AdS/CFT and (N=4) Scattering Amplitudes At Strong Coupling (SINP)

PLAN: Lec. 1: What is the AdS/CFT correspondence?
Statement of the duality for N=4 SYM. Dictionary between gauge theory and gravity (string theory). Where does the duality arise from? Checks of the duality.

Lec. 2: Application to scattering amplitudes in N=4 SYM at strong coupling.
BDS ansatz for n-gluon MHV scattering amplitude.
Check at strong coupling → minimal surface in AdS.
Dual conformal invariance and connection to mom. space Wilson loops.

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Lec. 1: What Is The AdS/CFT correspondence?

The AdS/CFT correspondence has been a hugely influential development in theoretical physics relating the physics of quantum fields (such as gauge theories) to quantum (or in the large N limit, classical) gravity. This is conceptually rather amazing since there is no obvious connection between the two topics. Moreover, it is potentially very useful in giving us a new angle on strongly interacting QFTs (which we have very few tools to deal with). That is part of the reason I am delivering lectures in this QCD school – for insights into scattering amplitudes at strong coupling and how it has even taught us some things at weak coupling. Besides, there has also been a strong influence on the physics of QGP (transport properties etc.) and in AdS/QCD.

I will restrict myself to stating facts about (and motivating) the most well studied example of this duality. Namely, that between the most symmetric 4d QFT – $\mathcal{N}=4$ Super Yang-Mills and IIB string theory on a ten (but for most purposes five) dimensional spacetime $AdS_5 \times S^5$. The claim is about a complete quantum equivalence of these two theories with a precise statement of the dictionary between the two sides. By now this duality has also been generalised to various other QFTs (some closer to QCD) and corresponding string theories.

So, first a bit about the theories on both sides of the duality.

$\mathcal{N}=4$ Super Yang-Mills is a generalisation of ordinary Yang-Mills theory. But this theory has additional fermionic symmetries. Thus in addition to A_μ^a, we also have $\Psi^a_{\alpha I}$ ($I=1..4$) - fermions and Φ^a_I ($I=1...6$) - all in the adjoint of the gauge group (say $SU(N)$).

The SUSY of the theory leads to it being a UV finite theory. The exact β-function $\beta(g_{YM})=0$. Thus we have a line of fixed points of the RG - very unusual behaviour for a 4d CFT. (labelled by g_{YM})

(We can consider the large N limit of the theory in which case the relevant coupling to be held finite is the 't Hooft combination $\lambda = g_{YM}^2 N$.) The theory thus has superconformal invariance. $\supset SO(4,2)$

IIB string theory is a supersymmetric (again maximal) quantum theory of gravity. When formulated in $AdS_5 \times S^5$ we have a background spacetime which is a product in which there is a background metric as well as a five form flux (gen. of 2 form EM flux) on both S^5 - five dim. sphere and AdS_5.

AdS_5 is the maximally symmetric Lorentzian signature space time which has <u>constant</u> <u>negative</u> curvature (like a hyperboloid).

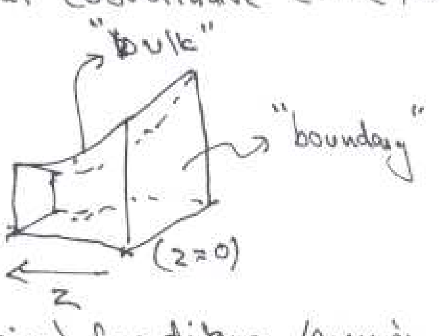
In suitable coordinates (Poincare) the metric on AdS_5 takes the form

→ 4-dim Minkowski space time

$$ds^2 = R^2 \left[\frac{\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2}{z^2} \right] \to \text{radial coordinate } (z \in (0,\infty))$$

radius of curvature.

The spacetime can be visualised as.



We can think of it as a 5 d spacetime built by foliating 4d mink. sp. time with an extra z direction ("warping") which sets the local scale (in the 5^{th} direction) for distances/energies.

There are two basic (dimensionless) parameters (g^2_{YM}, N) or (λ, N) in the gauge theory. There are also two basic parameters in the string theory. These are (R/l_s or $\sqrt{T}R$) and g_s or equiv. $G_N^{(5)}/l_5^3 \sim g_s^2$
(l_s: string scale; T: tension; g_s: string coupling)

The first statement of the equivalence is to establish a dictionary between these parameters. This reads as:

$$\boxed{\lambda \sim R^4/l_s^4 \qquad \text{and} \qquad \frac{\lambda}{N} \sim g_s.}$$

Notice some important facts: (a) At large N (λ fixed), $g_s \to 0$ or equivalently $G_N \to 0 \Rightarrow$ gravity ~~is~~ string theory is classical. Thus $1/N$ is a measure of quantum fluctuations in the bulk.

(b) $\lambda \gg 1 \Leftrightarrow$ R very large in string units $\Rightarrow$ curvature is small, we can further reduce (at large N) the classical string theory to classical gravity. (actually maximal supergravity)

(c) $\lambda \ll 1 \Leftrightarrow$ R small in string units – highly stringy but perturbative QFT!

Thus the regimes (b) + (c) are dual descriptions of inaccessible regions of parameter space of one or other side, in terms of simpler descriptions (classical GR/ pert. QFT).

The ~~[illegible]~~ map between the dynamics is of course the crucial element of the correspondence.

1) The gauge invt. operators e.g. $\mathrm{Tr}\, F^2_{\mu\nu}$, $\mathrm{Tr}[\Phi^k]$ $\Leftrightarrow$ fields in the string theory on $AdS_5 \times S^5$.
In particular, $T_{\mu\nu}$ (stress tensor) $\Leftrightarrow h_{\alpha\beta}$ (metric fluctuations), J_μ (global currents) $\Leftrightarrow A_\alpha$ (gauge)

Anomalous dimensions of operators ("spectrum" in a CFT) – $\Delta_i(\lambda)$ are given by the masses of the fields. (Planar limit $\Leftrightarrow$ classical string spec.)

Conserved currents (no anom. dim.) $\Leftrightarrow$ Gauge fields ("massless") in $AdS_5 \times S^5$.

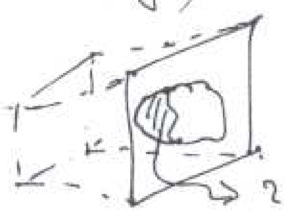
In the $\lambda \gg 1$ limit, most operators have large anomalous dimensions $\sim \sqrt{\lambda}$ and only those correspond to supergrav. modes survive. (have finite SUSY protected masses)

2) Correlation functions of gauge invariant operators $\leftrightarrow$ String amplitudes in AdS. In the $\lambda \gg 1$, planar limit where one has classical gravity

$$\underbrace{\left\langle e^{i\int J_A(x)\,\mathcal{O}_A(x)} \right\rangle_{SYM}}_{\text{operators} \leftrightarrow \text{supergrav. fields.}} \underset{(\lambda\gg 1)}{=} \int [\mathcal{D}\underbrace{\phi_A}_{\text{supergravity fields.}}]\, e^{i S^{(grav)}_{bulk}[\phi_A]}\Big|^{\text{saddle (i.e. classical)}}_{\phi_A \to J_A(x)}$$

This is a very powerful means to compute Corr. fns. in a strongly coupled field theory. It reduces to solving supergrav eqns w/ prescribed bdy. condns. ($\phi_A \to J_A(x)$) and evaluating the action on-shell. More generally, away from $\lambda \gg 1$ one would have to evaluate string amplitudes on AdS which we don't know how to do.

3) Wilson loops in the gauge theory are given at large λ by solving a minimal surface in AdS bounded by the loop.



2d world sheet

We need to find the Area of this surface. Then

$$\langle W_C[A..] \rangle = \left\langle e^{-S_{string}} \right\rangle\Big|_{\partial\Sigma = C}^{\text{saddle}}$$
$$= e^{-\underbrace{\frac{\sqrt{\lambda}}{2\pi} A_{min}}_{\substack{\text{String action} \\ \propto R^2.}}}$$

In $\mathcal{N}=4$ SYM since there is no dynamical confinement, the Wilson loops do not show area law behaviour – rather Coulomb ($1/r$) behaviour.

4) Quantities in the thermal gauge theory (Corr. fns, Wilson/Polyakov loops) etc. can also be computed – but now need to consider the background to be AdS-Schwarzschild BH geometry. Hawking temperature of BH $\longleftrightarrow$ temp. of gauge theory.

This is the set up used for studying the $\mathcal{N}=4$ version of QGP.

The origin of the AdS/CFT correspondence is the dual picture of objects called D-branes in string theory. These are extended defect like objects (solitons) in the string theory. They are of dim $(p+1)$ w/ p odd (≤ 9) in IIB theory. Thus, we have 1, 3, 5 branes etc.

The two pictures of D-branes are (a) as defects which carry a mass/unit volume and hence are black hole like solns. in the directions transverse to the directions of the $(p+1)$ dim worldvolume of the brane.
(b) can be described in terms of open strings (incl. gauge fields) living on this world volume.

There is a limit of these two descriptions for D3 branes $(p=3)$ where one has (a) $\rightarrow$
$AdS_5 \times S^5$. $\rightarrow$ world vol. + radial directions + (rest $\rightarrow S^5$)
(b) $\mathcal{N}=4$ SYM theory (from open strings) living on the D3 brane.

The equivalence of the two pictures (internal consistency of string theory) suggested the ~~also~~ AdS/CFT duality.

The duality has been checked in $\mathcal{N}=4$ SYM and other examples.
1) The spectrum of planar single trace ops. as a fn. of λ $\longleftrightarrow$ at large coupling w/ string techniques
2) Matching of various corr. fns.
3) Expected behaviour of SUSY Wilson loops which can be exactly computed in the QFT (localisation techniques)
4) Scattering Amplitudes.

Lec. 2: Indeed, Alday + Maldacena brilliantly captured the strong coupling $(\lambda \gg 1)$ physics of gluon scattering amplitudes in planar $\mathcal{N}=4$ SYM through a computation in AdS_5.
For the four gluon scattering amplitude they precisely reproduced the expectation from the BDS conjecture for such amplitudes. (including the various IR divergent pieces).

In the process, they also exploited a <u>dual</u> (super)conformal invariance (which had been independently noticed at weak coupling) which now seems to be one of the hidden symmetries of planar $\mathcal{N}=4$ SYM. Their method of solution by means of a Momentum Space Wilson loop computation ~~was~~ also led to people looking at <u>weak</u> coupling scattering amplitudes and realising they are related also to mom. space Wilson loops. It also led to important clarifications to the BDS ansatz which was shown to be a consequence of dual s.conf. invariance for 4 & 5 gluons. But it appears as if the ansatz is no longer true for 6 and higher gluons. It should be mentioned that interest in the BDS ansatz etc. stems from the fact that $\mathcal{N}=4$ SYM gluon scattering amplitudes give you a piece of the QCD gluon scatt. amp.

The colour ordered n-gluon (MHV) scatt. amplitude at L-loop order is IR divergent. $A_n^{(L)} \sim \frac{1}{\varepsilon^{2L}}$ → IR regulator $(4-2\varepsilon)$ dim. $(\varepsilon < 0)$

BDS essentially gave an ansatz (in terms of 1-loop amplitudes exponentiated) for the normalised

$$M_n^L(\varepsilon) = A_n^L(\varepsilon)/A_n^{(0)}$$ – depends only on kin. invts & ε.

In the case of 4 gluons, this took the form

ind. of kin.

$$M_4 = 1 + \sum_{L=1}^{\infty} (\lambda\mu^{2\varepsilon})^L M_4^{(L)} = A_{div,s} A_{div,t} \exp\left(\frac{f(\lambda)}{8}\left(\ln\frac{s}{t}\right)^2 + C\right)$$

Here $A_{div,s} = \exp\left(-\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right)$

Here. $f(x) = (x\partial_x)^2 f^{(-2)}(x)$ is the cusp anomalous dimension - an important quantity while. $(x\partial_x) g^{(-1)}(x) = g(x)$ is the collinear anomalous dimension and is the <u>same</u> fn. appearing in the finite part of the amplitude as well. It is known that for $\lambda \gg 1$, $f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \text{const} + O\left(\frac{1}{\sqrt{\lambda}}\right)$

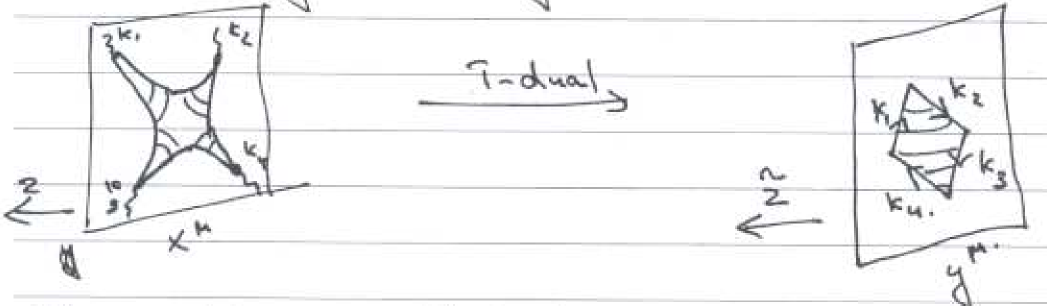
(At weak coupling $f(\lambda) = 8\left(\frac{\lambda}{16\pi^2}\right) - 16\zeta(2)\left(\frac{\lambda}{16\pi^2}\right)^2 + 176\zeta(4)\left(\frac{\lambda}{16\pi^2}\right)^3 + \ldots$

Alday + Maldacena argued that by using a "T-dual" set of variables for AdS_5 so that

$$ds^2 = R^2\left[\frac{\eta_{\mu\nu}dy^\mu dy^\nu + d\tilde{z}^2}{\tilde{z}^2}\right] \qquad (\tilde{z} = R^2/z)$$

(y^μ are T-dual to x^μ - exchanges position + momenta).

~~The string scattering calculation in AdS_5 corresp. to 4 gluon scatt.~~



The string worldsheet in the dual variables spans a closed loop on the $\tilde{z} = 0$ plane formed from the momenta k_i^μ concatenated with each other. (Mom. cons $\Rightarrow$ closed loop). Note that the sides of the loop are light like $(k_i^2 = 0)$.

In the $\lambda \gg 1$ limit the string amplitude is given by the saddle point surface (min. area surface) spanned by the bdy. loop. This was ingeniously calculated. There is a divergence from $\tilde{z} = 0$ (IR of the original theory) which can be regulated by either a bulk analogue of Dim. Reg. or by putting a cutoff at $\tilde{z} = \tilde{z}_1$

The former gives

$$A_4 = e^{iS}\Big|_{\text{min. area.}} \quad \text{where} \quad iS = 2iS_{div,s} + 2iS_{div,t} + \frac{\sqrt{\lambda}}{8\pi}\left(\ln\frac{s}{t}\right)^2 + \tilde{C}$$

where $$S_{div,s} = -\frac{1}{\varepsilon^2}\frac{1}{2\pi}\times\sqrt{\frac{\lambda\mu^{2\varepsilon}}{(-s)^{\varepsilon}}} - \frac{1}{\varepsilon}\frac{1}{4\pi}(1-\ln 2)\sqrt{\frac{\lambda\mu^{2\varepsilon}}{(-s)^{\varepsilon}}}$$

This is in precise agreement w/ the BDS ansatz with $f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + O(1)$ reproducing both the finite and the leading $\frac{1}{\varepsilon^2}$ IR pieces. It also gives a strong coupling prediction for the fn. $g(\lambda)$ $\approx \frac{\sqrt{\lambda}}{2\pi}\times(1-\ln 2)$

Note that the above calculation is identical to that of a Wilson loop (in the strong coupling limit) w/ light-like edges.

Inspired by this, people looked at the pert. calculation of light-like Wilson loops w/ separations $k_i^\mu = x_{i+1}^\mu - x_i^\mu$. They found that this reproduces pert. scattering amplitudes! This is a reflection of the dual s.conf. symmetry which through a Ward identity fixes the form of the scat. amplitudes for 4 + 5 gluons to the BDS ansatz while leaving that of 6 + higher gluons less constrained. It seems, in fact, that scatt. amp = Wilson loops $\neq$ BDS ansatz. (for any number n of gluons.)