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Statistical properties of nonlinear shell models of turbulence from linear advection models: rigorous results

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Abstract
In a recent paper it was proposed that for some *nonlinear* shell models of turbulence one can construct a *linear* advection model for an auxiliary field such that the scaling exponents of all the structure functions of the linear and nonlinear fields coincide. The argument depended on an assumption of continuity of the solutions as a function of a parameter. The aim of this paper is to provide a rigorous proof for the validity of the assumption. In addition we clarify here when the swap of a nonlinear model by a linear one will *not* work.

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1. Introduction

Shell models of turbulence [1, 2] serve a useful purpose in studying the statistical properties of turbulent fields due to their relative ease of simulation. In particular, shell models allowed accurate direct numerical calculation of the scaling exponents of their associated structure functions, including giving convincing evidence for their universality [3–8]. In contrast, simulations of the Navier–Stokes equations that model actual fluid turbulence are very much harder, and in addition one still does not know whether these equations in 3-dimensions are mathematically globally well posed. This problem does not exist in shell models [9], adding to their numerical attractiveness a possibility of proving various properties and results rigorously [9, 10].

Consider for example the Sabra shell model [8] which, like other shell models of turbulence, is a truncated description of the dynamics of Fourier modes, preserving some of the structure and conservation laws of the Navier–Stokes equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_n^2\right) u_n = \mathrm{i}[k_{n+1}u_{n+1}^* u_{n+2} - \delta k_n u_{n-1}^* u_{n+1} + (1-\delta)k_{n-1}u_{n-1}u_{n-2}] + f_n. \tag{1}$$

Here u_n, with $n = 0, 1, 2, \ldots$ and the boundary conditions $u_{-2} = u_{-1} = 0$, are the velocity modes restricted to 'wavevectors' $k_n = k_0\mu^n$ with k_0 determined by the inverse outer scale of turbulence. The model contains one additional parameter, δ, and it conserves two quadratic invariants (when the force and the dissipation terms are absent) for all values of δ. The first is the total energy $\sum_n |u_n|^2$ and the second is $\sum_n (-1)^n k_n^\alpha |u_n|^2$, where $\alpha = \log_\mu(1-\delta)$.

The scaling exponents are properties of the structure functions. To define the structure function we introduce an average over time according to

$$\langle A(t)\rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)\,\mathrm{d}t. \tag{2}$$

In practice, however, we take

$$\langle A(t)\rangle = \frac{1}{T}\int_0^T A(t)\,\mathrm{d}t, \tag{3}$$

for some T large enough, but finite. Our rigorous results also refer to this definition of an average over time. For values of viscosity ν small enough, and for a sufficiently large amplitude of the random force f_n there exists a large range of values k_n where numerical simulations show that structure functions follow a power-law behaviour. The low-order structure functions and the associated scaling exponents are

$$S_n^{(2)}(k_n) \equiv \langle u_n u_n^*\rangle \sim k_n^{-\zeta_2}, \tag{4}$$

$$S_n^{(3)}(k_n) \equiv \Im\langle u_{n-1} u_n u_{n+1}^*\rangle \sim k_n^{-\zeta_3}, \tag{5}$$

etc for higher order $S_n^{(p)}(k_n) \sim k_n^{-\zeta_p}$.

The values of the scaling exponents were determined accurately by direct numerical simulations. Besides ζ_3 which is exactly unity [6], all the other exponents ζ_p appear anomalous, differing from $p/3$. Numerical evidence is that the scaling exponents are also universal, i.e. they are independent of the forcing f_n as long as the latter is restricted to small n [2]. It was shown that for $0 < \delta < 1$ the leading scaling exponents are determined by the cascade of the energy invariant from large to small scales. For $1 < \delta < 2$ the second invariant $\sum_n k_n^\alpha |u_n|^2$ becomes important in determining the leading scaling exponents of the structure functions of u_n. In the bulk of this paper we consider the situation with $0 < \delta < 1$, but return to the other case in section 4, in order to clarify the role of invariants in determining the scaling properties.

In a recent publication further insight into the anomaly of the exponents of the nonlinear problem (for the field u_n) was sought by relating them to the scaling exponents of a *linear* model for a field w_n [15]. The linear model was constructed such that its scaling exponents would be the same as those of the nonlinear problem. The equations for this field are constructed under the following requirements: (i) the structure of the equations is obtained by linearizing the nonlinear problem and retaining only such terms that conserve the energy; (ii) the resulting equation is identical to the Sabra model when $w_n = u_n$; (iii) the energy is the only quadratic invariant for the passive field in the absence of forcing and dissipation. These requirements lead to the following linear model:

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = \frac{\mathrm{i}}{3}\Phi_n(\boldsymbol{u}, \boldsymbol{w}) - \nu k_n^2 w_n + f_n, \tag{6}$$

where the advection term is defined as

$$\Phi_n(\boldsymbol{u}, \boldsymbol{w}) = k_{n+1}[(1+\delta)u_{n+2}w^*_{n+1} + (2-\delta)u^*_{n+1}w_{n+2}]$$
$$+k_n[(1-2\delta)u^*_{n-1}w_{n+1} - (1+\delta)u_{n+1}w^*_{n-1}]$$
$$+k_{n-1}[(2-\delta)u_{n-1}w_{n-2} + (1-2\delta)u_{n-2}w_{n-1}], \tag{7}$$

together with $u_{-1} = u_{-2} = w_{-1} = w_{-2} = 0$. Observe that when $w_n = u_n$ this model reproduces the Sabra model and also that the total energy is conserved because $\sum_n \Im[\Phi_n(u, w)w^*_n] = 0$. The second quadratic invariant is not conserved by the linear model (6). Finally, both models have the same 'phase symmetry' in the sense that the phase transformations $u_n \to u_n \exp(\mathrm{i}\phi_n)$ and $w_n \to w_n \exp(\mathrm{i}\theta_n)$ leave the equations invariant iff $\phi_{n-1} + \phi_n = \phi_{n+1}$, $\theta_{n-1} + \theta_n = \theta_{n+1}$. This identical phase relationship guarantees that the nonvanishing correlation functions of both models have precisely the same forms. Thus, for example, the only second and third correlation functions in both models are those written explicitly in equations (4) and (5).

The advantage of the linear model is that the correlation functions are advanced in time by a linear propagator [11–14]. The linear model possesses 'statistically preserved structures' (SPSs) which are evident in the decaying problem equation (6) with $f_n = 0$. These are *left* eigenfunctions of eigenvalue 1 of the linear propagators for each order (decaying) correlation function [11]. For example, for the second order correlation function denote the propagator $P^{(2)}_{n,n'}(t|t_0)$; this operator propagates any initial condition $\langle w_n w^*_n\rangle(t_0)$ (with average over initial conditions, independent of the realizations of the advecting field u_n) to the decaying correlation function (with average over realizations of the advecting field u_n)

$$\langle w_n w^*_n\rangle(t) = P^{(2)}_{n,n'}(t|t_0)\langle w_{n'} w^*_{n'}\rangle(t_0). \tag{8}$$

The second order SPS, $Z^{(2)}_n$, is the left eigenfunction with eigenvalue 1:

$$Z^{(2)}_{n'} = Z^{(2)}_n P^{(2)}_{n,n'}(t|t_0). \tag{9}$$

Note that $Z^{(2)}_n$ is time independent even though the operator $P^{(2)}_{n,n'}(t|t_0)$ is time dependent. Each order correlation function is associated with another propagator $\boldsymbol{P}^{(p)}(t|t_0)$ and each of those has an SPS, i.e. a *left* eigenfunction $\boldsymbol{Z}^{(p)}$ of eigenvalue 1. These nondecaying eigenfunctions scale with k_n, $\boldsymbol{Z}^{(p)} \sim k_n^{-\xi_p}$, and the values of the exponents ξ_p are anomalous. Finally, it was shown that these SPSs are also the leading scaling contributions to the structure functions of the *forced* problem (6) [11,12]. Thus, *the scaling exponents of the linear problem are independent of the forcing* f_n, since they are determined by the SPS of the decaying problem.

To connect the linear model to the nonlinear problem one considers the system of two coupled equations:

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= \frac{\mathrm{i}}{3}\Phi_n(u,u) + \frac{\mathrm{i}\lambda}{3}\Phi_n(w,u) - \nu k_n^2 u_n + f_n,\\ \frac{\mathrm{d}w_n}{\mathrm{d}t} &= \frac{\mathrm{i}}{3}\Phi_n(u,w) + \frac{\mathrm{i}\lambda}{3}\Phi_n(w,w) - \nu k_n^2 w_n + \tilde{f}_n\end{aligned} \tag{10}$$

with λ being a real parameter and f_n and $\tilde{f}_n$ being different realizations of the same random force. Observe that for any $\lambda \neq 0$, the two equations in (10) exchange roles under the change $\lambda w_n \leftrightarrow u_n$. Thus, if the scaling exponents ξ_p and ζ_p of the two fields exist (i.e. a true scaling range exists), they must be the same for all $\lambda \neq 0$. For $\lambda = 0$ we recover the equations for the nonlinear and the linear models, equations (1) and (6). In [15] it was *assumed* that the scaling exponents of either field exhibit no jump in the limit $\lambda \to 0$. Indeed, [15] also presented

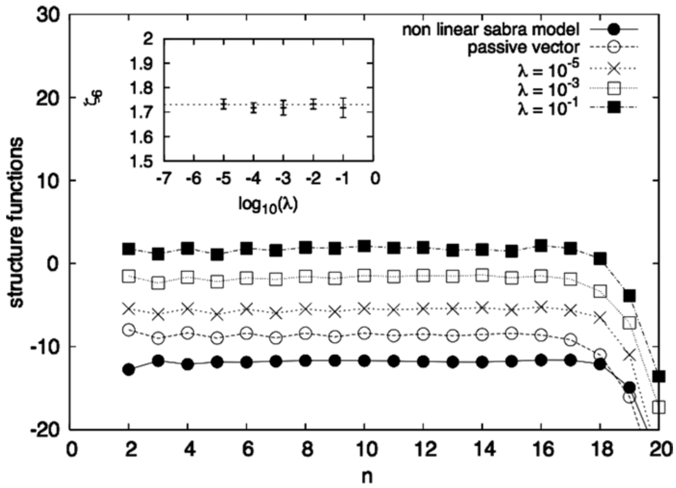

Figure 1. The 'compensated' sixth order structure function $S_n^{(6)}(k_n) \times k_n^{1.74}$ of the field w_n in equations (10) for $\lambda = 10^{-1}$, 10^{-3} and 10^{-5}, together with the 'compensated' sixth order structure function for the Sabra model (1) and for the linear model (6), respectively. The 'compensated' structure functions were all multiplied by $k_n^{\zeta_6}$ with the same value of ζ_6. The structure functions of the field u_n for $\lambda > 0$ are not shown since they are indistinguishable from those of the w_n. Inset: the scaling exponent ζ_6 of $S_n^{(6)}(k_n)$ as a function of λ.

strong evidence for the validity of this assumption (see figure 1), but no mathematical proof was provided.

The aim of this paper is to close this gap. The main result of our paper states that the solutions of the system (10) exist globally in time and depend continuously on the parameter λ, including the limit $\lambda \to 0$. In particular, we will show that *if* the structure functions of u_n and w_n exhibit the same scaling exponents for any $\lambda \neq 0$, they will also have the same scaling exponents in the limit $\lambda \to 0$ (including $\lambda = 0$). We would like to stress here that our rigorous results are valid for the structure functions, calculated over large, but finite, fixed interval of time, which is consistent with definition (3) of the long time average. In addition, we would like to note that the numerical results correspond to the equations with the stochastic implementation of the forcing, while our rigorous proofs deal with a deterministic force that depends on time. The statement and the proof of the main theorem will be given in section 3. The proof is based on the results on the global existence and uniqueness of solutions of equation (1), obtained previously in [9]. In the following section we present the necessary mathematical definitions and formulate the essential statement from [9].

2. Functional setting and previous analytic results

2.1. Functional setting

Following the classical treatment of the Navier–Stokes and Euler equations we re-write the Sabra shell model for an infinite vector $\boldsymbol{u} \equiv (u_0, u_1, \ldots)$

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + \frac{i}{3}\Phi(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \tag{11}$$

together with the initial conditions $\boldsymbol{u}(t=0)$. Introduce a Hilbert space H which is the space of infinite vectors equipped with the scalar product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$ defined as

$$(\boldsymbol{u},\boldsymbol{v})=\sum_{n=0}^{\infty}u_n v_n^*,\qquad |\boldsymbol{u}|=\left(\sum_{n=0}^{\infty}|u_n|^2\right)^{1/2}, \tag{12}$$

for every $\boldsymbol{u},\boldsymbol{v}\in H$. The space H is a space of all the velocity vectors having finite energy.

The linear operator $\boldsymbol{A}$, with a domain $D(\boldsymbol{A})$ dense in H, is a positive, definite diagonal operator defined through its action on the elements $\boldsymbol{u}$ by

$$\boldsymbol{A}\boldsymbol{u}=(k_0^2u_0,k_1^2u_1,\ldots), \tag{13}$$

where the eigenvalues k_n^2 satisfy $k_n=k_0\mu^n$. Using the fact that $\boldsymbol{A}$ is a positive definite operator, we can define the powers $\boldsymbol{A}^s$ of $\boldsymbol{A}$ for every $s\in\mathcal{R}$

$$\forall\boldsymbol{u}=(u_0,u_1,u_2,\ldots),\qquad \boldsymbol{A}^s u=(k_0^{2s}u_0,k_1^{2s}u_1,k_2^{2s}u_2,\ldots). \tag{14}$$

The space $D(\boldsymbol{A}^{s/2})$ is the domain of the operator $\boldsymbol{A}^{s/2}$ and we denote

$$V_s\equiv D(\boldsymbol{A}^{s/2})=\left\{\boldsymbol{u}=(u_0,u_1,u_2,\ldots),\ \sum_{j=0}^{\infty}k_j^{2s}|u_j|^2<\infty\right\}, \tag{15}$$

which are Hilbert spaces equipped with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_s=(\boldsymbol{A}^{s/2}\boldsymbol{u},\boldsymbol{A}^{s/2}\boldsymbol{v}),\qquad \forall\boldsymbol{u},\boldsymbol{v}\in D(\boldsymbol{A}^{s/2}), \tag{16}$$

and the norm $|\boldsymbol{u}|_s^2=(\boldsymbol{u},\boldsymbol{u})_s$, for every $\boldsymbol{u}\in D(\boldsymbol{A}^{s/2})$. Since V_s contains velocity vectors with s 'derivatives',

$$V_s\subseteq V_0=H\subseteq V_{-s},\qquad \forall s>0. \tag{17}$$

The case of $s=1$ is of a special interest for us. We denote $V=D(\boldsymbol{A}^{1/2})$ a Hilbert space equipped with a scalar product and a corresponding norm

$$((\boldsymbol{u},\boldsymbol{v}))=(\boldsymbol{A}^{1/2}\boldsymbol{u},\boldsymbol{A}^{1/2}\boldsymbol{v}),\qquad \|\boldsymbol{u}\|^2=((\boldsymbol{u},\boldsymbol{u})), \tag{18}$$

for every $\boldsymbol{u},\boldsymbol{v}\in V$. In addition the following relation holds:

$$k_1|\boldsymbol{u}|\leqslant\|\boldsymbol{u}\|. \tag{19}$$

In what follows we will need the interpolation inequality for the spaces V_s.

Lemma 1. Let $s>0$, then for all $\boldsymbol{u}\in V_s$ and $0<s'<s$

$$|\boldsymbol{u}|_{s'}\leqslant|\boldsymbol{u}|^{1-s'/s}|\boldsymbol{u}|_s^{s'/s}. \tag{20}$$

Proof. The lemma follows by a simple application of the Hölder inequality. ■

The bilinear operator $\frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{w})$ is defined above, cf equation (7), together with $\boldsymbol{\Phi}\equiv(\Phi_0,\Phi_1,\ldots)$. In [9] it was shown that such a definition of the bilinear operator makes $\boldsymbol{\Phi}$ an element of H whenever $\boldsymbol{u}\in H$ and $\boldsymbol{w}\in V$. For any $\boldsymbol{u},\boldsymbol{v}\in H$ and $\boldsymbol{w}\in V$ one proves [9]

$$|(\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})|\leqslant C|\boldsymbol{u}|\,|\boldsymbol{v}|\|\boldsymbol{w}\|, \tag{21}$$

for some positive constant C. In addition the conservation law is written in the present notation as

$$\Im(\boldsymbol{\Phi}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{w})=0. \tag{22}$$

All our rigorous results are valid for the deterministic forcing $\boldsymbol{f}$. Therefore, in order to account for the stochastic implementation of the forcing term we will assume that $\boldsymbol{f}$ depends on time, but always stays bounded in the H-norm. More precisely, define the space $L^\infty([0,T),H)$, for some $0<T\leqslant\infty$, as a space of functions of the time interval $[0,T)$ with values in H and the norm defined as

$$\|\boldsymbol{f}\|_\infty=\sup_{0\leqslant t<T}|\boldsymbol{f}(t)|. \tag{23}$$

In what follows, we will assume that the forcing term $\boldsymbol{f}$ satisfies $\boldsymbol{f}\in L^\infty([0,\infty),H)$.

2.2. Summary of previous results

In [9] equation (1) was studied, and the relevant results can be formulated as the following theorem.

Theorem CLT 06. The solution of equation (1) exists globally in time for any initial condition $\boldsymbol{u}(0)$ in H and for any $\boldsymbol{f}$ in $L^\infty([0,\infty), H)$. Moreover, the solutions are unique and the energy of the solution $\boldsymbol{u}(t)$ is bounded for all times:

$$\sup_{0\leqslant t\leqslant T} |\boldsymbol{u}(t)|^2 \leqslant K_0(k_0, \mu, \nu, \delta, \boldsymbol{u}(0), \boldsymbol{f}), \tag{24}$$

where the *a priori* constant K_0 depends on all the parameters of the equations, on the forcing and on the initial conditions.

If in addition we assume that the forcing $\boldsymbol{f} = (f_0, f_1, \ldots)$ acts on the finite number of shells, namely, if there exists $N \geqslant 0$, such that $f_n = 0$, for all $n \geqslant N$, then the solution $\boldsymbol{u} = (u_0, u_1, \ldots)$ has an exponentially decaying spectrum $|u_n|$ as a function of k_n and in particular for any $0 < t_0 \leqslant T$

$$\sup_{t_0\leqslant t\leqslant T} |\boldsymbol{u}(t)|_s^2 \leqslant K_s(k_0, \mu, \nu, \delta, \boldsymbol{u}(0), \boldsymbol{f}, t_0), \tag{25}$$

for any $s > 0$ and the *a priori* constants K_s depend on all the parameters of the equations, on the forcing and on the H-norm of the initial conditions (see definition (23)).

3. The main result

The main statement of this paper is that the system of equations (10) is globally well posed for all real λ and that the solutions depend continuously on the parameter λ. In particular, as $\lambda \to 0$ the solution of the system converges uniformly on any finite interval of time to the solution of the system with $\lambda = 0$. This statement is formulated as follows.

Theorem 1. Let $\boldsymbol{u}(0;\lambda), \boldsymbol{w}(0;\lambda)$ be given in H and the forces $\boldsymbol{f}, \tilde{\boldsymbol{f}}$ in $L^\infty([0,T], H)$. Denote by $\boldsymbol{u}(t;\lambda), \boldsymbol{w}(t;\lambda)$ the solutions of the coupled system (10). Then the following hold.

1. The solution of the system (10) exists globally in time for every λ. Moreover, energy of the solutions $|\boldsymbol{u}(t;\lambda)|^2$ and $|\boldsymbol{w}(t;\lambda)|^2$ is bounded for all times, where the bounds depend on all the parameters of the equations, λ, the forcing and the initial conditions.
2. For any $T > 0$,

$$\lim_{\lambda\to 0} \sup_{0\leqslant t\leqslant T} |\boldsymbol{u}(t;\lambda) - \boldsymbol{u}(t;0)|^2 = 0$$

and

$$\lim_{\lambda\to 0} \sup_{0\leqslant t\leqslant T} |\boldsymbol{w}(t;\lambda) - \boldsymbol{w}(t;0)|^2 = 0,$$

where $\boldsymbol{u}(t;0), \boldsymbol{w}(t;0)$ are the solutions of (10) with $\lambda = 0$.

3. If in addition we assume that the forces $f, \tilde{f}$ act only on the finite number of shells then for any $0 < t_0 \leqslant T$ and $s > 0$,

$$\lim_{\lambda\to 0} \sup_{t_0\leqslant t\leqslant T} |\boldsymbol{u}(t;\lambda) - \boldsymbol{u}(t;0)|_s^2 = 0$$

and

$$\lim_{\lambda\to 0} \sup_{t_0\leqslant t\leqslant T} |\boldsymbol{w}(t;\lambda) - \boldsymbol{w}(t;0)|_s^2 = 0,$$

where $\boldsymbol{u}(t;0), \boldsymbol{w}(t;0)$ are the solutions of (10) with $\lambda = 0$.

Proof. Part 1 of this theorem follows from defining a new variable $\boldsymbol{q}(t;\lambda) \equiv \boldsymbol{u}(t;\lambda)+\lambda\boldsymbol{w}(t;\lambda)$. This variable satisfies equation (1) with a forcing $\boldsymbol{f}+\lambda\tilde{\boldsymbol{f}}$. Accordingly theorem CLT06 provides the proof that $\boldsymbol{q}(t;\lambda)$ exists globally in time for every λ. Next observe that the system of equations (10) can be rewritten in the form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{\mathrm{i}}{3}\Phi_n(\boldsymbol{q},\boldsymbol{u})) - \nu k_n^2 u_n + f_n,$$
$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = \frac{\mathrm{i}}{3}\Phi_n(\boldsymbol{q},\boldsymbol{w}) - \nu k_n^2 w_n + \tilde{f}_n. \tag{26}$$

This form of writing shows that the the fields $\boldsymbol{u}$ and $\boldsymbol{w}$ satisfy *linear* diffusion advection equations advected by a smooth field $\boldsymbol{q}$. Accordingly both fields remain smooth for all time and all λ. In addition, using relation (24) one is able to derive the bounds for the energies of the solutions.

To prove part 2 of the theorem we need first proposition 1.

Proposition 1. Denote by $\boldsymbol{q}(t;0)$ the solution of the Sabra model (1) with initial data $\boldsymbol{q}(0;0)$ in H. This is also the solution of the first of equations (10) when $\lambda = 0$. Then

$$\lim_{\lambda\to 0}\sup_{0\leqslant t\leqslant T}|\boldsymbol{q}(t;\lambda)-\boldsymbol{q}(t;0)| = 0.$$

Moreover, if $\boldsymbol{f}$ and $\tilde{\boldsymbol{f}}$ act only on finite number of shells, then for any $0 < t_0 \leqslant T$ and $s > 0$ we have

$$\lim_{\lambda\to 0}\sup_{t_0\leqslant t\leqslant T}|\boldsymbol{q}(t;\lambda)-\boldsymbol{q}(t;0)|_s = 0.$$

Proof. Let us denote $\Delta\boldsymbol{q} = \boldsymbol{q}(t;\lambda)-\boldsymbol{q}(t;0)$. Then $\Delta\boldsymbol{q}$ satisfies the equation

$$\frac{\mathrm{d}\Delta\boldsymbol{q}}{\mathrm{d}t} + \nu\boldsymbol{A}\Delta\boldsymbol{q} - \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;\lambda),\Delta\boldsymbol{q}) - \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\Delta\boldsymbol{q},\boldsymbol{q}(t;\lambda)) + \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\Delta\boldsymbol{q},\Delta\boldsymbol{q}) = \lambda\tilde{\boldsymbol{f}}, \tag{27}$$
$$\Delta\boldsymbol{q}(0) = \lambda\boldsymbol{w}(0). \tag{28}$$

Take now the inner product in H of the above equation with $\Delta\boldsymbol{q}$. Computing the real part and using equation (22) we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\boldsymbol{q}|^2 + \nu\|\Delta\boldsymbol{q}\|^2 - \Re\left(\frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\Delta\boldsymbol{q},\boldsymbol{q}(t;\lambda),\Delta\boldsymbol{q}\right) = \Re(\lambda\tilde{\boldsymbol{f}},\Delta\boldsymbol{q}). \tag{29}$$

Using the Cauchy–Schwarz inequality and relation (21) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\boldsymbol{q}|^2 + \nu\|\Delta\boldsymbol{q}\|^2 \leqslant \left|\left(\frac{1}{3}\boldsymbol{\Phi}(\Delta\boldsymbol{q},\boldsymbol{q}(t;\lambda),\Delta\boldsymbol{q}\right)\right| + |(\lambda\tilde{\boldsymbol{f}},\Delta\boldsymbol{q})|$$
$$\leqslant C|\Delta\boldsymbol{q}|\|\Delta\boldsymbol{q}\||\boldsymbol{q}(t;\lambda)| + \lambda|\tilde{\boldsymbol{f}}||\Delta\boldsymbol{q}|. \tag{30}$$

Applying Young's inequality ($ab \leqslant a^2/2 + b^2/2$) twice and using inequality (19), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\boldsymbol{q}|^2 + \nu\|\Delta\boldsymbol{q}\|^2 \leqslant \frac{C^2}{\nu}|\Delta\boldsymbol{q}|^2|\boldsymbol{q}(t;\lambda)|^2 + \frac{\lambda^2}{\nu k_1^2}|\tilde{\boldsymbol{f}}|^2 + \frac{\nu}{2}\|\Delta\boldsymbol{q}\|^2.$$

Using the fact that relation (24) holds for any $t \geqslant 0$ and the fact that $\boldsymbol{f} \in L^\infty([0,\infty),H)$ (see definition (23)), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\boldsymbol{q}|^2 \leqslant \frac{2K_0^2C^2}{\nu}|\Delta\boldsymbol{q}|^2 + \frac{2\lambda^2}{\nu k_1^2}\|\tilde{\boldsymbol{f}}\|_\infty^2.$$

By Gronwall's inequality we conclude that

$$|\Delta \boldsymbol{q}(t)|^2 \leqslant \mathrm{e}^{\frac{2K_0^2C^2}{\nu}t}|\Delta \boldsymbol{q}(0)|^2 + \frac{\lambda^2\|\tilde{\boldsymbol{f}}\|_\infty^2}{K_0^2C^2k_1^2}(\mathrm{e}^{\frac{2K_0^2C^2}{\nu}t} - 1)$$

$$= \mathrm{e}^{\frac{2K_0^2C^2}{\nu}t}\lambda^2|\boldsymbol{w}(0)|^2 + \frac{\lambda^2\|\tilde{\boldsymbol{f}}\|_\infty^2}{K_0^2C^2k_1^2}(\mathrm{e}^{\frac{2K_0^2C^2}{\nu}t} - 1). \tag{31}$$

Therefore, for any $T > 0$,

$$\lim_{\lambda\to 0}\sup_{0\leqslant t\leqslant T}|\Delta \boldsymbol{q}(t)|^2 = 0,$$

and the first statement of the proposition follows.

The second statement of the proposition follows from the boundedness of $\boldsymbol{q}(t;\lambda)$ and $\boldsymbol{q}(t;0)$ in higher order norms after some fixed short transient period of time $t_0 > 0$ (provided that the forces $\boldsymbol{f}$ and $\tilde{\boldsymbol{f}}$ act on the finite number of shells as required by theorem CLT06) and interpolation inequality (20). ■

Finally, we are ready to finish the proof of the main theorem. Let us fix $t_0 > 0$, $s \geqslant 0$, and show that

$$\lim_{\lambda\to 0}\sup_{t_0\leqslant t\leqslant T}|\boldsymbol{w}(t;\lambda) - \boldsymbol{w}(t;0)|_s^2 = 0. \tag{32}$$

Denote $\Delta \boldsymbol{w} = \boldsymbol{w}(t;\lambda) - \boldsymbol{w}(t;0)$. The function $\Delta \boldsymbol{w}$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta \boldsymbol{w} + \nu \boldsymbol{A}\Delta \boldsymbol{w} - \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;\lambda), \boldsymbol{w}(t;\lambda)) + \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;0), \boldsymbol{w}(t;0)) = 0. \tag{33}$$

We rewrite it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta \boldsymbol{w} + \nu \boldsymbol{A}\Delta \boldsymbol{w} - \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0), \boldsymbol{w}(t;0)) - \frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;\lambda), \Delta \boldsymbol{w}) = 0, \tag{34}$$

and as before, taking the inner product in H with $\Delta \boldsymbol{w}$, computing the real part and using equation (22), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta \boldsymbol{w}|^2 + \nu\|\Delta \boldsymbol{w}\|^2 - \Re\left(\frac{\mathrm{i}}{3}\boldsymbol{\Phi}(\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0), \boldsymbol{w}(t;0))\right), \Delta \boldsymbol{w}) = 0.$$

Subsequently applying Young's inequality and inequality (21), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta \boldsymbol{w}|^2 + \nu\|\Delta \boldsymbol{w}\|^2 \leqslant \left|\left(\frac{\mathrm{i}}{3}\boldsymbol{\Phi}\left(\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0), \boldsymbol{w}(t;0)\right), \Delta \boldsymbol{w}\right)\right|$$

$$\leqslant C\|\Delta \boldsymbol{w}\||\boldsymbol{w}(t;0)||\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0)|$$

$$\leqslant \frac{C^2}{2\nu}|\boldsymbol{w}(t;0)|^2|\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0)|^2 + \frac{\nu}{2}\|\Delta \boldsymbol{w}\|^2. \tag{35}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta \boldsymbol{w}|^2 \leqslant \frac{C^2}{\nu}|\boldsymbol{w}(t;0)|^2|\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0)|^2,$$

and integrating over $(0, t)$ we conclude that

$$|\Delta \boldsymbol{w}(t)|^2 \leqslant \frac{C^2M}{\nu}t\sup_{0\leqslant t\leqslant T}|\boldsymbol{q}(t;\lambda) - \boldsymbol{q}(t;0)|^2,$$

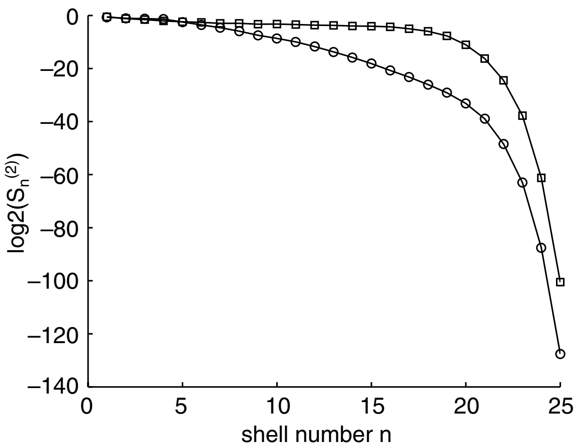

Figure 2. Double-logarithmic plot of the second order structure functions $\langle u_n u_n^* \rangle$ and $\langle w_n w_n^* \rangle$ of the system of equations (10) for $\delta = 1.25$ and $\lambda = 0$.

where we used the fact that $\Delta \boldsymbol{w}(0) = 0$ and $|\boldsymbol{w}(t; 0)|^2 \leqslant M$, for some constant M, as was stated in part 1 of the main theorem. It follows from proposition 1 that $\sup_{0 \leqslant t \leqslant T} |\boldsymbol{q}(t; \lambda) - q(t; 0)|^2 \to 0$, as $\lambda \to 0$. Hence we may conclude that

$$\lim_{\lambda \to 0} \sup_{0 \leqslant t \leqslant T} |\Delta \boldsymbol{w}(t)|^2 = 0.$$

By virtue of interpolation inequality (20) one can follow steps as in the proof of proposition 1 to show that the higher order norms of $\Delta \boldsymbol{w}$ also converge to 0 uniformly in the time interval $t_0 \leqslant t \leqslant T$, as $\lambda \to 0$. To finish the proof, one can observe that

$$|\boldsymbol{u}(t; \lambda) - \boldsymbol{u}(t; 0)| = |\boldsymbol{q}(t; \lambda) - \boldsymbol{q}(t; 0) - \lambda \boldsymbol{w}(t; \lambda)| \leqslant |\boldsymbol{q}(t; \lambda) - \boldsymbol{q}(t; 0)| + \lambda |\boldsymbol{w}(t; \lambda)|, \quad (36)$$

and from the fact that $|\boldsymbol{w}(t; \lambda)|$ is globally bounded in time it follows from proposition 1 that

$$\lim_{\lambda \to 0} \sup_{0 \leqslant t \leqslant T} |\boldsymbol{u}(t; \lambda) - \boldsymbol{u}(t; 0)| = 0. \qquad \blacksquare$$

3.1. Consequences for structure functions

Using the form of equations (26) we conclude that for f_n and $\tilde{f}_n$ being different realizations of the same random force, whenever the structure functions of u_n and w_n exhibit the same scaling exponents for all finite λ, they must also exhibit the same scaling exponents for $\lambda = 0$.

4. When can the nonlinear model exhibit scaling exponents that are different from the linear model?

In this section we turn our attention to situations when the nonlinear and the linear models cannot exhibit the same scaling exponents. The theorem as stated and proven still holds, but as we shall see this is a situation for which the two fields u_n and w_n cannot exhibit the same scaling exponents for all $\lambda \neq 0$. A case in point is the same nonlinear Sabra model with $1 < \delta < 2$. In figure 2 we show the second order structure functions $\langle u_n u_n^* \rangle$ and $\langle w_n w_n^* \rangle$ obtained from simulating equations (10) for $\lambda = 0$ and $\delta = 1.25$. As expected, the scaling of w_n is influenced by the cascade of energy, whereas that of u_n by the cascade of the second invariant. As a result the scaling exponents are distinctly different. The same system of equations for $\lambda = 1$ is

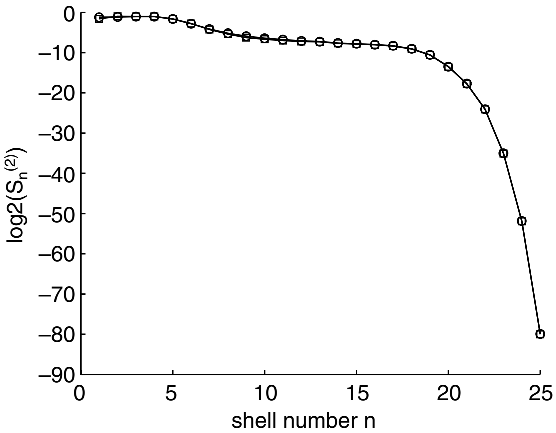

Figure 3. Double-logarithmic plot of the second order structure functions $\langle u_n u_n^* \rangle$ and $\langle w_n w_n^* \rangle$ of the system of equations (10) for $\delta = 1.25$ and $\lambda = 1$.

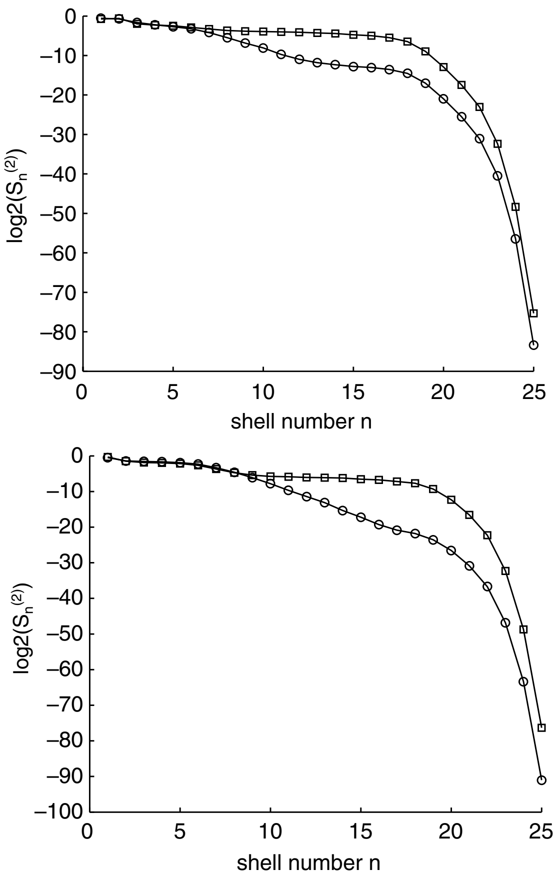

Figure 4. Double-logarithmic plot of the second order structure functions $\langle u_n u_n^* \rangle$ and $\langle w_n w_n^* \rangle$ of the system of equations (10) for $\delta = 1.25$ and $\lambda = 0.1$ (upper panel) and $\lambda = 0.01$ (lower panel).

symmetric in w_n and u_n. Indeed, in figure 3 we show the result of simulations for $\lambda = 1$, for which the second order structure function of the two fields is identical. Now however we cannot expect that this identity will persist for $\lambda \to 0$. In figure 4 we show the results of simulations for the same system of equations for $\lambda = 0.1$ and $\lambda = 0.01$.

To understand the results of the simulations we note that when $\lambda \neq 0$ the second invariant of the equation for u_n is destroyed, and one could think that the scaling of u_n should be dominated by the energy invariant. This is certainly true for $\lambda = 1$. But now when λ decreases towards zero, we should reconsider the system of equations (10) by renaming $\tilde{w}_n = \lambda w_n$. Substituting these re-named variables into equations (10) and re-arranging, we read as

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= \frac{\mathrm{i}}{3}\Phi_n(u,u) + \frac{\mathrm{i}}{3}\Phi_n(\tilde{w},u) - \nu k_n^2 u_n + f_n,\\ \frac{\mathrm{d}\tilde{w}_n}{\mathrm{d}t} &= \frac{\mathrm{i}}{3}\Phi_n(u,\tilde{w}) + \frac{\mathrm{i}}{3}\Phi_n(\tilde{w},\tilde{w}) - \nu k_n^2 \tilde{w}_n + \lambda \tilde{f}_n.\end{aligned} \tag{37}$$

Thus, the net result of the transformation is that the equations for u_n and $\tilde{w}_n$ are the same, but the forcing of w_n becomes weaker and weaker as $\lambda \to 0$. Accordingly, while the second invariant is still destroyed as a true invariant for any value of λ, for small λ the strength of the term $\frac{\mathrm{i}}{3}\Phi_n(\tilde{w},u)$ diminishes, allowing a cross-over behaviour in the scaling of u_n, precisely as we see in figure 4.

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