} \tag{3.10}$$

Thanks to the assumptions $4\mu c_0^2 h^2 \le \nu$ and (3.2), and Young's inequality, we have

$$\begin{aligned}
-\mu(I_h(w_1), w_1) &= -\mu(I_h(w_1) - w_1, w_1) - \mu\left\|w_1\right\|_{L^2}^2\\
&\le \mu\left\|I_h(w_1) - w_1\right\|_{L^2}\left\|w_1\right\|_{L^2} - \mu\left\|w_1\right\|_{L^2}^2\\
&\le \mu c_0 h\left\|w_1\right\|_{L^2}\left\|w_1\right\|_{H^1} - \mu\left\|w_1\right\|_{L^2}^2\\
&\le \mu c_0^2 h^2\left\|w_1\right\|_{H^1}^2 - \frac{3\mu}{4}\left\|w_1\right\|_{L^2}^2\\
&\le \frac{\nu}{4}\left(\left\|w_1\right\|_{L^2}^2 + \left\|\nabla w_1\right\|_{L^2}^2\right) - \frac{3\mu}{4}\left\|w_1\right\|_{L^2}^2\\
&\le \frac{\nu}{4}\left\|\nabla w_1\right\|_{L^2}^2 - \frac{5\mu}{8}\left\|w_1\right\|_{L^2}^2.
\end{aligned} \tag{3.11}$$

Taking the L^2-inner product of (3.6a) with $A_0 w = -\Delta w$, and using the orthogonality properties (2.3a) and (2.3b), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla w\right\|_{L^2}^2 + \nu\left\|A_0 w\right\|_{L^2}^2 + (B(w,w), A_0 u) = \int_\Omega \xi (A_0 w\cdot \mathbf{e}_2)\,\mathrm{d}x\\
- \mu(I_h(w_1), \Delta w_1).
\end{aligned} \tag{3.12}$$

Using Hölder's inequality and Ladyzhenskaya's inequality (2.5), we get

$$\begin{aligned}
|(B(w,w), A_0 u)| &\le \left\|A_0 u\right\|_{L^2}\left\|w\right\|_{L^4}\left\|\nabla w\right\|_{L^4}\\
&\le c\left\|A_0 u\right\|_{L^2}\left\|w\right\|_{L^2}^{1/2}\left\|w\right\|_{V_0}\left\|w\right\|_{H^2}^{1/2}.
\end{aligned}$$

Thanks to Remark 2.3, we have

$$\begin{aligned}
|(B(w,w), A_0 u)| &\le c\left\|A_0 u\right\|_{L^2}\left\|w\right\|_{L^2}^{1/2}\left(\left\|w\right\|_{L^2}^2 + \left\|\nabla w\right\|_{L^2}^2\right)^{1/2}\\
&\quad\times\left(\left\|w\right\|_{L^2}^2 + \left\|A_0 w\right\|_{L^2}^2\right)^{1/4}\\
&\le c\left\|A_0 u\right\|_{L^2}\left\|w\right\|_{L^2}^{1/2}\left(\left\|w\right\|_{L^2} + \left\|\nabla w\right\|_{L^2}\right)\left(\left\|w\right\|_{L^2}^{1/2} + \left\|A_0 w\right\|_{L^2}^{1/2}\right)\\
&\le c\left\|A_0 u\right\|_{L^2}\Big(\left\|w\right\|_{L^2}^2 + \left\|w\right\|_{L^2}\left\|\nabla w\right\|_{L^2} + \left\|w\right\|_{L^2}^{3/2}\left\|A_0 w\right\|_{L^2}^{1/2}\\
&\quad+\left\|w\right\|_{L^2}^{1/2}\left\|\nabla w\right\|_{L^2}\left\|A_0 w\right\|_{L^2}^{1/2}\Big).
\end{aligned}$$

Since $A_0 w = -\Delta w$ in periodic boundary conditions, we also have

$$\|\nabla w\|_{L^2}^2 = \int_\Omega \nabla w \cdot \nabla w \, \mathrm{d}x\mathrm{d}y = \int_\Omega w A_0 w \, \mathrm{d}x\mathrm{d}y \leq \|w\|_{L^2} \|A_0 w\|_{L^2} .$$

Thus, Young's inequality and Lemma 2.5 imply

$$\begin{aligned}
|(B(w,w), A_0 u)| &\leq c \|A_0 u\|_{L^2} \left(\|w\|_{L^2}^2 + \|w\|_{L^2}^{3/2} \|A_0 w\|_{L^2}^{1/2} + \|w\|_{L^2} \|A_0 w\|_{L^2} \right) \\
&\leq \frac{\nu}{20} \|A_0 w\|_{L^2}^2 + c \|A_0 u\|_{L^2} \left(1 + \frac{\|A_0 u\|_{L^2}^{1/3}}{\nu^{1/3}} + \frac{\|A_0 u\|_{L^2}}{\nu} \right) \|w\|_{L^2}^2 \\
&\leq \frac{\nu}{20} \|A_0 w\|_{L^2}^2 + c \|A_0 u\|_{L^2} \left(1 + \frac{\|A_0 u\|_{L^2}^{1/3}}{\nu^{1/3}} + \frac{\|A_0 u\|_{L^2}}{\nu} \right) \\
&\quad \times \left(\|w_1\|_{L^2}^2 + \|\nabla w_1\|_{L^2}^2 \right) .
\end{aligned} \tag{3.13}$$

Also, Young's inequality and Lemma 2.5 yield

$$\begin{aligned}
\left| \int_\Omega \xi (A_0 w \cdot \mathbf{e}_2) \, \mathrm{d}x \right| &= \left| \int_\Omega \xi \, \Delta w_2 \, \mathrm{d}x \right| \\
&\leq \|\nabla w_2\|_{L^2} \|\nabla \xi\|_{L^2} \\
&\leq \frac{\kappa}{20} \|\nabla \xi\|_{L^2}^2 + \frac{c}{\kappa} \|\nabla w_2\|_{L^2}^2 \\
&\leq \frac{\kappa}{20} \|\nabla \xi\|_{L^2}^2 + \frac{c}{\kappa} \|w_2\|_{L^2} \|\Delta w_2\|_{L^2} \\
&\leq \frac{\kappa}{20} \|\nabla \xi\|_{L^2}^2 + \frac{\nu}{20} \|\Delta w_2\|_{L^2}^2 + \frac{c}{\nu\kappa^2} \|w_2\|_{L^2}^2 \\
&\leq \frac{\kappa}{20} \|\nabla \xi\|_{L^2}^2 + \frac{\nu}{20} \|\Delta w_2\|_{L^2}^2 + \frac{c}{\nu\kappa^2} \|\nabla w_1\|_{L^2}^2 .
\end{aligned} \tag{3.14}$$

Using (1.10), Young's inequality and the assumption that $4\mu c h^2 \leq \nu$, we have

$$\begin{aligned}
-\mu(I_h(w_1), -\Delta w_1) &= \mu(I_h(w_1) - w_1, \Delta w_1) - \mu \|\nabla w_1\|_{L^2}^2 \\
&\leq \mu c_0 h \|w_1\|_{H^1} \|\Delta w_1\|_{L^2} - \mu \|\nabla w_1\|_{L^2}^2 \\
&\leq \frac{\mu^2 c_0 h^2}{2\nu} \left(\|w_1\|_{L^2}^2 + \|\nabla w_1\|_{L^2}^2 \right) + \frac{\nu}{2} \|\Delta w_1\|_{L^2}^2 \\
&\quad - \mu \|\nabla w_1\|_{L^2}^2 \\
&\leq \frac{\nu}{2} \|\Delta w_1\|_{L^2}^2 + \frac{\mu}{8} \|w_1\|_{L^2}^2 - \frac{7\mu}{8} \|\nabla w_1\|_{L^2}^2 .
\end{aligned} \tag{3.15}$$

Thanks to the Poincaré inequality (2.6a) (see Remark 2.4) we have $\|\nabla w_2\|_{L^2}^2 \geq \lambda_1 \|w_2\|_{L^2}^2$ and $\|\nabla \xi\|_{L^2}^2 \geq \lambda_1 \|\xi\|_{L^2}^2$. This implies that

$$\begin{aligned} \nu \|\nabla w\|_{L^2}^2 + \kappa \|\nabla \xi\|_{L^2}^2 &= \nu \|\nabla w_1\|_{L^2}^2 + \frac{\nu}{2} \|\nabla w_2\|_{L^2}^2 + \frac{\nu}{2} \|\nabla w_2\|_{L^2}^2 + \kappa \|\nabla \xi\|_{L^2}^2 \\ &\geq \nu \|\nabla w_1\|_{L^2}^2 + \frac{\nu}{2} \|\nabla w_2\|_{L^2}^2 + \frac{\nu\lambda_1}{2} \|w_2\|_{L^2}^2 + \kappa\lambda_1 \|\xi\|_{L^2}^2 . \end{aligned} \tag{3.16}$$

Since $\nu \|A_0 w\|_{L^2}^2 \geq 0$, it follows from equations (3.7) and (3.12) and estimates (3.8)–(3.11) and (3.13)–(3.15) and (3.16):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{V_0}^2 + \|\xi\|_{L^2}^2\right) + \frac{\min\{\nu,\kappa\}}{8}\left(\|\nabla w_1\|_{L^2}^2 + \frac{\|\nabla w_2\|_{L^2}^2}{2} + \frac{\lambda_1}{2}\|w_2\|_{L^2}^2\right. \\ &\quad \left. + \lambda_1 \|\xi\|_{L^2}^2\right) \leq \frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{V_0}^2 + \|\xi\|_{L^2}^2\right) + \frac{\min\{\nu,\kappa\}}{8}\left(\|\nabla w\|_{L^2}^2 + \|\nabla \xi\|_{L^2}^2\right) \\ &\quad \leq (\alpha(t) - \mu)\left(\|w_1\|_{L^2}^2 + \|\nabla w_1\|_{L^2}^2\right), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \alpha(t) :=& \frac{c}{\kappa\lambda_1} + \frac{c}{\nu\kappa^2} + \frac{c}{\kappa}\|\theta(t)\|_{L^\infty}^2 + c\|\nabla u(t)\|_{L^2}\left(1 + \frac{\|\nabla u(t)\|_{L^2}}{\nu}\right) \\ &+ c\|A_0 u(t)\|_{L^2}\left(1 + \frac{\|A_0 u(t)\|_{L^2}^{1/3}}{\nu^{1/3}} + \frac{\|A_0 u(t)\|_{L^2}}{\nu}\right). \end{aligned} \tag{3.18}$$

Since by assumption (u,θ) is a solution that is contained in the global attractor of (3.1), by Proposition 2.9 and Proposition 2.10, we conclude that there exist a positive constants $K_1 = K_1(\nu,\kappa,\lambda_1,L)$ such that for all $t \in \mathbb{R}$

$$\alpha(t) \leq K_1. \tag{3.19}$$

Then, assumption (3.2) implies that $\mu - \alpha(t) \geq \frac{\mu}{2}$, for all $t \geq 0$. Thus, thanks to the Poincaré inequality (2.6), inequality (3.17) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{V_0}^2 + \|\xi\|_{H_1}^2\right) + \gamma\left(\|w\|_{V_0}^2 + \|\xi\|_{H_1}^2\right) \leq 0, \tag{3.20}$$

where $\gamma = \min\left\{\frac{\nu}{16}, \frac{\nu\lambda_1}{16}, \frac{\kappa\lambda_1}{8}, \frac{\mu}{2}\right\}$. By Gronwall's inequality, it follows that

$$\|w(t)\|_{V_0}^2 + \|\xi(t)\|_{H_1}^2 \leq \left(\|w(0)\|_{V_0}^2 + \|\xi(0)\|_{H_1}^2\right) e^{-\gamma t}, \tag{3.21}$$

for every $t \geq 0$. Next we prove the second part of the theorem.

Taking the L^2-inner product of (3.6b) with $A_1\xi = -\Delta\xi$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\xi\|_{L^2}^2 + \kappa\|\Delta\xi\|_{L^2}^2 + (\mathcal{B}(w,\theta), -\Delta\xi) + (\mathcal{B}(v,\xi), -\Delta\xi) \le (w_2, -\Delta\xi). \tag{3.22}$$

Using the Cauchy–Schwarz and Young inequalities, and Lemma 2.5, we have

$$\begin{aligned} |(w_2, -\Delta\xi)| \le \|w_2\|_{L^2}^2\|\Delta\xi\|_{L^2}^2 &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\|w_2\|_{L^2}^2 \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\|\nabla w_1\|_{L^2}^2 . \end{aligned} \tag{3.23}$$

Using Hölder's inequality, Ladyzhenskaya's inequality (2.5) and Lemma 2.5, we get

$$\begin{aligned} |(\mathcal{B}(w,\theta), -\Delta\xi)| &\le \|w\|_{L^4}\|\nabla\theta\|_{L^4}\|\Delta\xi\|_{L^2} \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\|w\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\|w\|_{L^2}\|w\|_{H^1}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\|w\|_{L^2}\left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right)^{1/2} \\ &\quad \times \|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{c}{\kappa}\left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}\|w\|_{L^2}\right)\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{\nu}{20}\|\nabla w\|_{L^2}^2 + \frac{c}{\kappa}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\ &\quad \times \left(1 + \frac{1}{\nu}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\right)\|w\|_{L^2}^2 \\ &\le \frac{c}{\kappa}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\left(1 + \frac{1}{\nu}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\right) \\ &\quad \times \left(\|w_1\|_{L^2}^2 + \|\nabla w_1\|_{L^2}^2\right) + \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + \frac{\nu}{20}\|\nabla w\|_{L^2}^2 . \end{aligned} \tag{3.24}$$

The Hölder inequality and Ladyzhenskaya inequality (2.5) also yield

$$\begin{aligned} |(\mathcal{B}(v,\xi), \Delta\xi))| &\le \|v\|_{L^4}\|\nabla\xi\|_{L^4}\|\Delta\xi\|_{L^2} \\ &\le c\|v\|_{L^2}^{1/2}\|v\|_{V_0}^{1/2}\|\nabla\xi\|_{L^2}^{1/2}\|\Delta\xi\|_{L^2}^{3/2} \\ &\le \frac{\kappa}{20}\|\Delta\xi\|_{L^2}^2 + c\|v\|_{L^2}^2\|v\|_{V_0}^2\|\nabla\xi\|_{L^2}^2 . \end{aligned} \tag{3.25}$$

Since $\nu\|A_0 w\|_{L^2}^2 \ge 0$, we conclude from equations (3.7), (3.12) and (3.22), and estimates (3.8)–(3.11), (3.13)–(3.15) and (3.23)–(3.25), that

$$\frac{d}{dt}\left(\|w\|_{V_0}^2+\|\nabla\xi\|_{L^2}^2\right)+\frac{\min\{\nu,\kappa\}}{8}\left(\|\nabla w\|_{L^2}^2+\|\Delta\xi\|_{L^2}^2\right)$$
$$\le(\tilde{\alpha}(t)-\mu)\left(\|w_1\|_{L^2}^2+\|\nabla w_1\|_{L^2}^2\right)+2\|v\|_{L^2}^2\|v\|_{V_0}^2\|\nabla\xi\|_{L^2}^2, \tag{3.26}$$

where

$$\begin{aligned}\tilde{\alpha}(t):=&\frac{c}{\kappa\lambda_1}+\frac{c}{\kappa}+c\|\nabla u(t)\|_{L^2}\left(1+\frac{\|\nabla u(t)\|_{L^2}}{\nu}\right)\\&+c\|A_0u(t)\|_{L^2}\left(1+\frac{\|A_0u(t)\|_{L^2}^{1/3}}{\nu^{1/3}}+\frac{\|A_0u(t)\|_{L^2}}{\nu}\right)\\&+\frac{c}{\kappa}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\left(1+\frac{1}{\nu}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\right)\\&\le\alpha(t)+\frac{c}{\kappa}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\left(1+\frac{1}{\nu}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\right),\end{aligned} \tag{3.27}$$

where $\alpha(t)$ is defined in (3.18). Since (u,θ) is the reference solution that is assumed to be contained in the global attractor of (2.4), then by Proposition 2.10, there exists a constant $K_2=K_2(\nu,\kappa,\lambda_1,L)$ such that, for all $t\in\mathbb{R}$,

$$\tilde{\alpha}(t)\le K_1+K_2, \tag{3.28}$$

where K_1 is a constant defined in (3.19). By the first part of the theorem, $v(t)$ is a global solution of (1.9) that belongs to $C([0,T],V_0)$ for any $T>0$. Moreover, assumption (3.4) implies that $\|u(t)-v(t)\|_{V_0}^2\to 0$, as $t\to\infty$. Then, by Proposition 2.10

$$\|v(t)\|_{V_0}^2\le K_3, \tag{3.29}$$

for some constant $K_3=K_3(\nu,\kappa,\lambda_1,L)$, for all $t\ge 0$.

Now, assumption (3.4), equation (3.26) and estimates (3.27)–(3.29) yield that $\mu-\alpha(t)\ge\frac{\mu}{2}>0$, for $t\ge 0$, and thus

$$\frac{d}{dt}\left(\|w\|_{V_0}^2+\|\nabla\xi\|_{L^2}^2\right)+\frac{\min\{\nu,\kappa\}}{8}\left(\|\nabla w\|_{L^2}^2+\|\Delta\xi\|_{L^2}^2\right)$$
$$+\frac{\mu}{2}\left(\|w_1\|_{L^2}^2+\|\nabla w_1\|_{L^2}^2\right)\le cK_3^2\|\nabla\xi\|_{L^2}^2. \tag{3.30}$$

Thanks to Poincaré inequalities (2.6a) and (2.6b), by a similar argument as in (3.16), we have

$$\frac{d}{dt}\left(\|w\|_{V_0}^2+\|\nabla\xi\|_{L^2}^2\right)+\frac{\min\{\nu,\kappa\}}{8}\left(\|\nabla w_1\|_{L^2}^2+\frac{\|\nabla w_2\|_{L^2}^2}{2}+\frac{\lambda_1}{2}\|w_2\|_{L^2}^2+\lambda_1\|\nabla\xi\|_{L^2}^2\right)$$
$$+\frac{\mu}{2}\left(\|w_1\|_{L^2}^2+\|\nabla w_1\|_{L^2}^2\right)\le cK_3^2\|\nabla\xi\|_{L^2}^2,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{V_0}^2+\|\nabla\xi\|_{L^2}^2\right)+\min\left\{\frac{\nu}{16},\frac{\nu\lambda_1}{16},\frac{\kappa}{8},\frac{\mu}{2}\right\}\left(\|w\|_{V_0}^2+\|\nabla\xi\|_{L^2}^2\right) \\ \leq cK_3^2\,\|\nabla\xi\|_{L^2}^2\,. \tag{3.31}$$

Next, we observe that since $\nu\,\|\nabla w\|_{L^2}^2 \geq 0$, under the assumption (3.4) on μ, equations (3.7b) and (3.17) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{V_0}^2+\|\xi\|_{L^2}^2\right)+\frac{\kappa}{8}\,\|\nabla\xi\|_{L^2}^2 \leq 0. \tag{3.32}$$

Integrating (3.32) over the interval $(t, t+\tau)$ and using estimate (3.21), we conclude that

$$\frac{\kappa}{8}\int_t^{t+\tau}\|\nabla\xi(s)\|_{L^2}^2\,\mathrm{d}s \to 0, \quad \text{as} \quad t\to\infty, \tag{3.33}$$

for every $\tau > 0$. Now we apply the general Gronwall Lemma 2.6 to equation (3.31), while taking in Lemma 2.6 $\alpha(t)=\min\left\{\frac{\nu}{16},\frac{\nu\lambda_1}{16},\frac{\kappa}{8},\frac{\mu}{2}\right\}$ and $\beta(t)=cK_3^2\,\|\nabla\xi(t)\|_{L^2}^2$, we conclude that

$$\left(\|w(t)\|_{V_0}^2+\|\nabla\xi(t)\|_{L^2}^2\right)\to 0, \tag{3.34}$$

at an exponential rate, as $t\to\infty$. That is,

$$\|u(t)-v(t)\|_{V_0}^2+\|\theta(t)-\eta(t)\|_{V_1}^2 \to 0,$$

at an exponential rate, as $t\to\infty$. □

Remark 3.2 We note that, based on the analysis above, in the case where the observational measurements are not error free, one can estimate the error between the solution of the algorithm and the exact reference solution of the system in terms of the error in the measurements (see, e.g., Foias et al. 2000). When the observational data contain stochastic noise, an estimate on the expected value of the error (in time) in terms of the variance of the noise in the measurements can be obtained following the steps in Bessaih et al. (2015). Moreover, in the case of discrete spatio-temporal (discrete in time and space) measurements with systematic error, one can follow the work in Foias et al. (2000) and obtain estimates on the error (in appropriate norm) in time.

Theorem 3.3 *Suppose that I_h satisfies the approximation property (1.11) and the symmetry property (1.2). Let $(u(t),\theta(t))$, for $t\geq 0$, be a strong solution in the global attractor of (2.4).*

(1) Let $T>0$, $v_0\in V_0$ and $\eta_0\in H_1$. Suppose that $\mu>0$ is large enough such that condition (3.2) holds, and $h>0$ is small enough such that $2\mu c_0^2h^2\leq\frac{\nu}{16}$. Then,

(3.1) has a unique solution (v, η) that satisfies the regularity properties (3.3). Moreover, the solution (v, η) depends continuously on the initial data in the $V_0 \times H_1$ norm, and it satisfies

$$\|u(t) - v(t)\|_{V_0}^2 + \|\theta(t) - \eta(t)\|_{H_1}^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

(2) *Let $T > 0$, $v_0 \in V_0$ and $\eta_0 \in V_1$. Suppose that $\mu > 0$ is large enough such that condition (3.4) holds, and suppose that $h > 0$ is small enough such that $2\mu c_0^2 h^2 \le \frac{\nu}{16}$. Then, (3.1) has a unique strong solution (v, η) that satisfies the regularity properties (3.5).*
Moreover, the strong solution (v, η) depends continuously on the initial data, in the $V_0 \times V_1$ norm, and it satisfies

$$\|u(t) - v(t)\|_{V_0}^2 + \|\theta(t) - \eta(t)\|_{V_1}^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

Proof The proof of this theorem is identical to the proof of Theorem 3.1 except for estimates (3.11) and (3.15). We will reproduce the adequate versions of these two estimates here.

When the interpolant operator I_h satisfies (1.11), using Young inequality, instead of the treatment in (3.11) we have

$$\begin{aligned}
-\mu(I_h(w_1), w_1) &= -\mu(I_h(w_1) - w_1, w_1) - \mu \|w_1\|_{L^2}^2 \\
&\le \mu \|I_h(w_1) - w_1\|_{L^2} \|w_1\|_{L^2} - \mu \|w_1\|_{L^2}^2 \\
&\le \mu c_0 h \|w_1\|_{H^1} \|w_1\|_{L^2} + \mu c_0^2 h^2 \|w_1\|_{H^2} \|w_1\|_{L^2} - \mu \|w_1\|_{L^2}^2 \\
&\le 2\mu c_0^2 h^2 \|w_1\|_{H^1}^2 + \frac{\mu}{8} \|w_1\|_{L^2}^2 + \frac{2\mu^2 c_0^4 h^4}{\nu} \|w_1\|_{H^2}^2 + \frac{\nu}{8} \|w_1\|_{L^2}^2 \\
&\quad - \mu \|w_1\|_{L^2}^2 .
\end{aligned}$$

Recall that, (see Remark 2.4), $\|w_1\|_{H^2}^2 = \|w_1\|_{H^1}^2 + 2\|\Delta w_1\|_{L^2}^2$. Thus, thanks to the assumption $2\mu c_0^2 h^2 \le \frac{\nu}{16}$, we get

$$\begin{aligned}
-\mu(I_h(w_1), w_1) &\le \left(2\mu c_0^2 h^2 + \frac{2\mu^2 c_0^4 h^4}{\nu}\right) \|w_1\|_{H^1}^2 + \frac{\nu}{8} \|w_1\|_{L^2}^2 - \frac{7}{8}\mu \|w_1\|_{L^2}^2 \\
&\quad + \frac{2\mu^2 c_0^4 h^4}{\nu} \|\Delta w_1\|_{L^2}^2 \\
&\le \left(\frac{\nu}{8} + \frac{\nu}{128}\right) \|w_1\|_{H^1}^2 + \frac{\nu}{8} \|w_1\|_{L^2}^2 - \frac{7}{8}\mu \|w_1\|_{L^2}^2 \\
&\quad + \frac{\nu}{128} \|\Delta w_1\|_{L^2}^2
\end{aligned}$$

$$\leq \frac{\nu}{4} \|w_1\|_{H^1}^2 - \frac{7}{8}\mu \|w_1\|_{L^2}^2 + \frac{\nu}{128} \|\Delta w_1\|_{L^2}^2$$
$$= \frac{\nu}{4} \|\nabla w_1\|_{L^2}^2 + \left(\frac{\nu}{4} - \frac{7}{8}\mu\right) \|w_1\|_{L^2}^2 + \frac{\nu}{128} \|\Delta w_1\|_{L^2}^2 .$$

Assumption (3.2) implies that

$$-\mu(I_h(w_1), w_1) \leq \frac{\nu}{4} \|\nabla w_1\|_{L^2}^2 - \frac{3}{4}\mu \|w_1\|_{L^2}^2 + \frac{\nu}{128} \|\Delta w_1\|_{L^2}^2 . \tag{3.35}$$

Instead of the treatment in (3.15), by using Young inequality, we have

$$\begin{aligned} -\mu(I_h(w_1), -\Delta w_1) &= \mu(I_h(w_1) - w_1), \Delta w_1) - \mu \|\nabla w_1\|_{L^2}^2 \\ &\leq \mu c_0 h \|w_1\|_{H^1} \|\Delta w_1\|_{L^2} + \mu c_0^2 h^2 \|w_1\|_{H^2} \|\Delta w_1\|_{L^2} \\ &\quad -\mu \|\nabla w_1\|_{L^2}^2 \\ &\leq \frac{2\mu^2 c_0^2 h^2}{\nu} \|w_1\|_{H^1}^2 + \frac{2\mu^2 c_0^4 h^4}{\nu} \|w_1\|_{H^2}^2 + \frac{\nu}{4} \|\Delta w_1\|_{L^2}^2 \\ &\quad -\mu \|\nabla w_1\|_{L^2}^2 . \end{aligned}$$

Since $\|w_1\|_{H^2}^2 = \|w_1\|_{H^1}^2 + 2\|\Delta w_1\|_{L^2}^2$, the assumption $2\mu c_0^2 h^2 \leq \frac{\nu}{16}$ yields

$$\begin{aligned} -\mu(I_h(w_1), -\Delta w_1) &\leq \left(\frac{\mu}{16} + \frac{\nu}{512}\right) \|w_1\|_{H^1}^2 + \left(\frac{\nu}{512} + \frac{\nu}{4}\right) \|\Delta w_1\|_{L^2}^2 \\ &\quad - \mu \|\nabla w_1\|_{L^2}^2 \\ &\leq \frac{3\nu}{8} \|\Delta w_1\|_{L^2}^2 + \left(\frac{\mu}{16} + \frac{\nu}{512}\right) \|w_1\|_{L^2}^2 + \left(\frac{\mu}{16} + \frac{\nu}{512} - \mu\right) \\ &\quad \times \|\nabla w_1\|_{L^2}^2 . \end{aligned}$$

Thus, condition (3.2) implies that

$$-\mu(I_h(w_1), -\Delta w_1) \leq \frac{3\nu}{8} \|\Delta w_1\|_{L^2}^2 + \frac{\mu}{8} \|w_1\|_{L^2}^2 - \frac{7\mu}{8} \|\nabla w_1\|_{L^2}^2 . \tag{3.36}$$

The rest of the proof of the theorem follows the proof of Theorem 3.1 while replacing (3.11) and (3.15) by (3.35) and (3.36), respectively. □

Acknowledgements The work of A.F. is supported in part by the NSF Grant DMS-1418911. The work of E.L. is supported in part by the ONR Grant N0001416WX01475 and the ONR Grant N0001416WX00796. The work of E.S.T. is supported in part by the ONR Grant N00014-15-1-2333 and the NSF Grants DMS-1109640 and DMS-1109645.

References

Albanez, D., Nussenzveig-Lopes, H., Titi, E.S.: Continuous data assimilation for the three-dimensional Navier–Stokes-α model. Asymptot. Anal. **97**(1–2), 139–164 (2016)

Altaf, M.U., Titi, E.S., Gebrael, T., Knio, O., Zhao, L., McCabe, M.F., Hoteit, I.: Downscaling the 2D Bénard convection equations using continuous data assimilation. Computational Geosciences (COMG) (2015). arXiv:1512.04671

Azouani, A., Titi, E.S.: Feedback control of nonlinear dissipative systems by finite determining parameters: a reaction-diffusion paradigm. Evol. Equ. Control Theory **3**(4), 579–594 (2014)

Azouani, A., Olson, E., Titi, E.S.: Continuous data assimilation using general interpolant observables. J. Nonlinear Sci. **24**(2), 277–304 (2014)

Bessaih, H., Olson, E., Titi, E.S.: Continuous assimilation of data with stochastic noise. Nonlinearity **28**, 729–753 (2015)

Cao, C., Kevrekidis, I., Titi, E.S.: Numerical criterion for the stabilization of steady states of the Navier–Stokes equations. Indiana Univ. Math. J. **50**, 37–96 (2001)

Charney, J., Halem, J., Jastrow, M.: Use of incomplete historical data to infer the present state of the atmosphere. J. Atmos. Sci. **26**, 1160–1163 (1969)

Constantin, P., Foias, C.: Navier–Stokes Equations. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL (1988)

Farhat, A., Jolly, M.S., Titi, E.S.: Continuous data assimilation for the 2D Bénard convection through velocity measurements alone. Phys. D **303**, 59–66 (2015)

Farhat, A., Lunasin, E., Titi, E.S.: Abridged continuous data assimilation for the 2D Navier–Stokes equations utilizing measurements of only one component of the velocity field. J. Math. Fluid Mech. **18**(1), 1–23 (2015)

Farhat, A., Lunasin, E., Titi, E.S.: A note on abridged continuous data assimilation for the 3D subgrid scale α-models of turbulence, Preprint

Farhat, A., Lunasin, E., Titi, E.S.: Data assimilation algorithm for 3D Bénard convection in porous media employing only temperature measurements. J. Math. Anal. Appl. **438**(1), 492–506 (2016)

Foias, C., Manley, O., Temam, R.: Attractors for the Bénard problem: existence and physical bounds on their fractal dimension. Nonlinear Anal. Theory Methods Appl. **11**, 939–967 (1987)

Foias, C., Mondaini, C., Titi, E.S.: A discrete data assimilation scheme for the solutions of the 2D Navier–Stokes equations and their statistics. SIAM J. Appl. Dyn. Syst. **15**(4), 2109–2142 (2000)

Gesho, M., Olson, E., Titi, E.S.: A computational study of a data assimilation algorithm for the two-dimensional Navier–Stokes equations. Commun. Comput. Phys. **19**(4), 1094–1110 (2016)

Ghil, M., Shkoller, B., Yangarber, V.: A balanced diagnostic system compatible with a barotropic prognostic model. Mon. Weather Rev. **105**, 1223–1238 (1977)

Ghil, M., Halem, M., Atlas, R.: Time-continuous assimilation of remote-sounding data and its effect on weather forecasting. Mon. Weather Rev. **107**, 140–171 (1978)

Jones, D.A., Titi, E.S.: Determining finite volume elements for the 2D Navier–Stokes equations. Phys. D **60**, 165–174 (1992)

Leunberger, D.: An introduction to observers. IEEE. Trans. Autom. Control **16**, 596–602 (1971)

Lunasin, E., Titi, E.S.: Finite determining parameters feedback control for distributed nonlinear dissipative systems: a computational study, arXiv:1506.03709 [math.AP] (2015)

Markowich, P., Titi, E.S., Trabelsi, S.: Continuous data assimilation for the three-dimensional Brinkman–Forchheimer–Extended Darcy model. Nonlinearity **29**(4), 1292–1328 (2016)

Nijmeijer, H.: A dynamic control view of synchronization. Phys. D **154**, 219–228 (2001)

Robinson, J.C.: Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2001)

Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition, Applied Mathematical Sciences, vol. 68. Springer, New York (1997)

Temam, R.: Navier–Stokes Equations and Nonlinear Functional Analysis, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1995)

Temam, R.: Navier-Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, RI (2001), Reprint of the 1984 edition

Thau, F.E.: Observing the state of non-linear dynamic systems. Int. J. Control **17**, 471–479 (1973)