BENJAMINI-SCHRAMM TOPOLOGY

WHEN ARE TWO GRAPHS SIMILAR (CLOSE)?

Assume we store a huge graph on a computer

Vertices : $\underline{1 \dots N}$

we can ask the neighbors of k. THATS ALL!

We want to test $G_1 \cong G_2$.

Basically, all we can do is this: pick $R > 0$, pick $x_1, \dots, x_n \in \{1..N\}$ at random, explore the balls $B_R(x_i)$, make a statistics and compare!

Lets just use this statistic to start with.

G graph, $R>0$, (H,o) rooted graph

Let

$$P(G,R,(H,o)) = \mathbb{P}_{v\in V(G)}\left(B_R(v) \cong (H,o)\right)$$

D: (G_n) Benjamini-Schramm converges, if $\forall R, (H,o)$

$P(G_n, R, (H,o))$ converges.

Examples:

P_n path of length n

C_n circle

P_n, C_n also!

$C_n \times C_n$ torus

What is the limit?

[unfair question!]

$\lim P_n =$

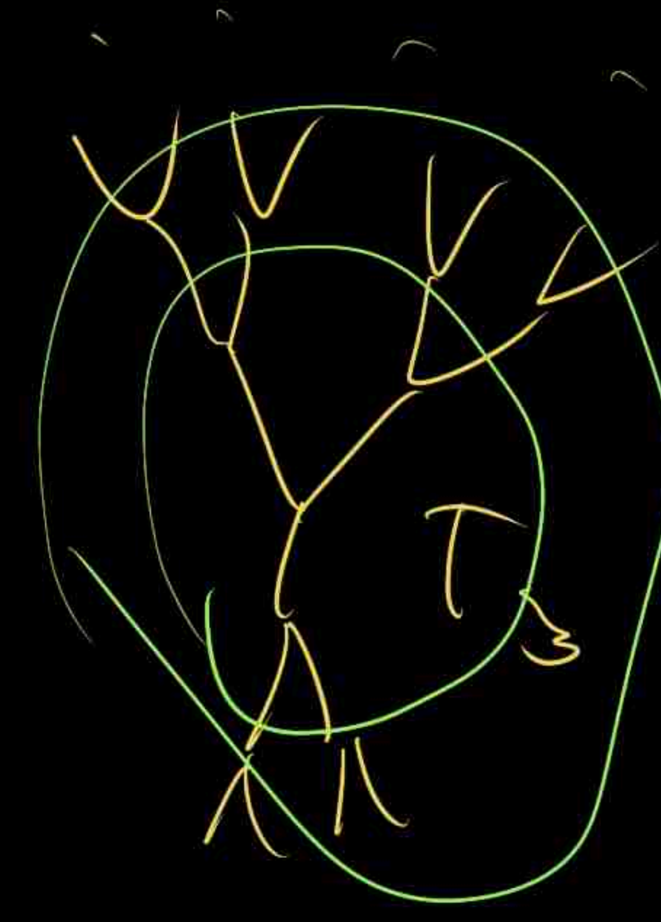
$\lim C_n =$

$\lim C_n \times C_n =$

$\mathbb{Z}^2 / n\mathbb{Z} \times n\mathbb{Z} \to \mathbb{Z}^2$

Q:

Let T_3 3-regular tree



What is

$G_n \to T_3$

$G_n = B_n(T_3)$

THIS DOES CONVERGE BUT <u>NOT</u> TO T_3!

E

NOW A BIT MORE ABSTRACTLY:

Fix some degree bound $\underline{D}$.

D: $\mathcal{G}_D$: space of rooted, connected graphs with degree bound D.

Metric on $\mathcal{G}_D$: $(G_1, o_1), (G_2, o_2) \in \mathcal{G}_D$

$$d((G_1, o_1), (G_2, o_2)) = \frac{1}{2^k}$$

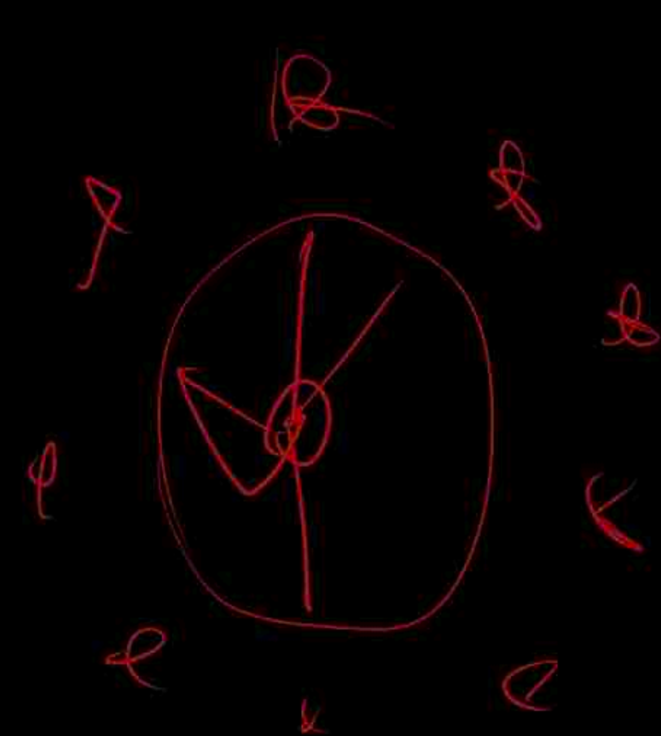
$$k = \max\{ r \,/\, B_r(G_1, o_1) \cong B_r(G_2, o_2) \}$$

E: This is a metric. $\mathcal{G}_D$ is compact and totally disconnected.

For $R > 0$, (H, v) rooted graph let

$$E(R, (H,v)) = \{ (G, o) \in \mathcal{G}_D \,/\, B_R(G, o) \cong (H, v) \}$$

E is open and closed.



RANDOM ROOTED GRAPHS

$x_n \to x \in K$

$\delta_{x_n} \xrightarrow{*} \delta_x$

$\int f \, d\delta_x = f(x)$

$f(x_n) \to f(x)$

K compact Borel space

$P(K) =$ space of Borel prob. measures on K

weak* convergence: $\mu_n \xrightarrow{*} \mu$ if $\forall f : K \to \mathbb{R}$ cont. map

$$\int f \, d\mu_n \to \int f \, d\mu$$

T: $P(K)$ is compact.

Quantum Button: Let G be a finite graph with max degree $\le D$. Choose $o \in V(G)$ uniformly at random.

Let $\lambda(G) = (G, o) \in P(\mathcal{G}_D)$

So we turn a deterministic graph into a prob measure on $\mathcal{G}_D$ (a random rooted graph)

D: G_n BS-converges, if $\lambda(G_n)$ weakly converges.

E cylindric set clopen $\Rightarrow$ X_E is continuous

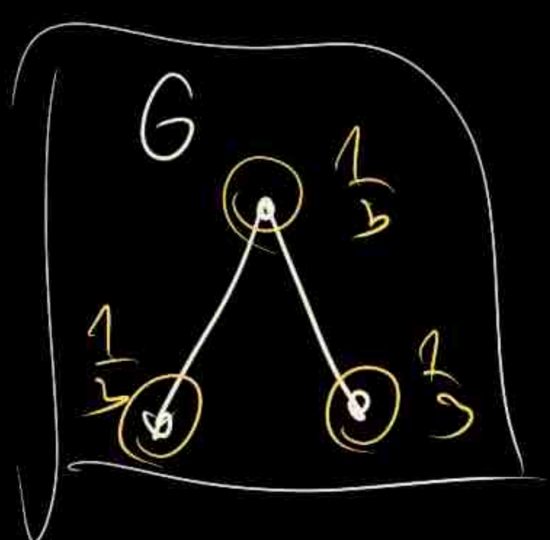
$$\mu_n(E) = \int X_E \, d\mu_n \to \int X_E \, d\mu = \mu(E)$$

These form a base.

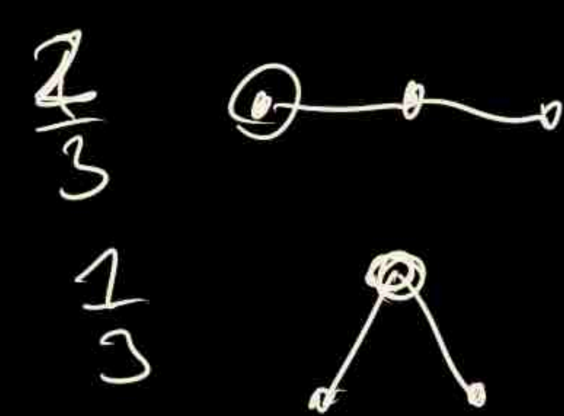
why do this abstract thing???

$$D: \lim G_n = \lim^{*} \lambda(G_n)$$

SO THE BS LIMIT OF GRAPHS IS A RANDOM ROOTED GRAPH!



$\lambda(G)$



$$G \sim G \overset{*}{\cup} G$$

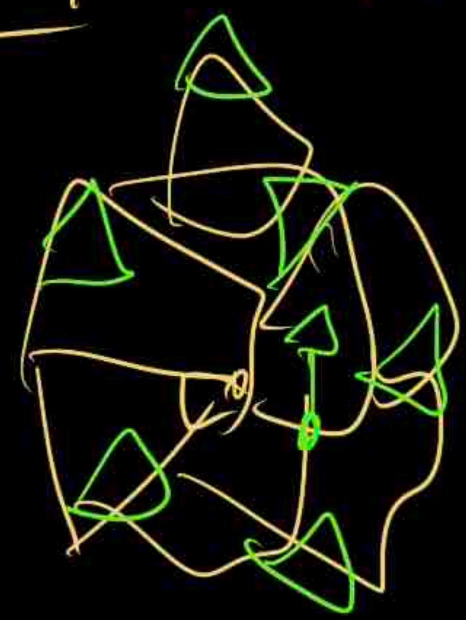
$$\lambda(G) = \lambda(G \overset{*}{\cup} G)$$

$$G_1, \quad G_2, \quad \dots \quad G_n, \quad \dots$$

$$x_1 \qquad \qquad x_n$$

WHAT CAN BE THE LIMIT?

NOT <u>ANY</u> MEASURE!



assume that 20% is on a triangle

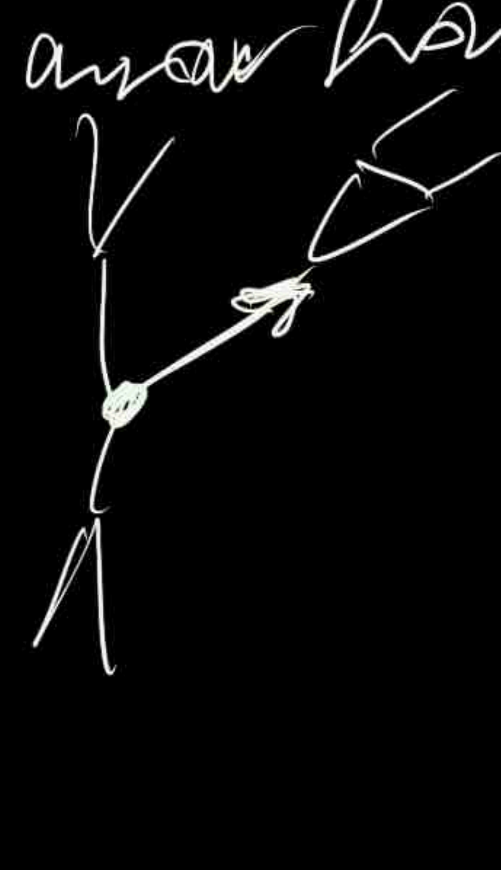
UNIMODULARITY: a property of random rooted graphs that every BS limit shares.

BASICALLY: If G is finite, pick a uniform random directed edge from G and reverse it $\Rightarrow$ stays uniform random!

Let (H, o) be a random rooted graph

$\lambda \in P(\mathcal{G}_D)$

$\vec{\mathcal{G}}_D$: rooted graphs with an arrow from the root!



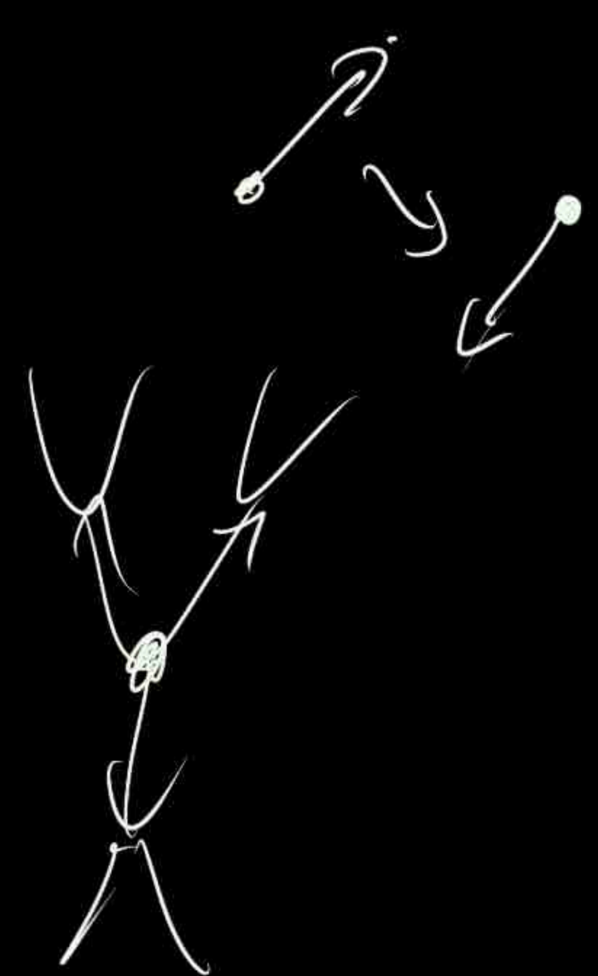
I can lift λ to a measure on $\vec{\mathcal{G}}_0$

I pick each neighbour with weight 1

$\vec{\lambda}$ is a measure on $\vec{\mathcal{G}}_0$

NOT a prob. measure.

λ is unimodular if reversing the arrow on $\vec{\lambda}$ gives back $\vec{\lambda}$.



You would think (maybe)

that if G is vertex transitive

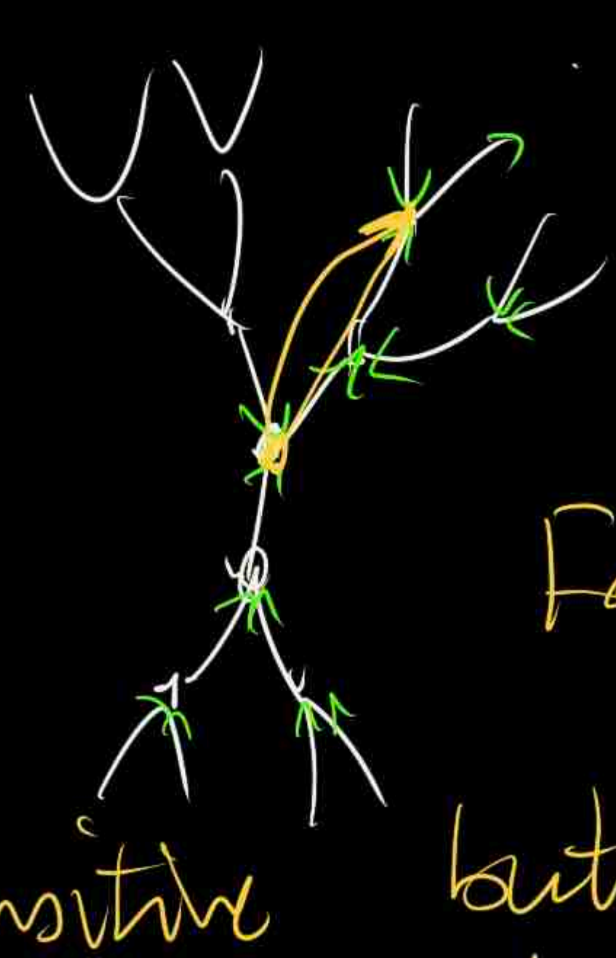
then δ_G is unimodular

NOT TRUE!

GRANDMOTHER GRAPH

T_3

ROOT



Connect x to its grandmother

Forget directions!

G is vertex transitive but δ_G is NOT unimodular!

If δ_G is unimodular $\Rightarrow$ G is vertex transitive E
G connected

percolation

edge percolation prob p

$\mathbb{Z}^2/n\mathbb{Z}\times n\mathbb{Z}$

I

G_n BS converges lim = percolated $\mathbb{Z}^2$

OPEN: IS EVERY UNIMOD. RANDOM ROOTED GRAPH A LIMIT OF FINITE GRAPHS?

IMMEDIATELY IMPLIES: EVERY F.G. GROUP IS SOFIC.

MASS TRANSPORT !!!

GENERAL QUESTION:

WHAT PARAMETERS ARE BS CONTINUOUS

$J(G)$ If $G_n \xrightarrow{BS}$ then $J(G_n)$ converges

EXAMPLES: – average degree
– degree distribution

NON-EXAMPLE: – Cheeger constant

G_n expanders

$G_n \cup G_n$

Spectrum is NOT continuous!

BUT the eigenvalue distribution IS!

G finite graph $A = \mathrm{Adj}(G)$ $\quad |G| = n$

b_i ONB for A eigenvalue λ_i

$$\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$$

Let $\mu_G = \frac{1}{|G|} \sum_i \delta_{\lambda_i}$

If max degree $\leq D$

Claim: If G_n BS converges $\Rightarrow$ μ_{G_n} weak* converges.

- μ_n weak* converges $\Leftrightarrow$ $\int x^k \, d\mu_n$ converges $\forall k$

$$\underline{\int x^k \, d\mu_{G_n}} = \frac{1}{|G_n|} \sum_{0}^{|G|-1} \lambda_i^k = \frac{1}{|G_n|} \operatorname{tr} A_{G_n}^k =$$

$$= \frac{1}{|G_n|} w_{k,n} =$$

w_k # of returning walks in G_n

$$\mathbb{E}\, w_{k,o,o}$$

$o \in G_n$ random

$$\boxed{A^k_{ij} = \#\text{ walks length } k,\ i \to j}$$

First nontrivial theorem

T (Lyons) Tree entropy is BS continuous on connected graphs.

$$\mathfrak{I}(G) = \frac{\log \#\,\text{spanning trees of } G}{|G|}$$

If G_n BS converges $\Rightarrow$ $\mathfrak{I}(G_n)$ converges

<u>STOP HERE</u>

$\mathcal{G}_D$

We proved:

T: If G_n BS converges $\Rightarrow$

$\Rightarrow$ μ_{G_n} weak* converges

Can we localize the eigenvalue distribution μ_G?

Convenient way to get a BS cont. parameter:

TAKE $f : \mathcal{G}_D \to \mathbb{R}, P(\mathbb{R}) \ldots$ continuous

NOW LET $$F(G) = \int_{\mathcal{G}_D} f \, d\lambda_G = \frac{1}{|G|} \sum_{o \in V(G)} f((G,o))$$

CLEARLY F is BS continuous!

G_n BS converges $\Leftrightarrow$ λ_{G_n} weak* converges

$F(G_n) = \int f \, d\lambda_{G_n}$ converges since f is cont.

$f((G,o)) = \begin{matrix} 1 \\ 0 \end{matrix}$ if o sits on a Δ

CAN YOU TURN THIS AROUND

LOCALIZATION?

YES for μ_G

spectral measure does the job

G is finite, $A = \text{Adj}\, G$, b_i ONB, λ_i

$$\mu_G = \frac{1}{|G|} \sum_{e}^{|G|-1} \delta_{\lambda_i}$$

Let $o \in V(G)$ $\quad x_o$

Let $\mu_{(G,o)} = \frac{1}{|G|} \sum_{i=0}^{|G|-1} \underbrace{\langle x_o, b_i \rangle^2}_{= b_i(o)^2} \delta_{\lambda_i}$

This is a localization on the finite level:

- $\mathbb{E}_{o \in G} \mu_{(G,o)} = \frac{1}{|G|} \sum_{o \in G} \mu_{(G,o)} = \mu_G$

- $\int x^k \, d\mu_{(G,o)} = w_{k,o,o}$

Works for the whole $\mathcal{G}_o$:

$\exists\, \mu_{(G,o)} \in P(\mathbb{R})$ such that $\forall k$

$$\int x^k \, d\mu_{(G,o)} = w_{k,o,o}$$

proof:

- let $(G,o) \in \mathcal{G}_o$ then

  $\exists$ finite graphs $(G_n, o_n) \to (G,o)$

- Consider $\mu_{(G_n,o)}$

  claim $\mu_{(G_n,o)}$ weakly converges

$\underset{\text{converges}}{\underset{\downarrow}{w_{G_n}(k,o,o_n)}} = \int x^k \, d\mu_{(G_n,o)}$ $\quad$ Let $\mu_{(G,o)} = \lim \mu_{(G_n,o)}$

Adding all up:

If G_n is BS convergent $G_n \xrightarrow{BS} (G,o)$

RANDOM

Then $\mu_{G_n} \xrightarrow{*} \mathbb{E}\,\mu_{(G,o)}$

WHAT ABOUT $\mu_G(\{0\}) = \dfrac{\#\{i \,/\, \lambda_i = 0\}}{|G|} = \dfrac{\dim \operatorname{Ker} A}{|G|}$

IS IT CONTINUOUS? YES W Y

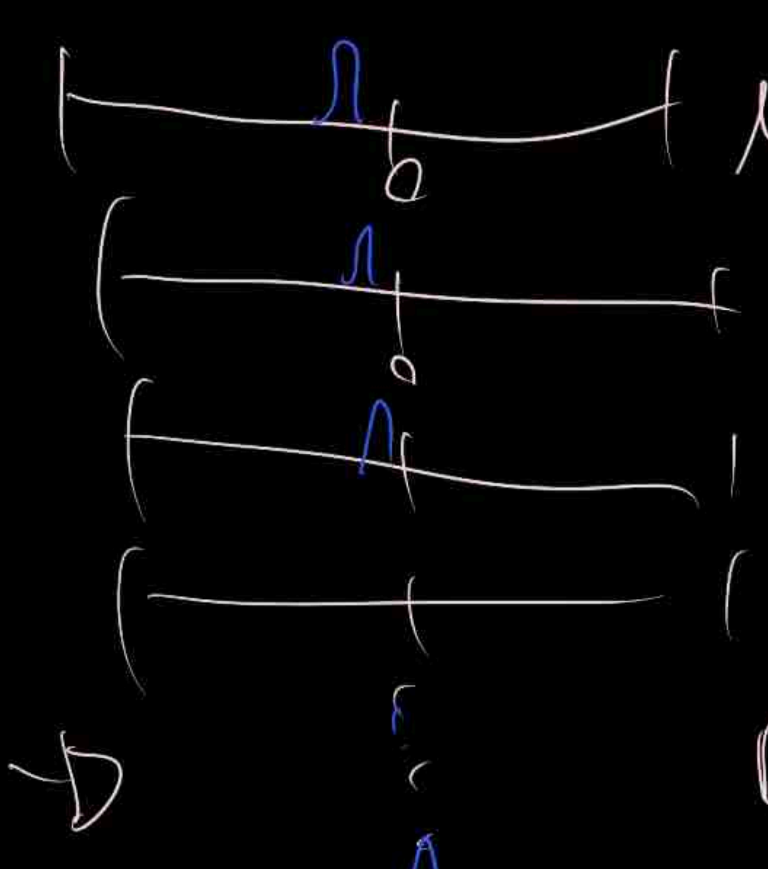
CAN YOU LOCALIZE IT? NO N ?

WHAT would be the natural localization?

$\mu_G(\{0\})$

$\mu_{(G,o)}(\{0\})$ COOL, but NOT continuous!

$G_n \to (G,o)$



μ_{G_n}

$\to$ 0

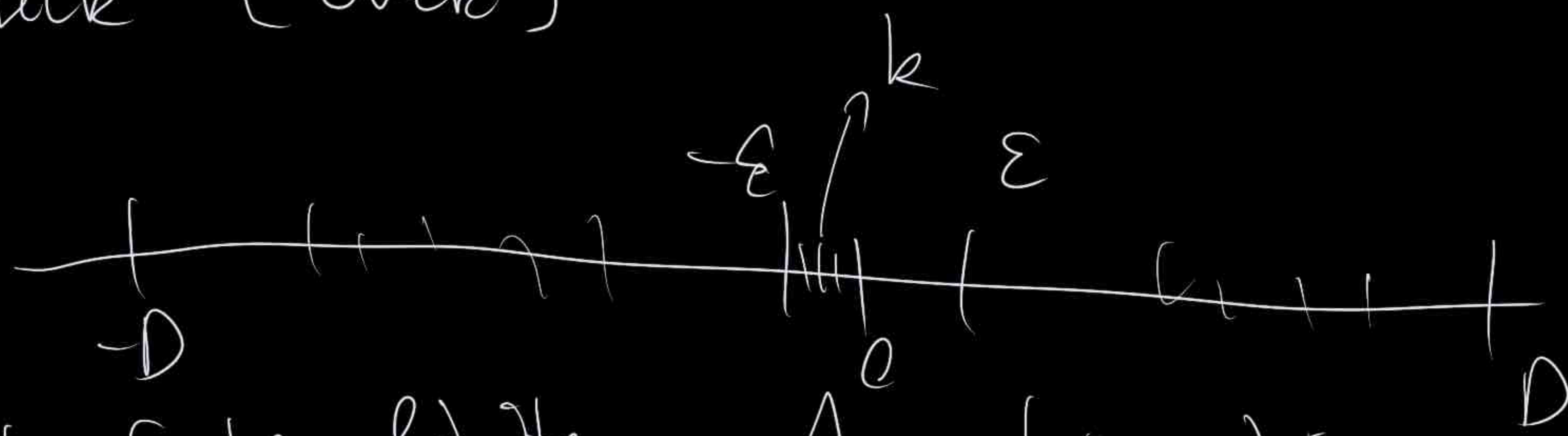
μ_G weakly conv.

$\delta_{\frac{1}{n}} \to \delta_0$

0 1

positive mass travels into 0

Trick (Lück)



Let G be finite, A, b_i, λ_i

Let $k = \#\{i \,/\, |\lambda_i| < \varepsilon,\ \lambda_i \neq 0\}$

$$1 \le \Big|\prod_{\substack{\lambda_i \neq 0 \\ \text{all} \\ \text{eigenvalues}}} \lambda_i\Big| \le \varepsilon^k \cdot D^{n-k} = \varepsilon^k D^n$$

$$\left(\frac{1}{\varepsilon}\right)^k \le D^n$$

$$k \le n \cdot \frac{\log D}{\log(1/\varepsilon)} \xrightarrow{\varepsilon \to 0} 0$$

$$\mu_G\big((-\varepsilon, \varepsilon) \setminus \{0\}\big) \le$$

So no mass travels to 0 ✓

T (Thom) Same for $\mu_G(\{r\})$ $\forall r$.

This is how you compute Betti numbers!

T (Lück) Let W be a finite complex, $\Gamma = \pi_1(W)$.

Let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots > \Gamma_n > \ldots$

$\Gamma_n \triangleleft \Gamma$ finite index, $\cap \Gamma_n = 1$

Then $\lim \frac{b_k(\widetilde{W}/\Gamma_n)}{|\Gamma : \Gamma_n|} = \beta_k^{(2)}(W)$.

OPEN!

we know, G_n BS cont., then

$\frac{\dim \operatorname{Ker}_{\mathbb{Q}}(A_n)}{|G_n|}$ converges

WHAT ABOUT $\mathbb{F}_p$?

$\frac{\dim \operatorname{Ker}_{\mathbb{F}_p}(A_n)}{|G_n|}$???

does it converge?

G_n converges $\Rightarrow$ $\mu_{G_n} \xrightarrow{*} \mu$, but what IS μ?

$$\int x^k d\mu = \lim \int x^k d\mu_{G_n} = \mathop{\mathbb{E}}_{o_n \in G_n} w(G_n, k, o_n, o_n)$$

Localize μ_{G_n}!

For a rooted graph (G, o) let

$\mu_{(G,o)} \in P(\mathbb{R})$ such that

$$\int x^k \, d\mu_{(G,o)} = w(G, k, o, o)$$

WHY does it exist? $\boxed{\text{FUNCT ANAL}}$

OR

Lets do it for finite G first

$$\mu_{(G,o)} = \frac{1}{|G|} \sum_{i=0}^{|G|-1} \underbrace{\langle X_o, b_i \rangle^2}_{= b_i(o)^2} \delta_{\lambda_i}$$

works:

$$\mu_G = \mathop{\mathbb{E}}_{o} \mu_{(G,o)} \qquad \int x^k d\mu_{(G,o)} = w \ldots$$

rooted spectral measure!

we localized μ_G

"Derivative"

$f : \mathcal{G}_0 \to \mathbb{R}$ $(*)$

$F = \mathbb{E} f(G) = \mathbb{E}_{o \in G} f((G,o)) = \int_{\mathcal{G}_0} f \, d\lambda_G$

If f is continuous on $\mathcal{G}_0$, then F is BS-cont ✓

Localization: F is given $\Rightarrow$ find f

$F : \mathrm{Sof} \to \mathbb{R}$ can you localize?

K compact, $\mu_i \in P(K)$

$F : P(K) \to \mathbb{R}$ has a derivative?

R-N derivative ...

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discrete Bettis have

cont. Bettis ???

BS convergence: covering towers ✓

T [Lück Approx] w/ finite complex ....

Can localize?

$\mu_G(\{0\})$ seems OK $\mu_{(G,e)}(\{0\})$

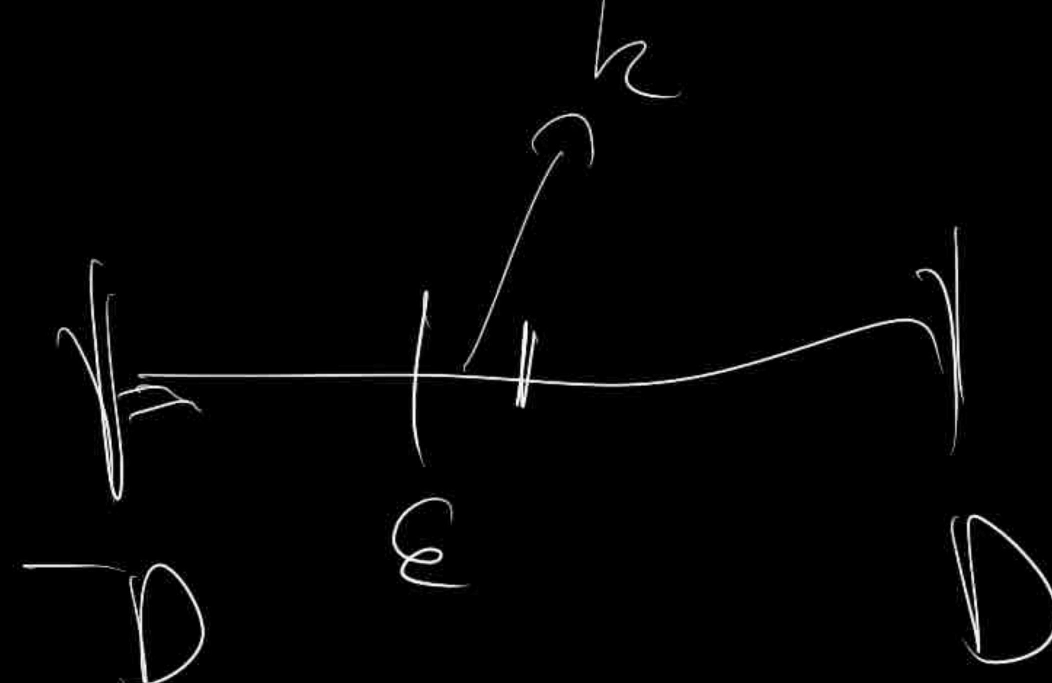
BUT ITS NOT CONT!!!!!

STILL,

T [behind Lück] G_n BS conv $\rightarrow$

$\Rightarrow \mu_{G_n}(\{0\})$ conv.

P: $\left|\prod_{\lambda_i \neq 0} \lambda_i\right| \geq 1$



$$\begin{array}{c} \wedge\!| \\ \varepsilon^k \cdot D^n \end{array}$$

$$D^n \geq \left(\frac{1}{\varepsilon}\right)^k \qquad k \leq n \cdot \log_{1/\varepsilon} D$$

$$\frac{k}{n} \leq \frac{\ln D}{\ln(1/\varepsilon)} \quad ✓$$

G_n BS conv $\Rightarrow$ $\dfrac{\dim(\ker_{\mathbb{Q}}(\mathrm{Adj}\, G_n))}{(G_n)}$ conv.

OPEN: over $\mathbb{F}_p$???