Distribution of Bipartite Entanglement of a Random Pure State

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J. Stat. Phys. 142, 403 (2011) (long version)

Plan

Plan:

- Bipartite entanglement of a random pure state

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- Reduced density matrix $\longrightarrow$ random matrix theory

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  Results

  Coulomb gas technique for large systems

  Phase transitions

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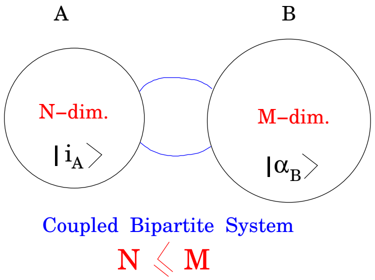
- Summary and Conclusions

Coupled Bipartite System



Bipartite quantum system $A \times B$: Hilbert space $H_A \otimes H_B$

subsystem A: dimension N (the system to study)

subsystem B: dimension $M \geq N$ ("environment")

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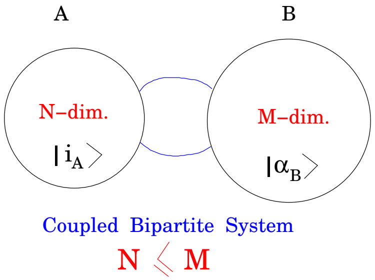

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subsystem A: dimension N (the system to study)

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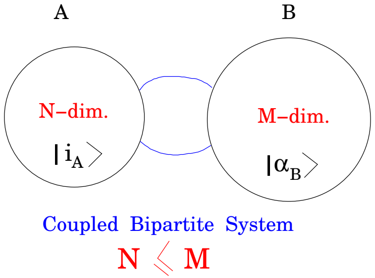
Large system: limit $N \gg 1$ and $M \gg 1$ with $M \approx N$.

Pure State and Reduced Density Matrix



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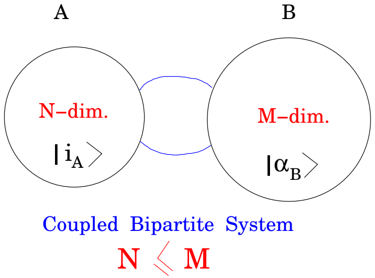

Any pure state of the full system: $\boxed{|\psi>= \sum_{i,\alpha} x_{i,\alpha}\, |i_A> \otimes |\alpha_B>}$

$X = [x_{i,\alpha}] \to (N \times M)$ rectangular Coupling matrix

Pure State of Bipartite System

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- $$\boxed{|\psi>=\sum_{i,\alpha} x_{i,\alpha}\,|i_A>\otimes|\alpha_B>}$$

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$$|\psi>=\sum_i a_i\,|i_A>\otimes\sum_\alpha b_\alpha\,|\alpha_B>=|\Phi_A>\otimes|\Phi_B>$$

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$\longrightarrow$ Fully separable (factorised)

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$$\boxed{\hat\rho=|\psi><\psi|} \text{ with } \mathrm{Tr}[\hat\rho]=1$$

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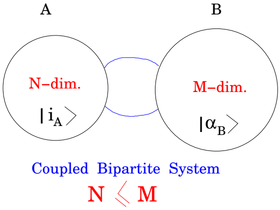
- Density matrix of the composite system

$$\boxed{\hat{\rho}=|\psi><\psi|} \text{ with } \mathrm{Tr}[\hat{\rho}]=1$$

- Pure state: $\hat{\rho}\neq\sum_k p_k|\psi_k><\psi_k| \to$ not a Mixed state

Reduced Density Matrix of subsystem A:



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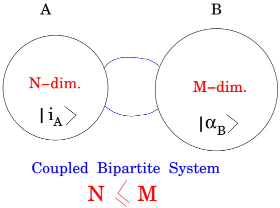

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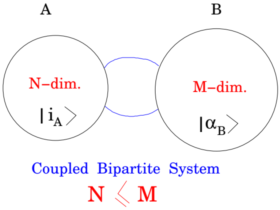

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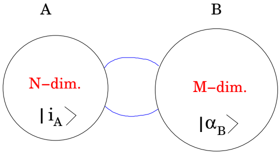

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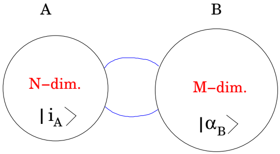

Coupled Bipartite System

$\mathbf{N} \lll \mathbf{M}$

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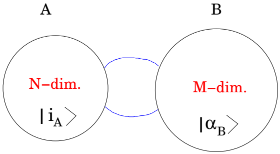

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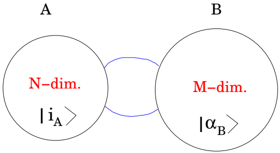

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  $$= \sum_{i,j=1}^{N} W_{i,j} \, |i_A >< j_A|$$

  where the $N \times N$ matrix: $\boxed{W = XX^\dagger}$

Eigenvalues of W:

- In the diagonal representation

$$\hat{\rho}_A = \sum_{i,j=1}^{N} W_{i,j} \, |i_A >< j_A| \to \sum_{i=1}^{N} \lambda_i |\lambda_i^A >< \lambda_i^A|$$

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(i) **Unentangled**

$\lambda_{i_0}=1,\ \lambda_j=0\ \forall j\neq i_0$

$|\psi\rangle=|\lambda_{i_0}^A\rangle\otimes|\lambda_{i_0}^B\rangle$

is **separable**

$\hat{\rho}_A=|\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A|$ is pure

(ii) **Maximally entangled**

$\lambda_j=1/N$ for all j

(all eigenvalues equal)

$|\psi\rangle$ is a superposition of all product states

$\hat{\rho}_A=\frac{1}{N}\sum_{i=1}^{N}|\lambda_i^A\rangle\langle\lambda_i^A|=\frac{1}{N}\mathbf{1}_A$ is completely mixed

Entanglement entropy

Subsystem A described by $\hat{\rho}_A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A\rangle\langle\lambda_i^A|$

- **Von Neuman entropy**: $S_{\mathrm{VN}} = -\mathrm{Tr}\left[\hat{\rho}_A \ln \hat{\rho}_A\right] = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i$

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- $S_{q\to 1} = S_{\rm VN}$ and $S_{q\to\infty} = -\ln(\lambda_{\max})$
- $\Sigma_2 = \exp[-S_{q=2}] \longrightarrow$ **Purity**

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(i) **Unentangled**
$\lambda_{i_0} = 1$, $\lambda_j = 0 \ \forall j \neq i_0$

$\hat{\rho}_A = |\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A|$ is pure

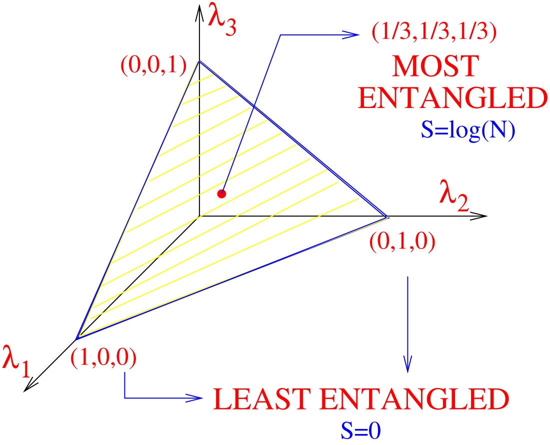
$S_{\rm VN} = 0 = S_q$ is **minimal**

(ii) **Maximally entangled**
$\lambda_j = 1/N$ for all j

$\hat{\rho}_A = \frac{1}{N}\sum_{i=1}^{N} |\lambda_i^A\rangle\langle\lambda_i^A|$ mixed

$S_{\rm VN} = \ln N = S_q$ is **maximal**

A Simple Diagram for N=3



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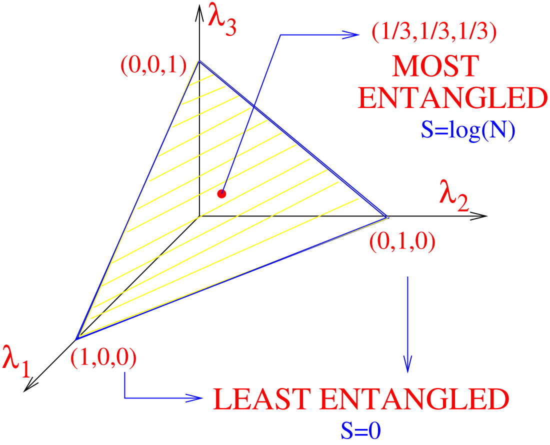

Random Pure State: Haar Measure

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• random pure state $\longrightarrow$ where $X=[x_{i,\alpha}]$ are **uniformly distributed** among the sets of $\{x_{i,\alpha}\}$ satisfying $\sum_{i,\alpha}|x_{i,\alpha}|^2=\mathrm{Tr}\rho=1$

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- Equivalently, for the $N\times M$ matrix

$$P(X)\propto\delta\left(\mathrm{Tr}(XX^{\dagger})-1\right)$$

Random Pure State: Haar Measure

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- Equivalently, for the $N \times M$ matrix

$$P(X) \propto \delta\left(\mathrm{Tr}(XX^\dagger) - 1\right)$$

- In the basis $|i^A\rangle$ of H_A: $\hat{\rho}_A = W = XX^\dagger$

$\longrightarrow$ Distribution of the eigenvalues λ_i of $\hat{\rho}_A$?

$\longrightarrow$ **Distribution of the entanglement entropies** S_{VN}, S_q?

Joint PDF of Eigenvalues

- the joint pdf $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ eigenvalues of the Wishart matrix

  $W = XX^\dagger$ ($X \to N \times M$ Gaussian random matrix)

  with an additional constraint $\boxed{\sum_{i=1}^{N} \lambda_i = 1}$

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- Joint distribution of Wishart eigenvalues (James '64):

$$\boxed{P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta}$$

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(Llyod & Pagels '88, Zyczkowski & Sommers '2001)

Joint PDF of Eigenvalues

- the joint pdf $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ eigenvalues of the Wishart matrix

  $W = XX^\dagger$ ($X \to N \times M$ Gaussian random matrix)

  with an additional constraint $\boxed{\sum_{i=1}^{N} \lambda_i = 1}$

- Joint distribution of Wishart eigenvalues (James '64):

$$\boxed{P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta} \times \delta\left(\sum_{i=1}^{N}\lambda_i - 1\right)$$

(Llyod & Pagels '88, Zyczkowski & Sommers '2001)

- Given this pdf, what is the distribution of the Renyi entropy:

  $S_q = \frac{1}{1-q} \ln\left[\sum_{i=1}^{N} \lambda_i^q\right]$

Random Pure State: Well Studied

- Average von Neumann entropy: $\langle S_{\mathrm{VN}} \rangle \to \ln N - \frac{N}{2M}$ for $M \sim N \gg 1$

$\longrightarrow$ **Near Maximal**

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- Laplace transform of the purity ($q = 2$) distribution for large N [Facchi et. al. (2008, 2010)]

Results: Entropy Distribution

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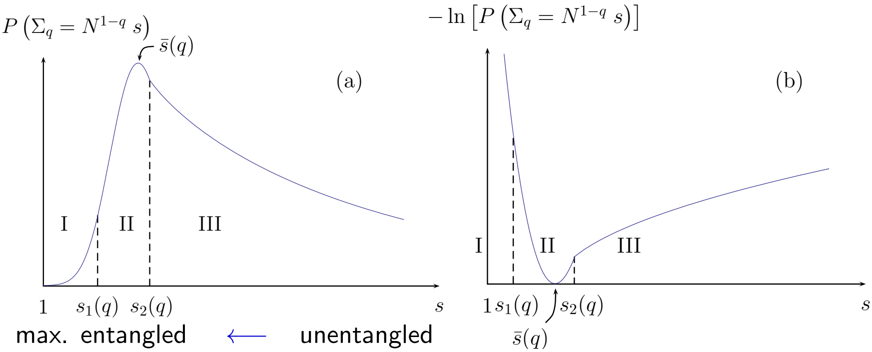
For $q = 2$, $\langle \Sigma_2 \rangle \approx \frac{2}{N}$ (purity); For $q \to 1$, $\langle S_{VN} \rangle \approx \ln N - \frac{1}{2}$

Results: pdf of $\Sigma_q = \sum_i \lambda_i^q$

$$P\left(\Sigma_q = N^{1-q}\,\mathbf{s}\right) \approx \begin{cases} \exp\left\{-\beta N^2 \Phi_I(s)\right\} & \text{for } 1 \le s < s_1(q) \\ \exp\left\{-\beta N^2 \Phi_{II}(s)\right\} & \text{for } s_1(q) < s < s_2(q) \\ \exp\left\{-\beta N^{1+\frac{1}{q}}\ \Psi_{III}(s)\right\} & \text{for } s > s_2(q) \end{cases}$$



Computation of the pdf of Σ_q: Coulomb gas method

- PDF of $\Sigma_q = \sum_i \lambda_i^q$

$$P(\Sigma_q, N) = \int P(\lambda_1, ..., \lambda_N)\ \delta\left(\sum_i \lambda_i^q - \Sigma_q\right)\left(\prod_i d\lambda_i\right)\ .$$

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- The joint pdf of the eigenvalues can be interpreted as a **Boltzmann weight** at inverse temperature β:

$$\begin{aligned} P(\lambda_1, ..., \lambda_N) &\propto \delta\Big(\sum_i \lambda_i - 1\Big) \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \\ &\propto \exp\left\{-\beta E\left[\{\lambda_i\}\right]\right\} \end{aligned}$$

where

$$E\left[\{\lambda_i\}\right] = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i<j} \ln|\lambda_i - \lambda_j| \quad (\text{with } \sum_i \lambda_i = 1)$$

$\longrightarrow$ effective energy of a 2D Coulomb gas of charges

Charge Density and Effective Energy

$$E\{\lambda_i\} = -\gamma \sum_{i=1}^{N} \ln \lambda_i - \sum_{i<j} \ln |\lambda_i - \lambda_j| \quad \text{with} \quad \sum_i \lambda_i = 1$$

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Continuous **charge density** in the large N limit (regimes **I**, **II**):

$$\{\lambda_i\} \longrightarrow \rho(x) = \frac{1}{N} \sum_i \delta\left(x - \lambda_i N\right)$$

$$P\left(\Sigma_q = N^{1-q}\, s, N\right) \propto \int \mathcal{D}[\rho]\; \exp\left\{-\beta N^2\, E_s[\rho]\right\}$$

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with **effective energy**

$$\begin{aligned} E_s[\rho] \;=\; & -\frac{1}{2} \int_0^\infty \int_0^\infty dx\, dx'\, \rho(x)\rho(x')\, \ln|x - x'| \\ & +\mu_0 \left(\int_0^\infty dx\, \rho(x) - 1 \right) \\ & +\mu_1 \left(\int_0^\infty dx\, x\, \rho(x) - 1 \right) + \mu_2 \left(\int_0^\infty dx\, x^q\, \rho(x) - s \right) \end{aligned}$$

where μ_0, μ_1 and μ_2 are Lagrange multipliers

Saddle Point Solution

$$E_s[\rho] = -\frac{1}{2}\int_0^\infty \int_0^\infty dx dx' \, \rho(x)\rho(x') \, \ln|x - x'| + \mu_0 \left(\int_0^\infty dx \, \rho(x) - 1 \right)$$
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- **Saddle point method for large N**:

$$P\left(\Sigma_q=N^{1-q}\,s,N\right)\propto\int\mathcal{D}\left[\rho\right]\exp\left\{-\beta N^2\,E_s\left[\rho\right]\right\}\propto\exp\left\{-\beta N^2 E_s\left[\rho_c\right]\right\}$$

where ρ_c minimizes the energy: $\left.\frac{\delta E_s}{\delta\rho}\right|_{\rho=\rho_c}=0$

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$$\int_0^\infty dx' \, \rho_c(x') \ln|x - x'| = \mu_0 + \mu_1 x + \mu_2 x^q \equiv V(x)$$

$V(x) \rightarrow$ **effective potential** for the charges.

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- Taking derivative with respect to x gives

$$\mathcal{P}\int_0^\infty dx'\, \frac{\rho_c(x')}{x-x'} = \mu_1 + q\,\mu_2 x^{q-1} = V'(x)$$

Steps for Computing the PDF of Σ_q

(i) find the **solution** $\rho_c(x)$ **of the integral equation**

$$\mathcal{P}\int_0^{\infty} dx' \, \frac{\rho_c(x')}{x-x'} = \mu_1 + q\,\mu_2 x^{q-1} = V'(x)$$

Tricomi's solution: density $\rho_c(x)$ with finite support $[L_1, L_2]$:

$$\rho_c(x) = \frac{1}{\pi\sqrt{x-L_1}\sqrt{L_2-x}}\left[C - \mathcal{P}\int_{L_1}^{L_2} \frac{dy}{\pi}\,\frac{\sqrt{y-L_1}\sqrt{L_2-y}}{x-y}\,V'(y)\right],$$

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(ii) for a given s, the unknown Lagrange multipliers μ_0, μ_1 and μ_2 are fixed by the **three conditions**:

$$\int_0^\infty \rho_c(x)\,dx = 1\,,\ \int_0^\infty x\,\rho_c(x)\,dx = 1 \ \text{ and } \ \int_0^\infty x^q\,\rho_c(x)\,dx = s$$

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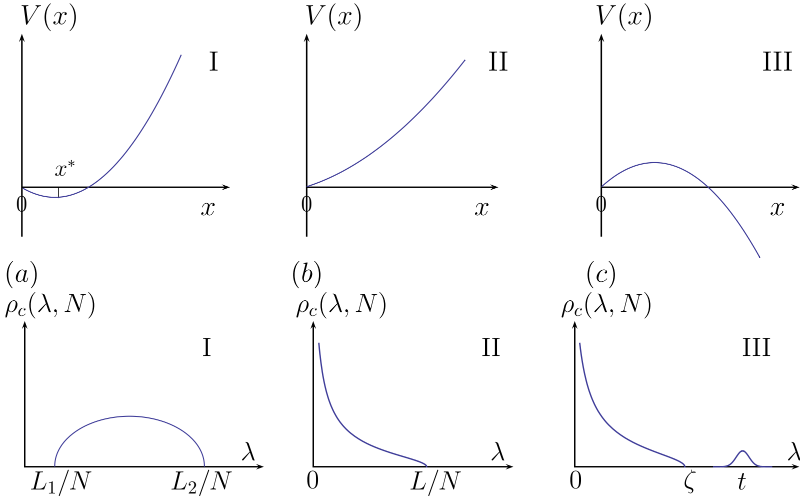
(iii) evaluate the **saddle point energy** $E_s[\rho_c]$

Regime I:
$\mathbf{1 < s < s_1}$
$(s_1(q=2) = 5/4\ )$

Regime II:
$\mathbf{s_1 < s < s_2}$
$(s_2(q=2) = 2 + \frac{2^{4/3}}{N^{1/3}}\ )$

Regime III:
$\mathbf{s > s_2}$
(weakly entangled)



$q = 2$: Purity

- **Regime I**: $1 < s < 5/4$

$$\rho_c(x) = \frac{\sqrt{L_2 - x}\,\sqrt{x - L_1}}{2\pi\,(s-1)}\,,$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_I(s)}\,, \quad \Phi_I(s) = -\frac{1}{4}\ln(s-1) - \frac{1}{8}$$

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- **Regime II**: $5/4 < s < 2 + \frac{2^{4/3}}{N^{1/3}}$

$$\rho_c(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}}\,(A + B\,x) \quad \text{with} \quad L = 2\left(3 - \sqrt{9-4s}\right)$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_{II}(s)}\,, \quad \Phi_{II}(s) = -\frac{1}{2}\ln\left(\frac{L}{4}\right) + \frac{6}{L^2} - \frac{5}{L} + \frac{7}{8}$$

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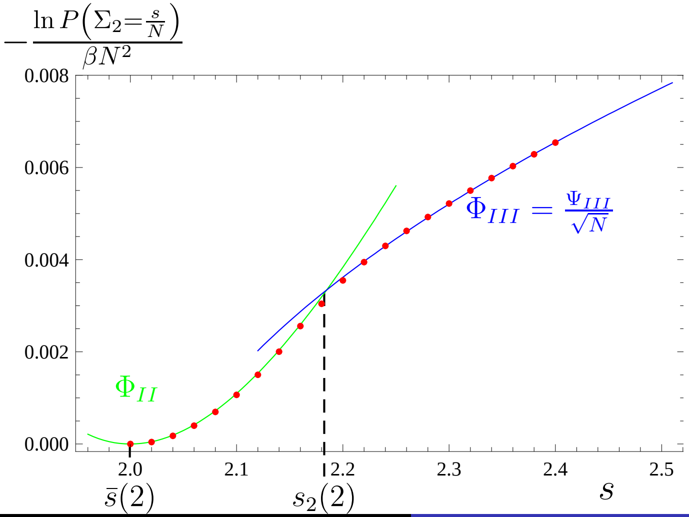
- **Regime III**: $s > 2 + \frac{2^{4/3}}{N^{1/3}}$ Continuous density with support $[0, \zeta]$ with $\zeta \approx \frac{4}{N}$ and separated $\lambda_{\max} \gg \zeta$: $\lambda_{\max} = t \approx \frac{\sqrt{s-2}}{\sqrt{N}}$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \approx e^{-\beta N^{\frac{3}{2}}\,\Psi_{III}(s)}\,, \quad \Psi_{III}(s) = \frac{\sqrt{s-2}}{2}$$

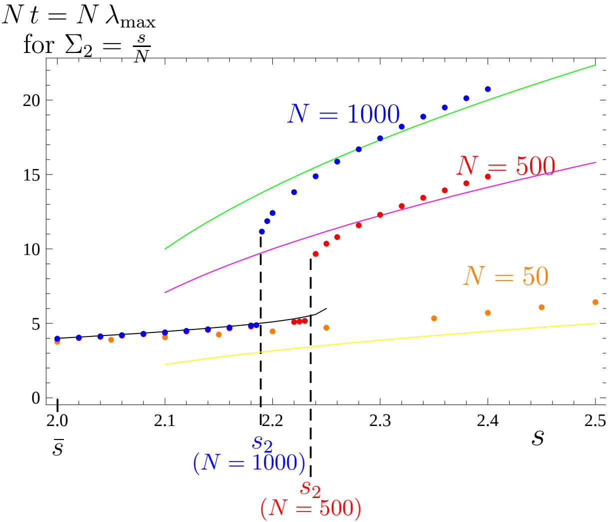
Numerical simulations

Monte Carlo Simulations (non-standard Metropolis algorithm):
pdf of Σ_2 (purity) for $N = 1000$ (second phase transition)



Numerical Simulations (2)

Jump of the maximal eigenvalue (rightmost charge) at $s = s_2$ for $q = 2$ and different values of N



The limit $q \to 1$ — von Neumann Entropy

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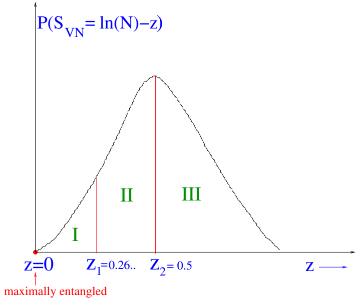
$$P\left(S_{VN} = \ln(N) - z\right) \approx \begin{cases} \exp\left\{-\beta N^2 \phi_I(z)\right\} & \text{for } 0 \le z < z_1 = \frac{2}{3} - \ln\left(\frac{3}{2}\right) = 0.26 \\ \exp\left\{-\beta N^2 \phi_{II}(z)\right\} & \text{for } 0.26.. < z < z_2 = 1/2 \\ \exp\left\{-\beta \frac{N^2}{\ln(N)}\, \phi_{III}(z)\right\} & \text{for } z > z_2 = 1/2 \end{cases}$$



The limit $q \to \infty$ — Maximal eigenvalue

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- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

$$\lambda_{\max} = \frac{4}{N} + 2^{4/3}\, N^{-5/3}\, Y_\beta$$

$Y_\beta \to$ random variable distributed via Tracy-Widom law

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- $S_{q\to\infty} = -\ln(\lambda_{\max})$

$$P\left(\lambda_{\max} = \frac{t}{N}\right) \approx \begin{cases} \exp\left\{-\beta N^2 \chi_I(t)\right\} & \text{for } 1 \le t < t_1 = \frac{4}{3} \\ \exp\left\{-\beta N^2 \chi_{II}(t)\right\} & \text{for } \frac{4}{3} < t < t_2 = 4 \\ \exp\left\{-\beta N\, \chi_{III}(t)\right\} & \text{for } t > t_2 = 4 \end{cases}$$

- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

$$\lambda_{\max} = \frac{4}{N} + 2^{4/3}\, N^{-5/3}\, Y_\beta$$

$Y_\beta \to$ random variable distributed via Tracy-Widom law

- For finite N and M, see P. Vivo (2011)

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Work in progress (C. Nadal, S.M with C. Pineda and T. Seligman)

Distribution of entropy for N finite but $M \gg N$ (large environment)?

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