Top eigenvalue of a random matrix: A tale of tails

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Sud, France

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First Appearence of Random Matrices

Biometrika, 20, 32-52 (1928)

THE GENERALISED PRODUCT MOMENT DISTRIBUTION IN SAMPLES FROM A NORMAL MULTIVARIATE POPULATION.

By JOHN WISHART, M.A., B.Sc. Statistical Department, Rothamsted Experimental Station.

1. *Introduction.*

For some years prior to 1915, various writers struggled with the problems that arise when samples are taken from uni-variate and bi-variate populations, assumed in most cases for simplicity to be normal. Thus "Student," in 1908*, by considering the first four moments, was led by K. Pearson's methods to infer the distribution of standard deviations, in samples from a normal population. His results, for comparison with others to be deduced later, will be stated in the form

$$dp = \frac{1}{\Gamma\left(\frac{N-1}{2}\right)} A^{\frac{N-1}{2}} . e^{-Aa} . a^{\frac{N-3}{2}} da \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

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SESSION IIB

INTERPRETATION OF LOW ENERGY NEUTRON SPECTROSCOPY

CHAIRMAN—W. W. Havens, Jr.

IIB1. DISTRIBUTION OF NEUTRON RESONANCE LEVEL SPACING.

E. P. WIGNER, *Princeton University*
Presented by E. P. Wigner

The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low energy levels is not possible, and we can only make

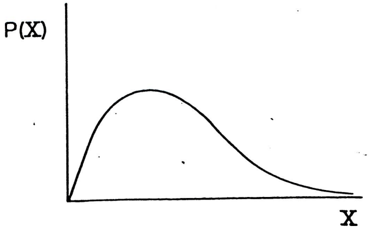

Fig. IIB1-1. Probability of a level spacing X.

our, that is a much more serious deviation and much

bility that two successive roots have a distance X,

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he
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vs
If
a-

the root shall be λ. All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

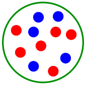
DISCUSSION

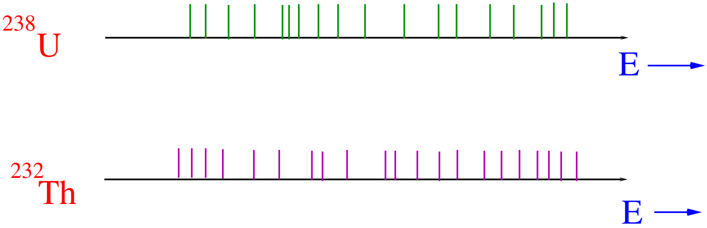
W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is given in S. S. Wilks book about statistics and I found it just by accident.

Random Matrices in Nuclear Physics

spectra of heavy nuclei





WIGNER ('50) : replace complex H by random matrix

DYSON, GAUDIN, MEHTA, ...

Applications of Random Matrices

Physics: nuclear physics, quantum chaos, disorder and localization, mesoscopic transport, optics/lasers, quantum entanglement, neural networks, gauge theory, QCD, matrix models, cosmology, string theory, statistical physics (growth models, interface, directed polymers...), ...

Mathematics: Riemann zeta function (number theory), Voiculescu's free probability theory, combinatorics and knot theory, determinantal points processes, integrable systems, ...

Statistics: multivariate statistis, principal component analysis (PCA), image processing, data compression, Bayesian model selection, ...

Information Theory: signal processing, wireless communications, ..

Biology: sequence matching, RNA folding, gene expression network ...

Economics and Finance: time series analysis,....

Recent Ref: The Oxford Handbook of Random Matrix Theory
ed. by G. Akemann, J. Baik and P. Di Francesco (2011)

Plan

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- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall

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- Similar 3-rd order phase transition in $2-d$ large N gauge theory

  Gross-Witten-Wadia transition (U(N) lattice)
  Durhuss-Olesen/Douglas-Kazakov transition (continuum Yang-Mills)

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- Summary and Conclusions

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & 0 \\ & x_{22} & & & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix} \qquad \Pr[x_{ii}=x] = (2\pi)^{-1/2}\exp[-x^2/2] \longrightarrow \text{GAUSSIAN}$$

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

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- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN

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[R.M. May, Nature, 238, 413 (1972)——Ecosystems]

[Cavagna et. al. 2000, Fyodorov & Williams 2007,——Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006, Marsh et. al. 2011,—String theory]

[Beltrani 2007, Dedieu & Malajovich, 2007, Houdre 2011—-Random Polynomials, Random Words

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More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

$$\boxed{P_N \sim \exp[-\beta\theta N^2]} \text{ for large } N$$

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

Spin glasses, Structural glasses, Supercooled liquids, String landscapes

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- Near a stationary point: $V(\{y_i\}) \approx \sum_{i,j} H_{i,j}(y_i - y_i^*)(y_j - y_j^*)$

Hessian matrix: $H_{i,j} \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

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- Eigenvalues of the Hessian (stability) matrix determines the nature of the stationary point:

$\Longrightarrow$ Local Minimum, Local Maximum & Saddles

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \to$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \to$ Local Minimum

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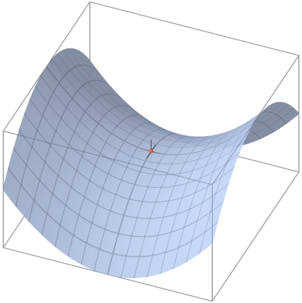
Two real eigenvalues: (λ_1, λ_2)

If $\lambda_1 < 0$ and $\lambda_2 < 0 \to$ Local Maximum

If $\lambda_1 > 0$ and $\lambda_2 > 0 \to$ Local Minimum

$$\left.\begin{array}{ll} \lambda_1 < 0, & \lambda_2 > 0 \\ \lambda_1 > 0, & \lambda_2 < 0 \end{array}\right\} \quad \to \text{Saddle}$$



Fraction of Local Maximum/Minimum in Random Hessian Model

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- $\longrightarrow$ Most of the stationary points $\to$ Saddles

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- where Z_N = Partition Function

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

Coulomb Gas Interpretation

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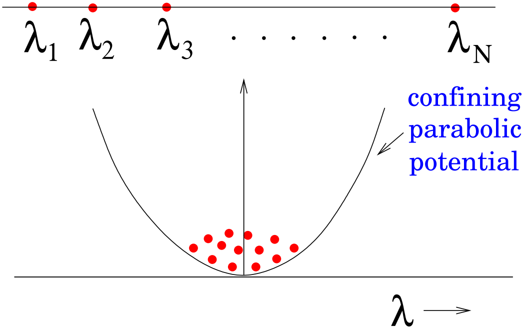

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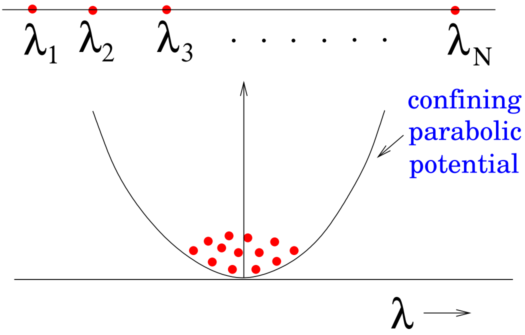

- Balance of energy $\Rightarrow N^2\,\lambda^2 \sim N^2$
- Typical eigenvalue: $\lambda_{\mathrm{typ}} \sim O(1)$ for large N

Spectral Density: Wigner's Semicircle Law

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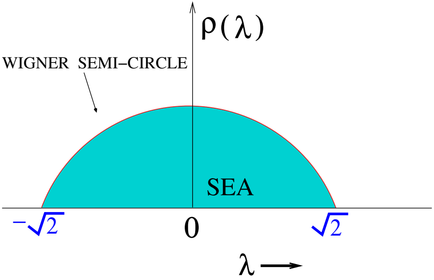

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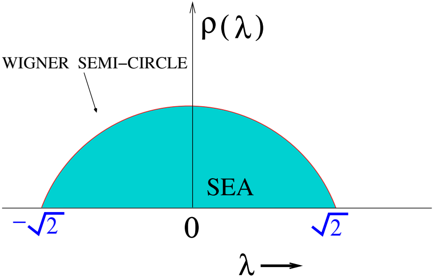
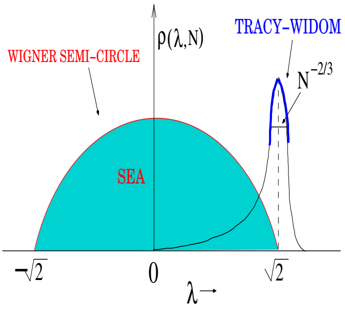
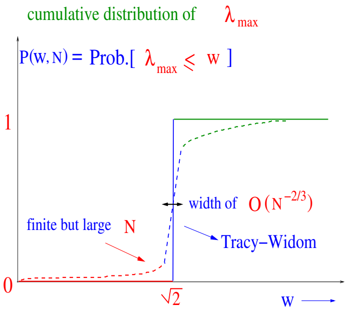


- $\langle \lambda_{\max} \rangle = \sqrt{2}$ for large N.

- $\lambda_{\max}$ fluctuates from one sample to another. $\mathrm{Prob}[\lambda_{\max}, N] = ?$

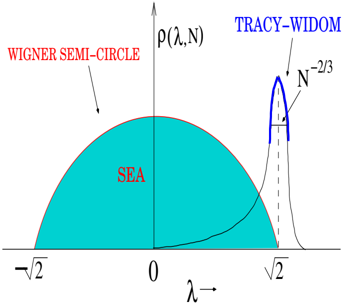


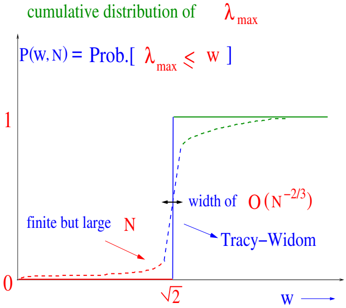
ρ(λ,N)
TRACY−WIDOM
WIGNER SEMI−CIRCLE
N−2/3
SEA
−√2
0
√2
λ→



cumulative distribution of λmax
P(w, N) = Prob.[λmax ≤ w]
1
0
width of O(N−2/3)
finite but large N
Tracy−Widom
√2
w

Distribution for $\lambda_{\max}$





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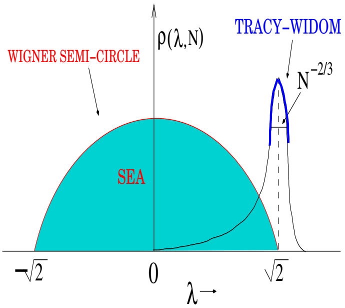

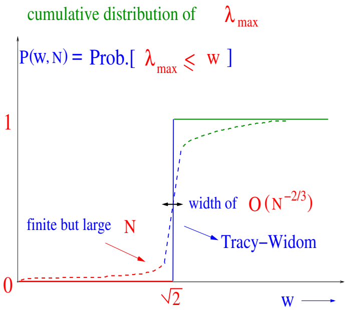

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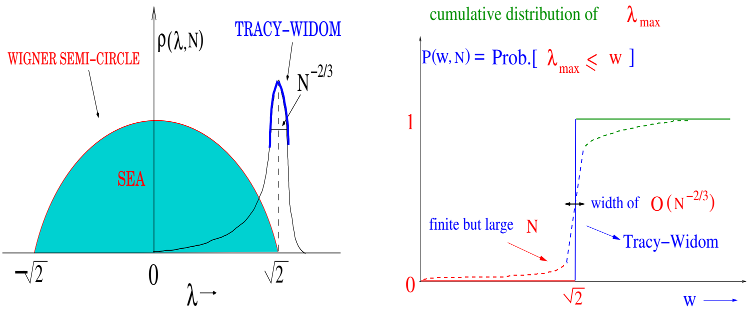

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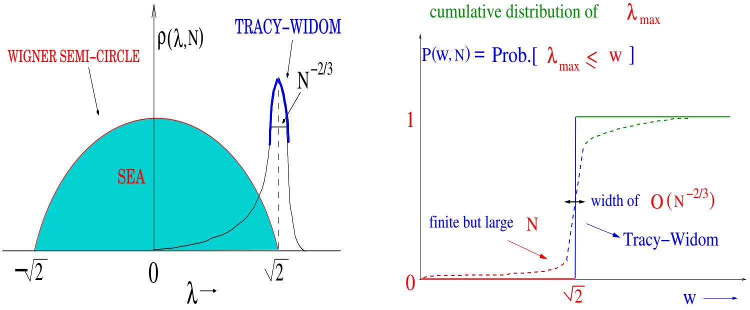

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- $F_\beta(x) \to$ obtained from solution of Painlevé-II equation

Tracy-Widom distribution (1994)

The scaling function $F_\beta(x)$ has the expression:

- $\beta = 1$: $F_1(x) = \exp\left[-\frac{1}{2}\int_x^\infty \left[(y-x)q^2(y) + q(y)\right]\, dy\right]$

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$\frac{d^2 q}{dy^2} = 2\, q^3(y) + y\, q(y)$ with $q(y) \to \mathrm{Ai}(y)$ as $y \to \infty$ $\to$ Painlevé-II

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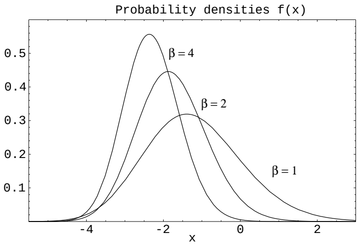

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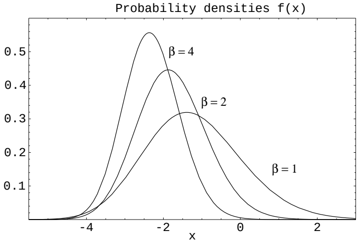

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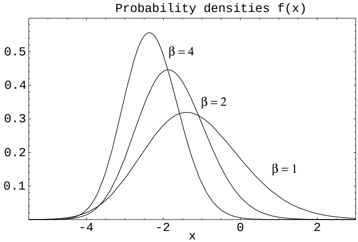

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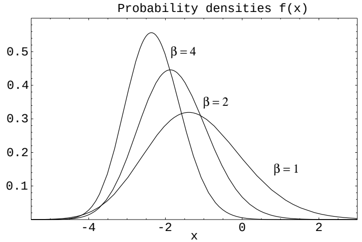

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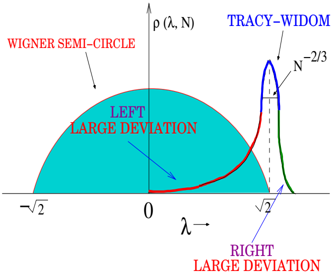
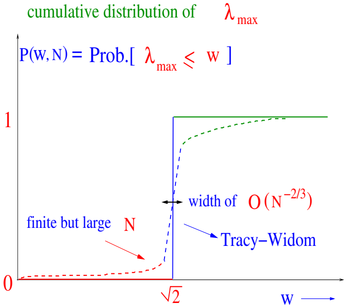
Applications: Growth models, Directed polymer, Sequence Matching .....

(Amir, Baik, Borodin, Borot, Calabrese, Comtet, Corwin, Deift, Dotsenko, Dumitriu, Edelman, Ferrari, Forrester, Johansson, Johnstone, Le doussal, Nadal, Nechaev, O'Connell, Péché, Prähofer, Quastel, Rains, Rambeau, Rosso, Sano, Sasamoto, Schehr, Spohn, Takeuchi, Virag, Vivo,..)

Probability of Large Deviations of $\lambda_{\max}$:





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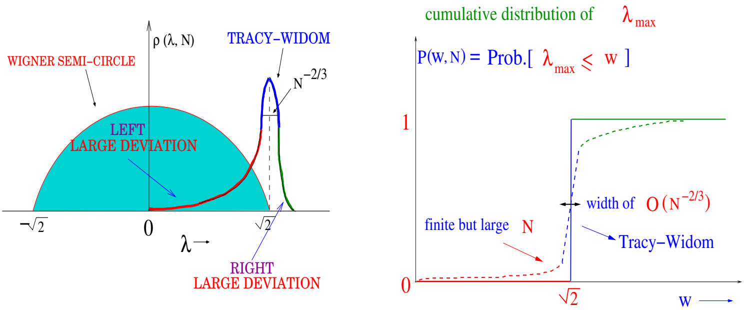

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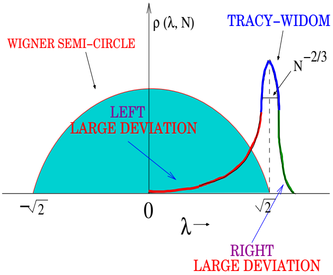

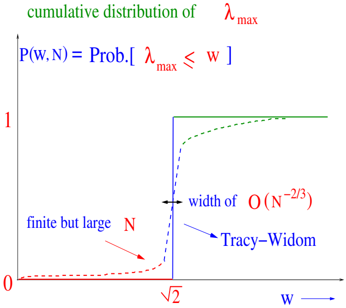

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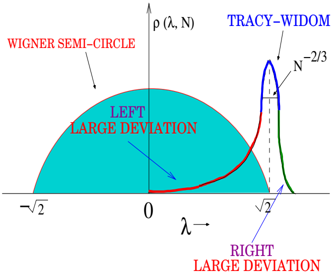

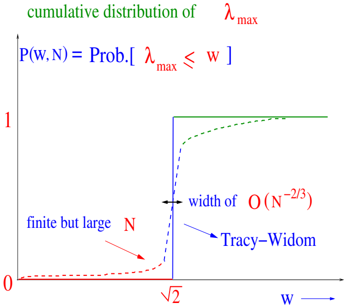

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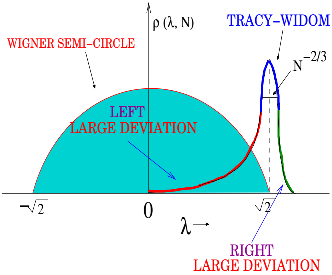

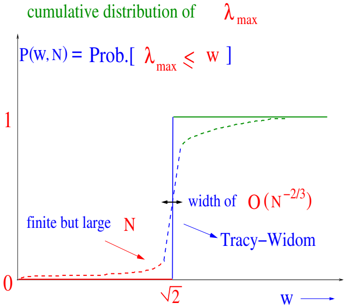

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  $\Rightarrow$ $w = 0$, i.e., fluctuation$=w - \sqrt{2} = -\sqrt{2}$

  large compared to typical $O(N^{-2/3})$

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- Let $M \rightarrow$ no. of heads in N tosses of an unbiased coin

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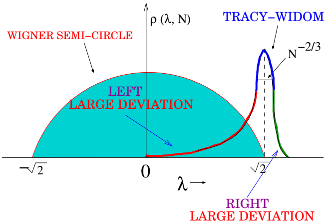
- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak

Large Deviation Tails of $\lambda_{\max}$: Summary of Results



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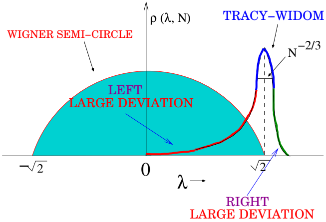

Prob. density of the top eigenvalue: $\mathrm{Prob.}\,[\lambda_{\max} = w, N]$ behaves as:

$$\sim \quad \exp\left[-\beta N^2 \Phi_-(w)\right] \qquad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim \quad N^{2/3} f_\beta \left[\sqrt{2}\, N^{2/3} \left(w - \sqrt{2}\right)\right] \qquad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

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Using Coulomb gas + Saddle point method for large N:

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$$\begin{aligned}\Phi_-(w) &= \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} \\ &+ 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\Big] \quad \text{where} \quad w < \sqrt{2}\end{aligned}$$

Dean & S.M., PRL, 97, 160201 (2006)

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$$\Phi_+(w) = \frac{1}{2}w\sqrt{w^2-2} + \ln\left[\frac{w - \sqrt{w^2-2}}{\sqrt{2}}\right] \quad \text{where} \quad w > \sqrt{2}$$

S.M. & Vergassola, PRL, 102, 060601 (2009)

Ben Arous, Dembo & Guionnet, Prob. Th. Rel. Fields, 120, 1 (2001)

Distribution of $\lambda_{\max}$: Coulomb Gas Method

$$\mathrm{Prob}[\lambda_{\max} \le w,\ N] = \mathrm{Prob}[\lambda_1 \le w, \lambda_2 \le w, \ldots, \lambda_N \le w] = \frac{Z_N(w)}{Z_N(\infty)}$$

$$\boxed{Z_N(w) = \int_{-\infty}^{w} \ldots \int_{-\infty}^{w} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{N\sum_{i=1}^{N}\lambda_i^2 - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right\}\right]}$$

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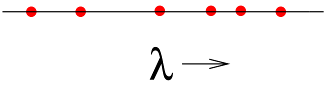
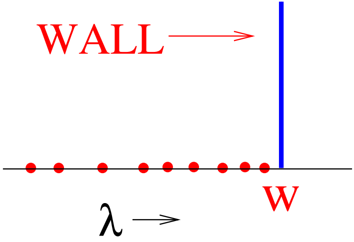
denominator

numerator





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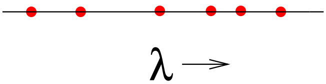

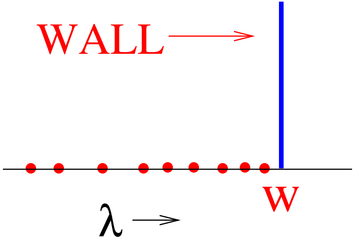

Setting up the Saddle Point Method

- $$\boxed{Z_N(w) \propto \int_{-\infty}^{w} \prod_i d\lambda_i \exp\left[-\beta N^2 E\left(\{\lambda_i\}\right)\right]}$$

$$E\left(\{\lambda_i\}\right) = \frac{1}{2N}\sum_i \lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|\lambda_j - \lambda_k|$$

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$$Z_N(w) \propto \int \mathcal{D}f(x)\, \exp\left[-\beta\, N^2 \left\{E\left[f(x)\right] + C\left(\int f(x)dx - 1\right)\right\} + O(N)\right]$$

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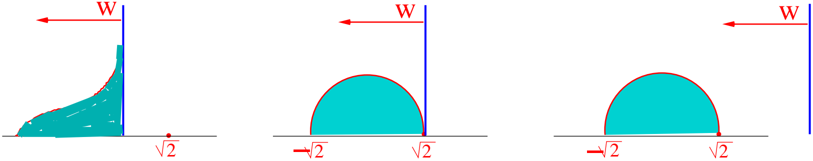
As we bring the wall from ∞

charge density $f_w(x)$ vs. x for different W

$W < \sqrt{2}$ $\qquad$ $W = \sqrt{2}$ $\qquad$ $W > \sqrt{2}$



pushed
(UNSTABLE)

critical

unpushed
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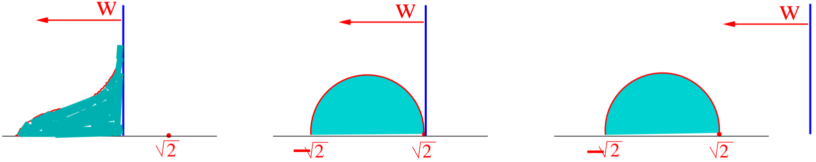

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- $\boxed{x = \mathcal{P}\int_{-\infty}^{w} \frac{f_w(y)\,dy}{x - y}}$ for $x \in [-\infty, w] \rightarrow$ Semi-Hilbert transform

$\rightarrow$ Inverse electrostatic problem $\rightarrow$ Given the potential x find the charge density $f_w(x)$ (though not quite!)

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- General method for solving such singular integral equations:
  - Resolvent Method $\rightarrow$ (Brezin, Itzykson, Parisi, Zuber, 1978)
  - Explicit solution (single support case) $\rightarrow$ (Tricomi , 1957)

Tricomi Solution

Assuming finite support of $f(x)$ over $[a, b]$

- $$\boxed{U(x) = \mathcal{P}\int_a^b \frac{f(y)\,dy}{x-y}}$$ for $x \in [a, b]$

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- General solution (Tricomi, '57):

$$\boxed{f(x) = -\frac{1}{\pi^2\sqrt{(b-x)(x-a)}}\left[\mathcal{P}\int_a^b \frac{\sqrt{(b-x')(x'-a)}}{x-x'}\,U(x')\,dx' + B\right]}$$

for $x \in [a, b]$ where $B \to$ arbitrary constant

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- In our problem, $U(x) = x$ and $b = w$ (wall position) and we assume $a = -L_1(w)$

Exact Saddle Point Solution

- Exact solution (Dean and S.M., 2006, 2008):

$$\boxed{f_w(x) = \frac{\sqrt{x + L_1(w)}}{2\pi\sqrt{w - x}}\ [w + L_1(w) - 2x]}$$

where $-L_1(w) \le x \le w$ and $L_1(w) = [2\sqrt{w^2+6} - w]/3$

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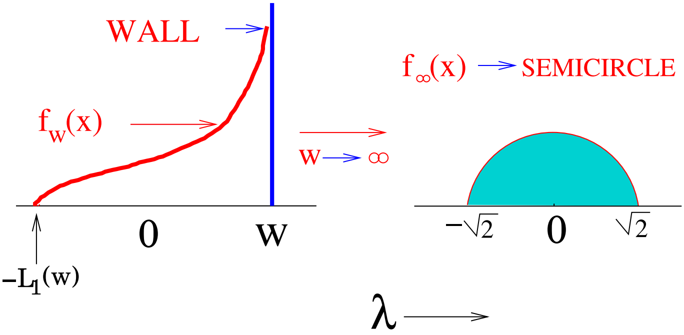

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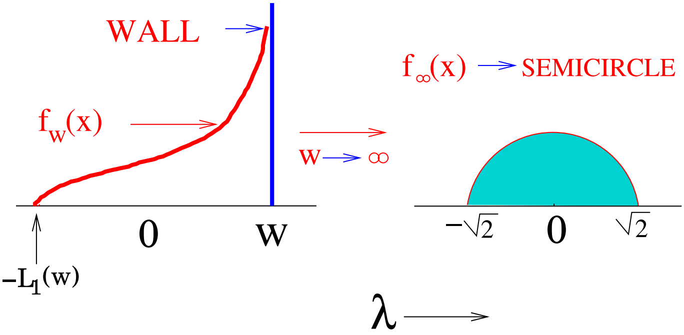

Left Large Deviation Function

$$\mathrm{Prob}[\lambda_{\max} \le w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp\left[-\beta N^2 \{S[f_w(x)] - S[f_\infty(x)]\}\right]$$
$$\sim \exp\left[-\beta\, N^2\, \Phi_-(w)\right]$$

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$\lim_{N\to\infty} -\frac{1}{N^2} \ln\left[P(w,N)\right] = \Phi_-(w) \to$ left large deviation function

physically $\Phi_-(w) =$ energy cost in pushing the Coulomb gas

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$$\begin{aligned}
\Phi_-(w) \quad &= \quad \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} \\
&+ \quad 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\Big] \quad \text{for} \quad w < \sqrt{2} \\
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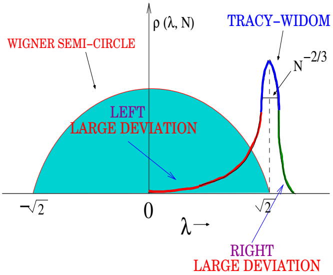
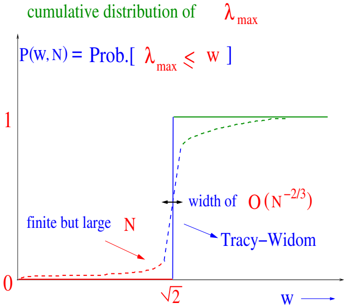
Note also that $\Phi_-(w) \approx \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3$ as $w \to \sqrt{2}$ from below

Matching the Left Tail of Tracy-Widom Distribution





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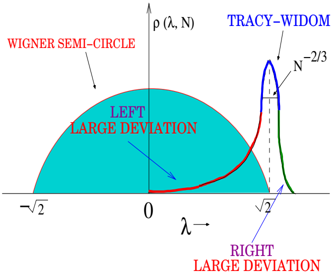

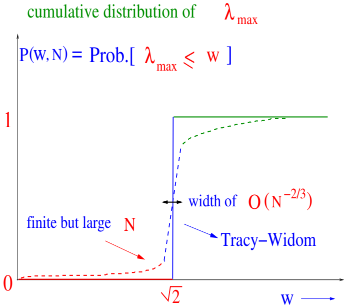

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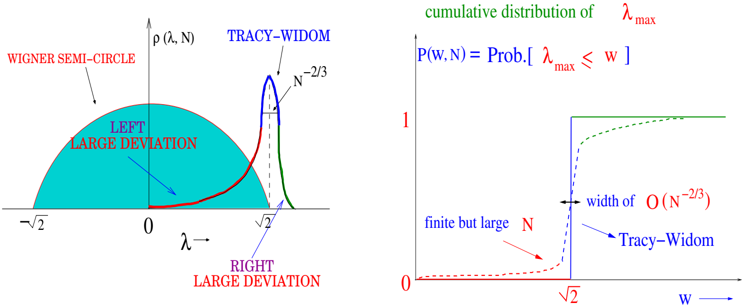

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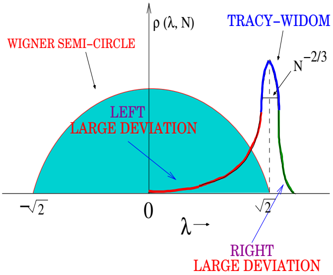

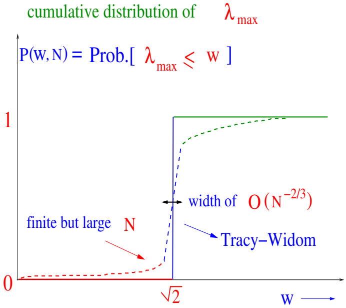

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- As $w \to \sqrt{2}$ from below, $\Phi_-(w) \to \frac{(\sqrt{2}-w)^3}{6\sqrt{2}} \Rightarrow$

$$\mathrm{Prob}[\lambda_{\mathrm{max}} \leq w, N] \approx \exp\left[-\frac{\beta}{24}\left|\sqrt{2}\, N^{2/3}\,(w-\sqrt{2})\right|^3\right]$$

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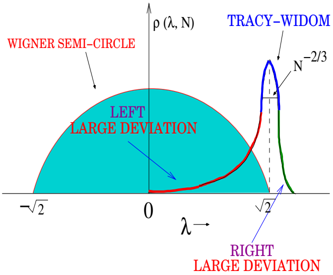

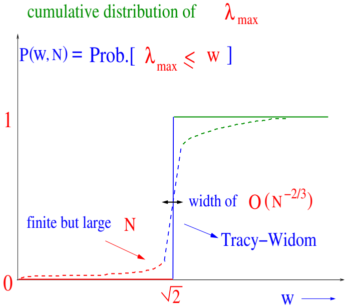

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- recovers the correct left tail of TW: $F_\beta(x) \sim \exp[-\frac{\beta}{24}\,|x|^3]$ as $x \to -\infty$

Right Large Deviation Function: $w > \sqrt{2}$

- For $w \geq \sqrt{2}$, saddle point solution of the charge density $f_w(x)$ sticks to the semi-circle form: $f_w(x) = \frac{1}{\pi}\sqrt{2-x^2}$ for all $w \geq \sqrt{2}$

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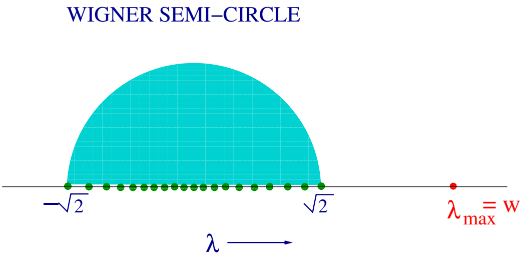
• For $w > \sqrt{2}$ the configuration of charges that dominate this pdf is one where the top eigenvalue $\lambda_{\max} >> $ all other eigenvalues



Pulled Coulomb Gas

$\Rightarrow P(\lambda_{\max} = w, N) \propto \exp[-\beta N \, \Delta E(w)/2] = \exp[-\beta \, N \, \Phi_+(w)]$

$$\boxed{\Delta E(w) = w^2 - 2 \int_{-\sqrt{2}}^{\sqrt{2}} \ln(w - \lambda) \, \rho_{sc}(\lambda) \, d\lambda}$$

$\Rightarrow$ energy cost in pulling a charge out of the Wigner sea

$\rho_{sc}(\lambda) = \frac{1}{\pi}\sqrt{2 - \lambda^2} \rightarrow$ Wigner density

Pulled Coulomb Gas

$$\Rightarrow P(\lambda_{\max} = w, N) \propto \exp[-\beta N\, \Delta E(w)/2] = \exp[-\beta\, N\, \Phi_+(w)]$$

$$\boxed{\Delta E(w) = w^2 - 2\int_{-\sqrt{2}}^{\sqrt{2}} \ln(w-\lambda)\, \rho_{sc}(\lambda)\, d\lambda}$$

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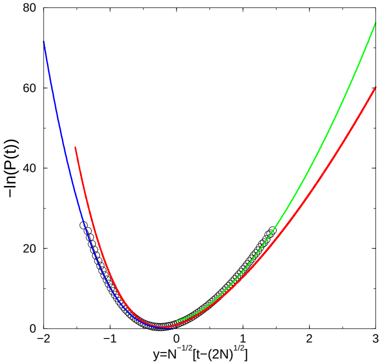
recovers the right tail of TW: $f_\beta(x) \sim \exp[-\frac{2\beta}{3}\, |x|^{3/2}]$ as $x \rightarrow \infty$

Comparison with Simulations:



$N \times N$ real Gaussian matrix ($\beta = 1$): $N = 10$

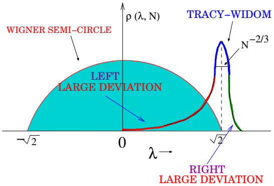
circles $\rightarrow$ simulation points

red line $\rightarrow$ Tracy-Widom

blue line $\rightarrow$ left large deviation function ($\times N^2$)

green line $\rightarrow$ right large deviation function ($\times N$).

Summary and Generalizations



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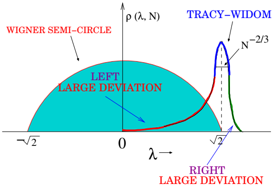

Prob. density of the top eigenvalue: $\mathrm{Prob.}\,[\lambda_{\max} = w, N]$ behaves as:

$$\sim \quad \exp\left[-\beta N^2 \Phi_-(w)\right] \qquad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim \quad N^{2/3} f_\beta \left[\sqrt{2}\, N^{2/3} \left(w - \sqrt{2}\right)\right] \qquad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \quad \exp\left[-\beta N \Phi_+(w)\right] \qquad \text{for} \quad w - \sqrt{2} \sim O(1)$$

3-rd Order Phase Transition

Cumulative prob. of $\lambda_{\rm max}$:

$$P(w,N) = \frac{Z_N(w)}{Z_N(\infty)} \approx \begin{cases} \exp\left\{-\beta N^2 \Phi_-(w)\right\} & \text{for } w < \sqrt{2} \quad \text{pushed} \\ 1 - A\,\exp\left\{-\beta N \Phi_+(w)\right\} & \text{for } w > \sqrt{2} \quad \text{unpushed} \end{cases}$$

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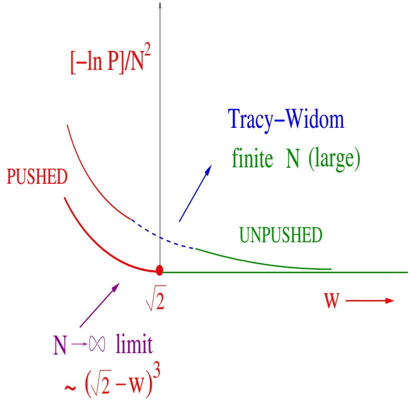
3-rd derivative $\to$ discontinuous

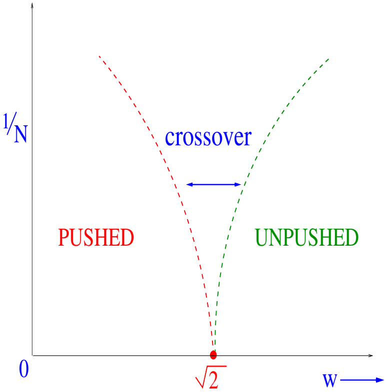
- Crossover: $N \to \infty$, $w \to \sqrt{2}$ keeping $(w - \sqrt{2})\, N^{2/3}$ fixed

$$P(w,N) \to F_\beta\left[\sqrt{2}\, N^{2/3}\, \left(w - \sqrt{2}\right)\right] \to \text{Tracy-Widom}$$

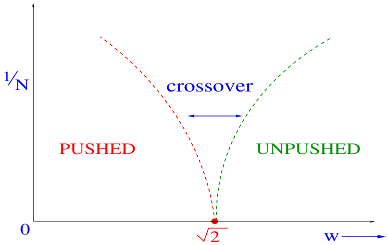
Analogue of the double-scaling limit in large N gauge theory

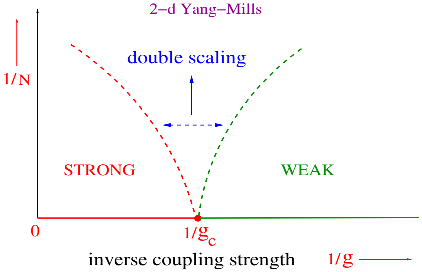
Phase Diagram





3-rd Order Phase Transition: 2-d YM





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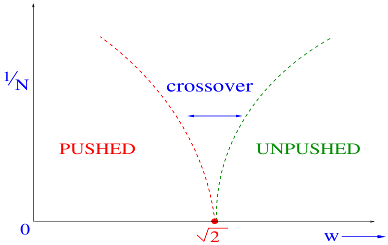

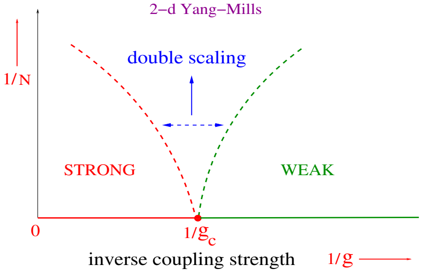

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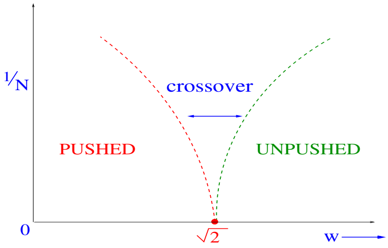

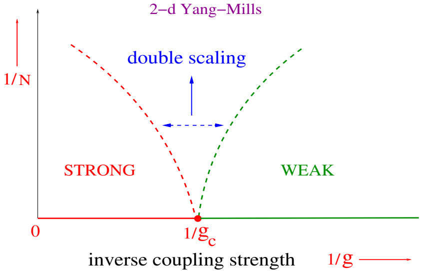

Similar 3-rd order phase transition in 2-d Yang-Mills theory for large N

- Lattice (Wilson action with U(N)): Gross & Witten, Wadia, '80
- Continuum (Heat-Kernel action): Darhuus & Olesen/Douglas & Kazakov

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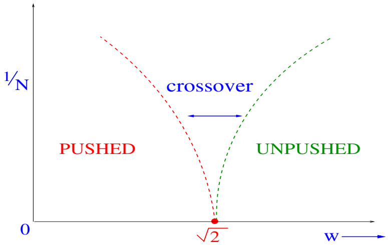

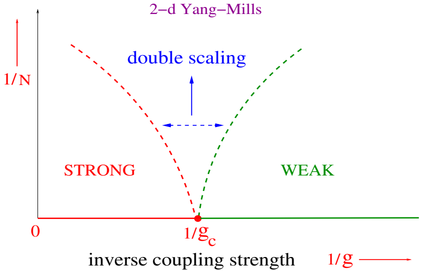

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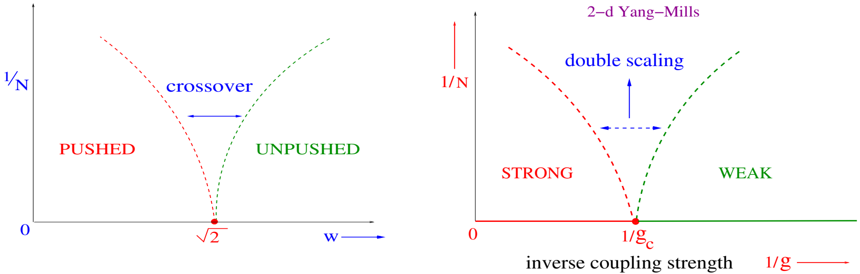

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Tracy-Widom $\equiv$ crossover function in the double scaling regime where

$$(w - \sqrt{2})\, N^{2/3} \to \text{fixed}$$

Periwal-Shevitz '90 (Wilson/Unitary), Gross-Matytsin '94 (heat kernel)

Generalization to Wishart Matrices

- $W = X^\dagger X \to (N \times N)$ square covariance matrix (Wishart, 1928)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2} N \,\mathrm{Tr}(X^\dagger X)\right]$

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$$\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \xrightarrow[N\to\infty]{} \rho(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4-\lambda}{\lambda}}$$

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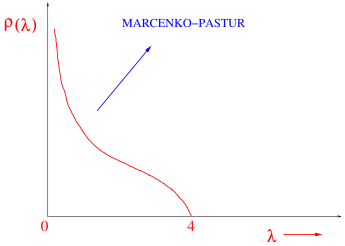
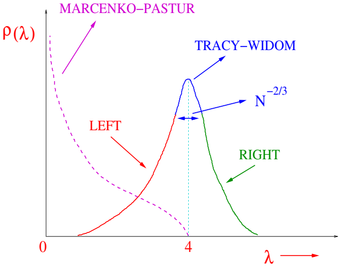


Distribution of $\lambda_{\max}$



- $\langle \lambda_{\max} \rangle = 4$ (as $N \to \infty$)

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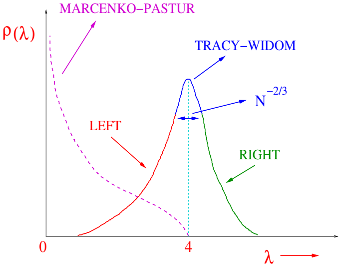

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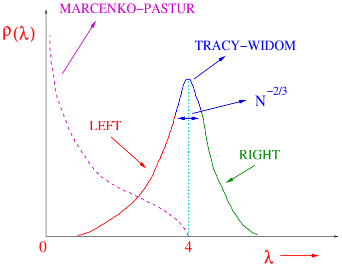

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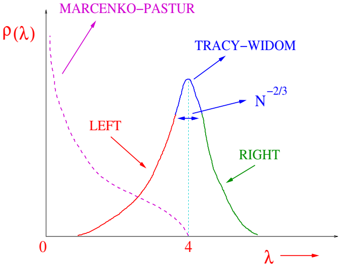

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- $\Psi_-(w)$ and $\Psi_+(w)$ $\to$ computed exactly

(Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009)

Exact Left and Right Large Deviation Functions

Using Coulomb gas + Saddle point method for large N:

- Left large deviation function:

$$\boxed{\Psi_-(w) = \ln\left[\frac{2}{w}\right] - \frac{w-4}{8} - \frac{(w-4)^2}{64} \quad w \leq 4}$$

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- Right large deviation function:

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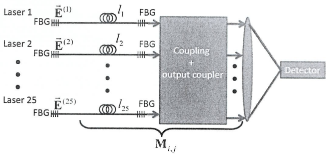
(S.M. and Vergassola, 2009)

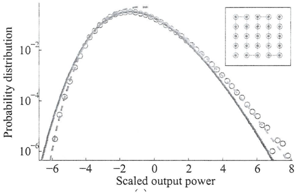
Experimental Verification with Coupled Lasers

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500, 000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.





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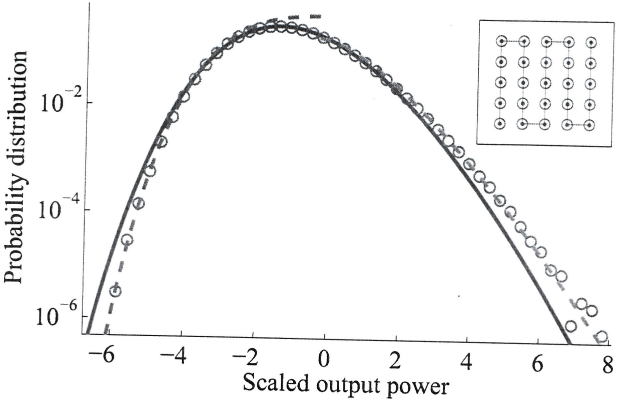
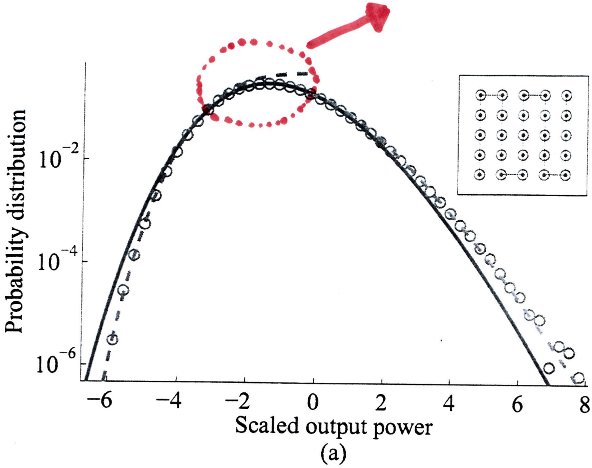


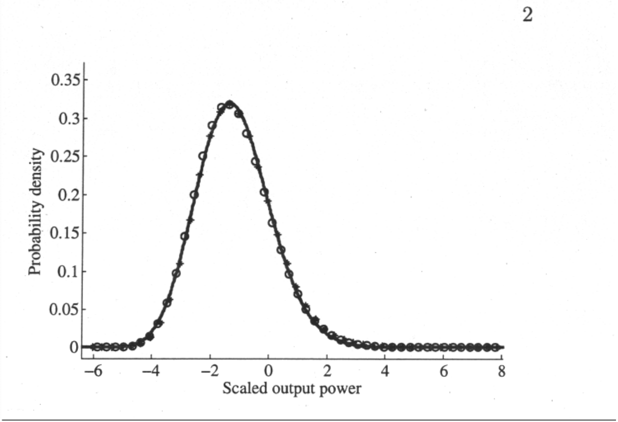
(a)

Experimental Verification with Coupled Lasers



Experimental Verification with Coupled Lasers

Fridman et. al. arXiv: 1012.1282



Other Problems with 3-rd Order Phase Transitions

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- Non-Intersecting Brownian Motions and Random Matrices

  $\rightarrow$ relation to 2-d Yang-Mills gauge theory

  Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008)

  Forrester, S.M. & Schehr, Nucl. Phys. B 844, 500 (2011)

Conclusions

- 3-rd order phase transition $\Rightarrow$ ubiquitous

  charge droplet with square-root edge hits a hard wall

- Close to the critical point (double-scaling) $\Rightarrow$ Painlevé-II

Collaborators

Students: C. Nadal (Oxford Univ., UK)

J. Randon-Furling (Univ. Paris-1, France)

Collaborators:

- O. Bohigas (LPTMS, Orsay, France)
- A. Comtet (LPTMS, Orsay, France)
- K. Damle and V. Tripathi (Tata Institute, Bombay, India)
- D.S. Dean (LPT, Toulouse, France)
- A. Lakshminarayan (IIT Chennai, India)
- A. Scardichio (ICTP, Trieste, Italy)
- G. Schehr (LPT, Orsay, France)
- S. Tomsovic (Washington State Univ., US)
- M. Vergassola (Institut Pasteur, Paris, France)
- P. Vivo (LPTMS, Orsay, France)
- Recent collaboration with G. Borot and B. Eynard (Saclay, France)

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Left Large Deviation: Beyond the Leading Order

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where explicit expressions for $\Phi_1(w)$, ϕ_β and $\Phi_2(\beta, w)$ were obtained recently (Borot, Eynard, S.M., & Nadal, JSTAT, P11024 (2011))

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For $\beta = 1$, 2 and 4

$\rightarrow$ agrees with Baik, Buckingham and DiFranco (2008)

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- As a bonus, our method also provides a 'simpler' derivation of TW distribution for $\beta = 2$ (Nadal and S.M., 2011)