Adding all up:

If $G_n$ is BS convergent $G_n \xrightarrow{BS} (G,o)$
RANDOM

Then $\mu_{G_n} \xrightarrow{*} \mathbb{E}\,\mu_{(G,o)}$

---

WHAT ABOUT $\mu_G(\{0\}) = \dfrac{\# i / \lambda_i = 0}{|G|} = \dfrac{\dim \ker A}{|G|}$

IS IT CONTINUOUS? YES N Y

CAN YOU LOCALIZE IT? NO N ?

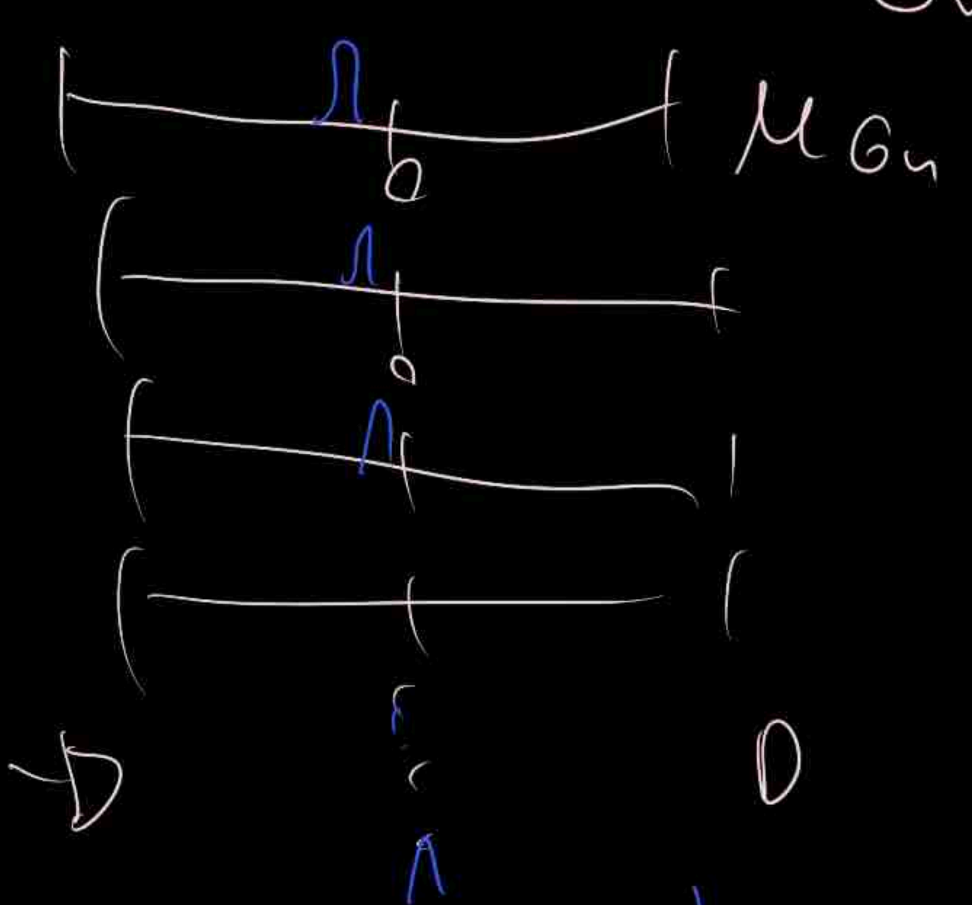
---

WHAT would be the natural localization?

$\mu_G(\{0\})$

$\mu_{(G,o)}(\{0\})$ COOL, but NOT continuous!

$G_n \to (G,o)$



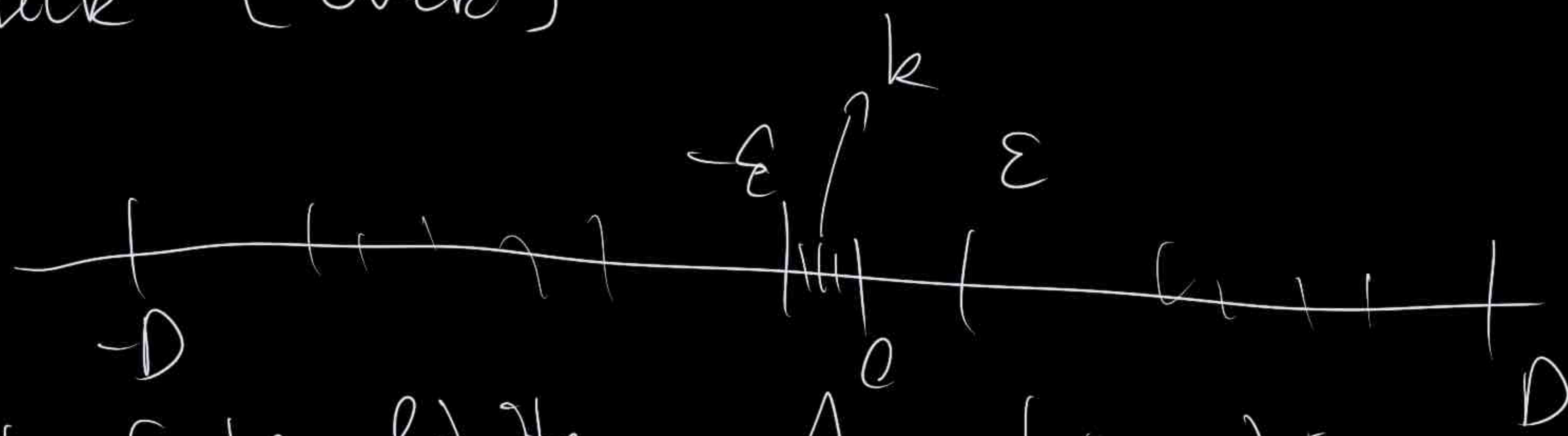

$\mu_G$ weakly conv.

$\delta_{\frac{1}{n}} \to \delta_0$

0 1

positive mass travels into 0

Trick (Luck)




Let $G$ be finite, $A_G$, $b_i$, $\lambda_i$

Let $k = \#\{i \mid |\lambda_i| < \varepsilon, \lambda_i \neq 0\}$

$$1 \le \Big|\prod_{\substack{\lambda_i \neq 0 \\ \text{all} \\ \text{eigenvalues}}} \lambda_i\Big| \le \varepsilon^k \cdot D^{n-k} = \varepsilon^k D^n$$

$$\left(\frac{1}{\varepsilon}\right)^k \le D^n$$

$$k \le n \cdot \frac{\log D}{\log(1/\varepsilon)} \xrightarrow{\varepsilon \to 0} 0$$

$$\mu_G\big((-\varepsilon,\varepsilon) \setminus \{0\}\big) \le$$

So no mass travels to 0 ✓

T(Thom) Same for $\mu_G(\{x\})$ $\forall x$.

This is how you compute Betti numbers!

T (Lück) Let $W$ be a finite complex, $\Gamma = \pi_1(W)$.

Let $\Gamma = \Gamma_0 > \Gamma_1 > \dots > \Gamma_n > \dots$

$\Gamma_n \triangleleft \Gamma$ finite index, $\bigcap \Gamma_n = 1$

Then $\displaystyle \lim \frac{b_k(\widetilde{W}/\Gamma_n)}{|\Gamma : \Gamma_n|} = \beta_k^{(2)}(W).$

---

OPEN!

We know, $G_n$ BS cont., then

$$\frac{\dim \operatorname{Ker}_{\mathbb{Q}}(A_n)}{|G_n|} \text{ converges}$$

WHAT ABOUT $\mathbb{F}_p$?

$$\frac{\dim \operatorname{Ker}_{\mathbb{F}_p}(A_n)}{|G_n|} \quad ???$$

does it converge?

$G_n$ converges $\Rightarrow$ $\mu_{G_n} \xrightarrow{*} \mu$, but what IS $\mu$?

$$\int x^k d\mu = \lim \int x^k d\mu_{G_n} = \mathop{\mathbb{E}}_{o_n \in G_n} w(G_n, k, o_n, o_n)$$

Localize $\mu_{G_n}$!

For a rooted graph $(G, o)$ let

$\mu_{(G,o)} \in P(\mathbb{R})$ such that

$$\int x^k d\mu_{(G,o)} = w(G, k, o, o)$$

WHY does it exist? $\boxed{\text{FUNCT ANAL}}$

OR

Lets do it for finite $G$ first

$$\mu_{(G,o)} = \frac{1}{|G|} \sum_{i=0}^{|G|-1} \underbrace{\langle X_o, b_i \rangle^2}_{= b_i(o)^2} \delta_{\lambda_i}$$

works:

$$\mu_G = \mathop{\mathbb{E}}_{o} \mu_{(G,o)} \qquad \int x^k d\mu_{(G,o)} = w\ldots$$

rooted spectral measure!

we localized $\mu_G$

"Derivative"

$f : \mathcal{G}_0 \to \mathbb{R}$ (*)

$F = \mathbb{E} f(G) = \underset{o \in G}{\mathbb{E}} f((G,o)) = \int_{\mathcal{G}_0} f \, d\nu_G$

If $f$ is continuous on $\mathcal{G}_0$, then $F$ is BS-cont ✓

Localization: $F$ is given $\Rightarrow$ find $f$

$F : \mathrm{Sof} \to \mathbb{R}$ can you localize?

$K$ compact, $\mu_i \in P(K)$

$F : P(K) \to \mathbb{R}$ has a derivative?

R-N derivative ...

discrete Bettis have

cont. Bettis ???

BS convergence: covering towers ✓

T [Lück Approx] w/ finite complex ...

Can localize ?

$\mu_G(\{0\})$ seems OK $\mu_{(G,e)}(\{0\})$

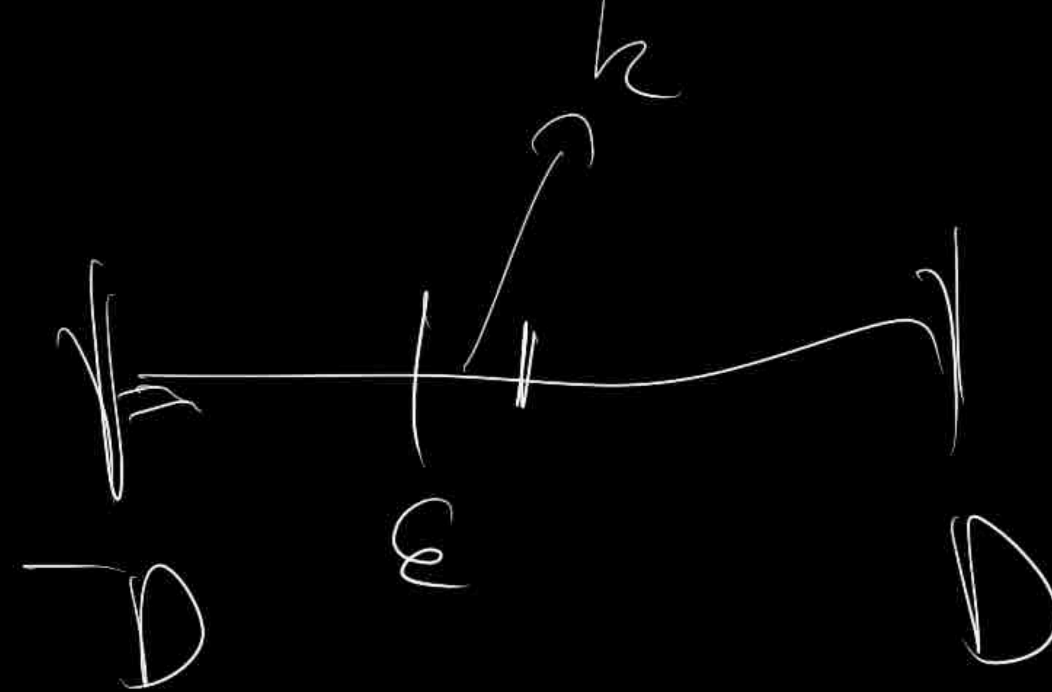
BUT ITS NOT CONT!!!!!

STILL,

T [behind Lück] $G_n$ BS conv $\to$

$\Rightarrow$ $\mu_{G_n}(\{0\})$ conv.

P: $\left|\prod_{\lambda_i \neq 0} \lambda_i\right| \geq 1$



$\wedge\!|$

$\varepsilon^k \cdot D^n$

$D^n \geq \left(\frac{1}{\varepsilon}\right)^k$

$k \leq n \cdot \log_{1/\varepsilon} D$

$\frac{k}{n} \leq \frac{\ln D}{\ln(1/\varepsilon)}$ ✓

$G_n$ BS conv $\Rightarrow$ $\dfrac{\dim(\ker_{\mathbb{Q}}(\mathrm{Adj}\, G_n))}{|G_n|}$ conv.

<u>OPEN: over $\mathbb{F}_p$ ???</u>

$\Gamma$ countable group, $\{0,1\}^\Gamma$ product topology

$H < \Gamma \qquad \chi_H \in \{0,1\}^\Gamma$

$\boxed{\mathrm{Sub}_\Gamma} = \{H < \Gamma\}$ endowed with $\tau$ $\quad$ closed in $\{0,1\}^\Gamma$

$H_n \to H \iff \forall g \in H \quad g \in H_n$ for $n > n_0$

$\qquad\qquad\qquad\quad g \notin H \quad g \notin H_n$ for $n > n_0$

$\mathrm{Sub}_\Gamma$ compact (closed in $\{0,1\}^\Gamma$)

---

$\Gamma = \langle S \rangle \qquad S = S^{-1}$ finite

$\mathrm{Cay}(\Gamma, S)$ Cayley graph

$$x \xrightarrow{s} sx \qquad \forall x \in \Gamma \quad \forall s \in S$$

$H \leq \Gamma \qquad \Gamma/H = \{gH \,/\, g \in \Gamma\}$

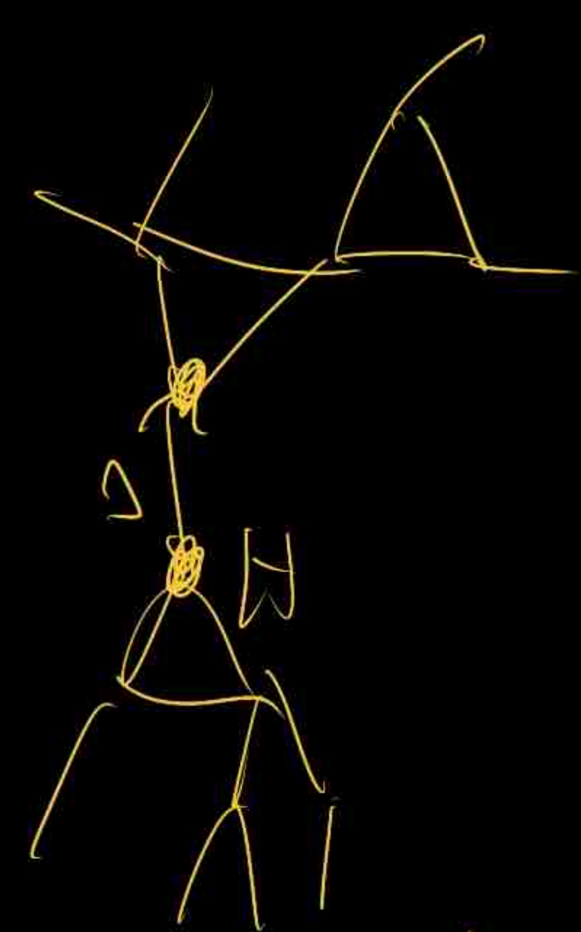
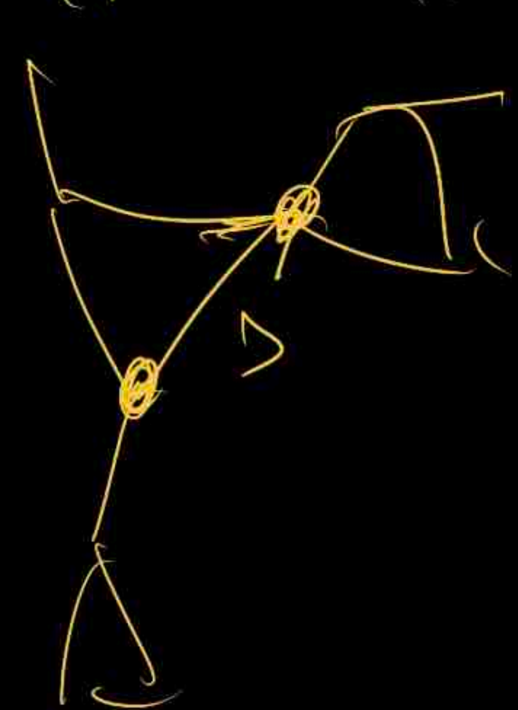
cosets $\quad gH \xrightarrow{s} sgH$

$\mathrm{Sch}(\Gamma, H, S)$ Schreier graph

$Sch(\Gamma/H, S)$ is naturally rooted at $H$

$\mathbb{E}$: $\underset{\underset{\mathrm{Sub}_\Gamma}{\cap}}{H_n} \to H \iff Sch(\Gamma/H_n, S) \underset{\underset{\mathcal{G}_0}{\cap}}{\longrightarrow} Sch(\Gamma/H, S)$

NOTE: we slightly changed the notion of a graph! Its edge labeled!

| Schreier graphs | Subgroups |
| --- | --- |
| $\mathrm{Sch}(\Gamma/H, S)$ $\longleftrightarrow$ | $H < \Gamma$ |
| $\mathcal{G}_{D,S}$ $\longleftrightarrow$ | $\mathrm{Sub}_\Gamma$ |
| $\mathrm{Sch}(\Gamma/H_n, S) \to \mathrm{Sch}(\Gamma/H, S)$ in $\mathcal{G}_{D,S}$ $\longleftrightarrow$ | $H_n \to H$ in $\mathrm{Sub}_\Gamma$ |
| unimodular random rooted Schreier graph $\lambda \in P(\mathcal{G}_{D,S})$ means: | $\mathrm{Sch}(\Gamma/H, S)$ |
| $\lambda$-random $(G, o)$ $\longleftrightarrow$ |  |
| move the root with a fixed [illegible] you get $\lambda$ back | move the root $\Downarrow$ $\mathrm{Sch}(\Gamma/H^g/S)$ |
|  | |

TRANSLATION:

unimodular random Schreier graph of $\Gamma$

$\Updownarrow$

conjugacy invariant random subgroup of $\Gamma$

---

D: A random subgroup $H < \Gamma$ with law $\lambda \in P(\mathrm{Sub}_\Gamma)$ is an Invariant Random Subgroup if $\forall g \in \Gamma$ law of $H^g = \lambda$.

$\Gamma \curvearrowright \mathrm{Sub}_\Gamma$ by conjugation

$\lambda$ is $\Gamma$-invariant measure.

Example: $\delta_N$ $\quad N \lhd \Gamma$ is an IRS

What else? $\quad H < \Gamma \quad |\Gamma : H| < \infty$

random conjugate of $H$ ✓

$|\Gamma : N_\Gamma(H)| < \infty$ $\quad$ 1

IS THERE ANYTHING ELSE???

convex combination take extremal ones (ergodic)

$$\begin{array}{c}\Gamma \times \Delta \\ \vee \\ H \times \Delta\end{array}$$

ANSWER: depends on $\Gamma$!!!

T: For $\Gamma = SL_n(\mathbb{Z})$ $(n \geq 3)$ this is ALL!

(Stück-Zimmer) + 7 Samurai } $\boxed{\text{HARD}}$

Annals $\qquad$ Annals

$\Gamma$ Abelian $\quad$ measure on $\mathrm{Sub}_\Gamma$

We did not invent IRS. New: $\boxed{\text{BS convergence + IRS}}$

name IRS coined in Abert - Glasner - Virag

(Ian Biringer suggested IRS) + Vershik - Bowen -

IRS again

stabilizer = IRS

- zoo of IRS-es? of $F_d$ Bowen ⟿ Gaboriau-Le Maître, Carderi

countable equivalence relations

last paper: IRS

support? $\mathrm{Sub}_\Gamma$ $\Gamma = F_d$

isolated points

Can be that big!

- IRS rigidity
  - Vershik totally nonfree actions

    IRS-es of $S_\infty$ classified

  - Stuck-Zimmer Lie groups!
  - character rigidity Bekka - ... Bader

OK so one can prove things about IRS. But is it a good tool?

2 applications!

$SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ x & y & 1 \end{pmatrix}$$

$\mathbb{Z}^2$

by hand:

$\mathbb{Z}^2$ has no $SL_2(\mathbb{Z})$-invariant random subgroups!

FOR TODAY!

"zoo" of IRS-es in $F_d$

Bowen, Le Maître, Eisenmann–Glasner,
Le Maître,

T [Carderi–Gaboriau–Le Maître]

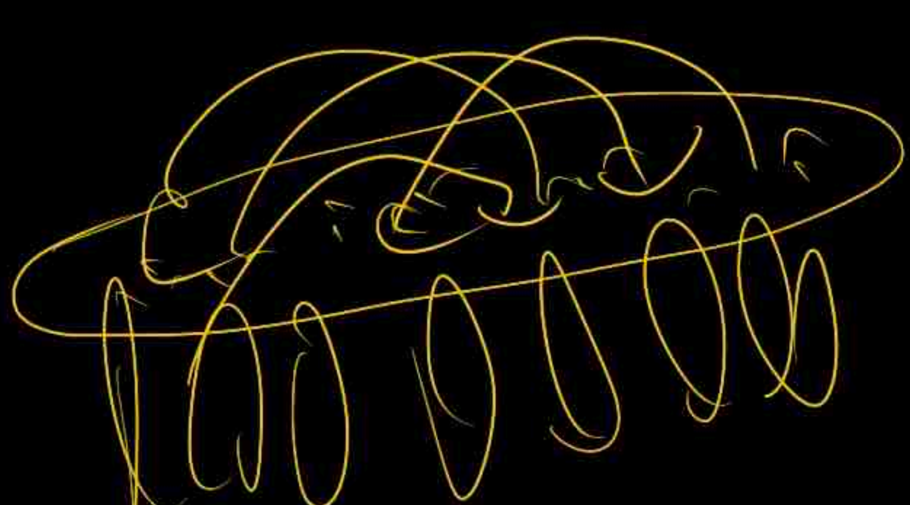
Dense totipotent free subgroups in full groups

Many results, but e.g.

$\Gamma = F_d$ $\mathrm{Sub}_\Gamma$ as a top. space has isolated points (finite index subgroups)

Remove them, no more.

$\exists$ ergodic IRS supported on this!



pmp. countable equivalence relation

# IRS RIGIDITY

"Few IRS-es"

- Vershik totally nonfree actions
  IRS-es of $S_\infty$ classified
- Stück-Zimmer Lie groups!
- character rigidity:
  Starts with Bekka [Inv. Math]

$Sym_f(\mathbb{N})$

$\mathbb{N}$

higher rank lattices

$SL_n(\mathbb{Z})$

T [Bader-Boutonnet-Houdayer-Peterson]
very general character rigidity
for higher rank lattices.

Note: IRS still wins for Lie groups

T [Stuck-Zimmer + Nevo]

If $G$ is semisimple real Lie group with property T, $H \leq G$ is IRS ergodic $H \neq 1, G$

Then $\exists$ lattice $\Gamma < G$

$H$ is a random conjugate of $\Gamma$.

$f : \Gamma \to \mathbb{C}$ is a <u>character</u> if

its conj. invariant, pos. definite

[$K(x, y) = f(x^{-1} y)$ is pos. def]

$H < \Gamma$ $\qquad X_H$ is pos. def

If $H$ is an IRS $\qquad \boxed{f_H(g) = \mathbb{P}(g \in H)}$

$$\mathbb{E}\, X_H$$