DYNAMICAL VIEWPOINT
AND
GROUP REPRESENTATIONS

HRI

September 2-12, 2010

Prof. Ravi S. Kulkarni

Manoj and Siddhartha have arranged this excellent meeting. Special thanks to Jürgen, who not only delivered excellent lectures on classification of finite simple groups, but also managed to get together some of the experts in the world on computational group theory. Let us give all of them a big hand.

In my experience, this is the first meeting of its kind in India. I hope, many of the visitors from abroad will come again, and the several aspects of finite simple groups and computational group theory and their applications will actively develop in India.

Another notable feature of this meeting was the team-spirit that the organizers and participants exhibited. Indeed, classification of finite simple groups, and computational group theory are two of the best examples of team spirit in the history of mathematics.

I am hardly a poet. But to congratulate the team spirit, I thought of a poem. I have borrowed the first two lines and the last line from Buddha's supposedly first lecture delivered at Saarnath, which you visited, where he enunciated his life's philosophy. In the middle, I have inserted something that eulogizes the team spirit, that is desirable in the community of mathematicians.

We are what we think,
With our thoughts, we create our worlds.
Hit upon the right thoughts,
and our lives are easy,
Not so right thoughts,
and our lives are messy.

But alas! our lives are far too fleeting,
Takes many lives, and many minds,
to come to a meeting,
To reach the Truth, and its Understanding
That is the Golden Rule!

Now I start with the mathematical theme of my lecture. It is **the dynamic viewpoint**.

There are many representation theories. Even just for finite groups, we have actions on finite sets, finite graphs, compact surfaces of finite genus,..., finite-dimensional linear representations. For Lie groups, there are finite and infinite dimensional real and complex representations. Then there are representation theories of associative algebras, Lie algebras, Jordan algebras,... . But behind all these representation theories, there is a common, conscious or subconscious theme of dynamics. It is actually elementary like Euclid's basic notions, but often not articulated.

I have summarised this viewpoint in two definitive papers:

1)Dynamical Types and Centralizers—- Jour. Ramanujan Math. Soc. (2007) vol. 22, pp. 57-74.

2)Dynamics of linear and affine maps—- Asian Journal of Math(2008) vol.12, No. 3,pp. 321-344.

This viewpoint has developed gradually, as I have myself developed in mathematics. This talk may be taken as a light-hearted introduction to these two papers.

1 Dynamical Viewpoint

Let X be a set.

Definition 1.1. *A dynamical system is a function*

$$f : X \to X.$$

Note: Domain f = Codomain f. So f may be iterated.

$$f^0 = Id_X, f^1 = f, f^2 = f \circ f, f^3 = f \circ f \circ f, ..., f^n = \underbrace{f \circ f \circ f ... \circ f}_{n \text{ times}}.$$

Definition 1.2. *Forward orbit*

$$O_f^+(x) = \{x, f(x), f^2(x), ...\}$$

Definition 1.3. *Full orbit*

$$O_f(x) = \{..., f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), ...\}$$

Note that in the definition of full orbit, ..., $f^{-2}(x), f^{-1}(x)$ may not be single elements, but subsets of X.
Then

$$X = \bigcup O_f(x),$$

a partition of X i.e. $\forall x, y \in X$ either

$$O_f(x) = O_f(y) \text{ or } O_f(x) \bigcap O_f(y) = \emptyset$$

Dynamically important notions

Definition 1.4. *Fixed point of $f : x \in X$ such that $f(x) = x$.*

Definition 1.5. *Periodic point of $f : x \in X$ such that for some $n \in \mathbb{N} := \{1, 2, ...\}$.*

Definition 1.6. *Invariant subsets : $Y \subseteq X$ is f-invariant if $f(Y) \subseteq Y$.*

• Equivalence
Let

$$f : X \to X, \; g : Y \to Y$$

be two dynamical systems.

Definition 1.7. *f is equivalent to g if there exists a bijection $h : X \to Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes.

In particular, let $X = Y$. Still, we do not ask that $h = id_X$, but allow h to be any bijection: $X \to X$. Let $\mathcal{M}_X = \{f :\to X \mid f \text{ a dynamical system}\}$. The above equivalence arises from a group-action. Then $\mathcal{M}_X$ is a magma, i.e., a semigroup with identity, under composition of functions, and $\mathcal{S}_X = \{f \in \mathcal{M}_X \mid f\text{is a bijection }\}$ is a group.
Then $\mathcal{S}_X$ acts on $\mathcal{M}_X$ as follows:

$$f \in \mathcal{M}_X \qquad\qquad h \in \mathcal{S}_X$$

Then

$$h \bullet f = h \circ f \circ h^{-1}.$$

Thus, $f, g \in \mathcal{M}_X$ are equivalent if and only if they are in the same orbit of $\mathcal{S}_X$ acting on $\mathcal{M}_X$.

The $\mathcal{S}_X$-action on $\mathcal{M}_X$ may be taken in the active sense, then we get the notion of "similarity (equivalence) of dynamical systems", or in a passive sense, i.e., an element in $\mathcal{S}_X$ is a "renaming" of a points of X , or a "change of co-ordinates" on X. The latter viewpoint is of importance in geometry or physics, where we wish to get invariants of a "space" independent of the auxiliary co-ordinatization of the space, which we may (have to) introduce to do (local) computations.

When f, g are actually in $\mathcal{S}_X$(which is a group), then their equivalence, i.e. similarity as dynamical systems, amounts to their conjugacy in $\mathcal{S}_X$.

Thus, the basic notion of conjugacy in groups has dynamic origin, so also are the notions of "centralizers" and "normalizers", as we shall see.

• Now let X be a set equipped with a structure σ. (Here, I do not wish to go into the set-theoretic issue of "what is a structure" on X.)
Let $\mathcal{S}_{(X,\sigma)} = \{f \in \mathcal{S}_X \mid f \text{ preserves } \sigma\}$.

Suppose also that we have a notion of "f preserves σ" for $f \in \mathcal{M}_X$.

Example 1.8. *Let the structure σ be a topology τ on the set X namely, τ is a set of subsets of X such that*

i) $X \in \tau, \emptyset \in \tau$

ii) arbitrary union of elements of τ is also an element of τ

iii) finite intersection of elements of τ is also an element of τ.

Then

"$f \in \mathcal{S}_X$ preserves σ" $\equiv f$ is a homeomorphism.

However it is up to us what we should mean by "$f \in \mathcal{M}_X$ preserves σ." We could take "$f \in \mathcal{S}_X$ preserves σ (i.e. τ)" to mean f carries "open sets" to "open sets". But I think, that you will agree that this is not a wise choice. Instead, "$f \in \mathcal{M}_X$ preserves τ "to mean "f^{-1}(open set) = open set" is a wise choice!

So suppose that we have a notion of when $f \in \mathcal{M}_X$ preserves σ.

Then let $\mathcal{M}_{(X,\sigma)} = \{f \in \mathcal{M}_X \mid f \text{ preserves } \sigma\}$. Again, we say

Definition 1.9. *$f, g \in \mathcal{M}_{(X,\sigma)}$ are equivalent if there exists $h \in \mathcal{S}_{(X,\sigma)}$ such that $h \circ f = g \circ h$.*

Again, as before, this equivalence relation arises from a group action:

$\mathcal{S}_{(X,\sigma)}$ acts on$\mathcal{M}_{(X,\sigma)}$ by $h \bullet f = h \circ f \circ h^{-1}$, and f and g in $\mathcal{S}_{(X,\sigma)}$ are equivalent iff they are in the same orbit of $\mathcal{S}_{(X,\sigma)}$.

• Thus , given (X, σ), a basic issue is : "understand" the abstract set $\mathcal{S}_{(X,\sigma)} \diagdown \mathcal{M}_{(X,\sigma)}$
Technically, the "understanding" is termed as:
"Find good parametrizations of $\mathcal{S}_{(X,\sigma)} \diagdown \mathcal{M}_{(X,\sigma)}$ "

Definition 1.10. *Parametrization := setting a bijection of $\mathcal{S}_{(X,\sigma)} \diagdown \mathcal{M}_{(X,\sigma)}$ (the abstract set) with a "known" set.*

The known set is often expressed in terms of "numbers", but it need not be!
We shall soon illustrate this.

2 Representation of a magma

Let $\mathcal{M}$ be an abstract magma, i.e., a semigroup with identity. For example, $\mathcal{M}$ could be an abstract group.

An action of $\mathcal{M}$ on a set X is, as usual, a function

$$\varphi : \mathcal{M} \times X \to X$$

satisfying

$i)$ $\varphi(m_1, \varphi(m_2, x)) = \varphi(m_1 m_2, x) \qquad \forall m_1, m_2 \in M, x \in X$

$ii)$ $\varphi(1, x) = x \qquad \forall x \in X.$

Equivalently, we have a representation of $\mathcal{M}$ on X, i.e. a homomorphism

$$\varphi_* : \mathcal{M} \to \mathcal{M}_X.$$

Of course $\varphi_*(\mathcal{M})$ is a sub-magma of $\mathcal{M}_X$.

Why do we consider abstract $\mathcal{M}$ rather than concrete $\varphi_*(\mathcal{M})$? Historically, Jordan, Klein, Lie, ... considered only "transformation groups", i.e., $\varphi_*(\mathcal{M})$'s. This is an important cognitive development on how we should formulate mathematical theories. Under the cogent advocacy of H. Weyl, (among others), "symmetry" is now considered as a category of mathematical thought, in addition to the age-old categories of mathematical thought, namely "space" and "number". David Mumford has called the passage such as from $\varphi_*(\mathcal{M})$ to $\mathcal{M}$, a reification process.

Some of the reasons, why we consider $\mathcal{M}$, rather than just $\varphi_*(\mathcal{M})$ are:

A) The same $\mathcal{M}$ can act on the same X by different ways - i.e. "φ" is an important ingradient of "$\varphi_*(\mathcal{M})$".

B) φ may not be injective. So $ker\varphi$ exists in $\mathcal{M}$, but does not show up in $\mathcal{M}$'s-action on X.

C)$\mathcal{M}$ may be injective, but $Y \subseteq X$ may be an invariant subset of X, so $\mathcal{M}$ acts on Y as well, but $\varphi\mid_Y$ may not be injective.

Thus, there arise a whole new set of problems. Instead of starting with (X, σ), and asking for the understanding of $\mathcal{S}_{(X,\sigma)} \diagdown \mathcal{M}_{(X,\sigma)}$, we ask: given $\mathcal{M}$, understand it through its actions - i.e. understand its representations as various X's - i.e. understand $Hom(\mathcal{M}, \mathcal{M}_X)$, with $\mathcal{M}$ fixed, and X varying.

For this understanding, it is important to introduce appropriate equivalence relations on $Hom(\mathcal{M}, \mathcal{M}_X)$.

There are two natural choices of equivalence relations. First note that

$\mathcal{S}_X$ acts on $Hom(\mathcal{M}, \mathcal{M}_X) : u \in \mathcal{S}_X, f : \mathcal{M} \to \mathcal{M}_X$, then $(u \bullet f)(g) = u \circ f(g) \circ u^{-1}$, where $\circ$ denotes composition in $\mathcal{M}_X$.

Let $\mathcal{S}_{\mathcal{M}}$ denote the group of all semi-group automorphisms of $\mathcal{M}$. Then $\mathcal{S}_{\mathcal{M}}$ also acts on $Hom(\mathcal{M}, \mathcal{M}_X) : u \in \mathcal{S}_{\mathcal{M}}$, then $u \bullet f = ufu^{-1}$.

Thus $\mathcal{S}_{\mathcal{M}} \times \mathcal{S}_X$ acts on $Hom(\mathcal{M}, \mathcal{M}_X)$.

First Equivalence Relation

Consider only the action of $\mathcal{S}_X$, on $Hom(\mathcal{M}, \mathcal{M}_X)$, i.e., $f, g \in Hom(\mathcal{M}, \mathcal{M}_X)$ are equivalent if and only if they are in the same $\mathcal{S}_X$- orbit. This is the usual equivalence relation of (Frobenius's) linear representation theory of groups.

Second Equivalence Relation

Consider the action of $\mathcal{S}_{\mathcal{M}} \times \mathcal{S}_X$ on $Hom(\mathcal{M}, \mathcal{M}_X)$ and the corresponding equivalence relation. Without going into further details, let me only mention, that this equivalence relation is important for Riemann's moduli problem of Riemann surfaces, which Gareth briefly mentioned.

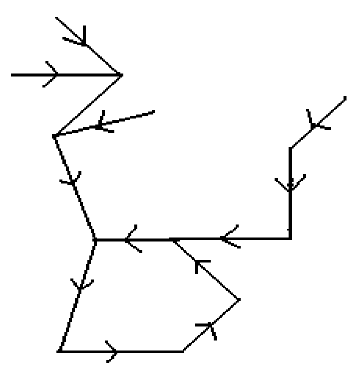
3 Understanding $\mathcal{S}_X \diagdown \mathcal{M}_X$, Permutation representations

Let X be a finite set, $\mid X \mid = n$. Let $f : X \to X$ be an element of $\mathcal{M}_X$.
Graph of $f := G(f)$.

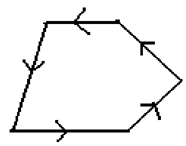
Let the vertex-set be X, and for $x, y \in X$ such that $f(x) = y$, let there be a directed edge $x \longrightarrow y$. Thus we get a labelled directed graph, which we denote by $G_0(f)$. Let $G(f)$ denote the underlying un-labelled directed graph.

Note: $\mid \mathcal{M}_X \mid = n^n$.

$G(f)$ may have several components. Each component looks like



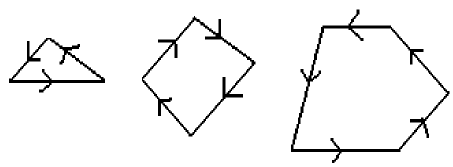
(The circuit



may reduce to a point).

Clearly $\mathcal{S}_X \diagdown \mathcal{M}_X$ is parametrized by the isomorphism-type of such graphs, i.e., f is equivalent to g if and only if $G(f)$ is isomorphic to $G(g)$ as directed graphs.

In the special case where $f \in \mathcal{S}_X, G(f)$ has the form



This amounts to the basic statement one learns in algebra-courses: two elements in $\mathcal{S}_X$ are conjugate if and only if they have the same cycle-type.

• Let W={0,1,2,3,...} be the semigroup of whole numbers where the binary operation is addition. The action $\varphi : W \times X \to X$, or what is the same thing representations $\varphi_* : W \to \mathcal{M}_X$ is determined by $f := \varphi_*(1)$. Thus,

$$Hom(W, \mathcal{M}_X) \approx \mathcal{M}_X$$

$$\mathcal{S}_X \diagdown Hom(W, \mathcal{M}_X) \approx \mathcal{S}_X \diagdown \mathcal{M}_X$$

•Let G be a group. Then $Hom(G, \mathcal{M}_X) = Hom(G, \mathcal{S}_X)$.
For $\alpha \in Hom(G, \mathcal{S}_X)$ attach a graph $\Gamma(\alpha)$ as follows:

$$\text{Vertex set} := X$$

For $x, y \in X$ there exists an edge $x \to y$ if and only if there exists $g \in G$ such that $g(x) = y$. This gives a labelled graph $\Gamma_\circ(\alpha)$. The underlying unlabelled graph is $\Gamma(\alpha)$.
The graph $\Gamma(\alpha)$ may not be connected.

Definition 3.1. *α is indecomposable* $:= \Gamma(\alpha)$ *is connected.*

Usual terminology: G acts transitively on X).

• As is standard, the abstract set of {indecomposable G-actions} is parametrized by conjugacy classes of subgroups of G.

Definition 3.2. *A transitive permutation group is called primitive if the corresponding conjugacy class of subgroups consists of maximal subgroups.*

What is the "dynamic" formulation of this notion?

A G- set is a set with a G-action.

Definition 3.3. *An indecomposable G-set is one where the corresponding action is indecomposable (transitive).*

Definition 3.4. *A morphism of two G-sets X, Y is a map $f : X \to Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes for all $u \in G$.

Let $Hom_G(X, Y) = Hom(X, Y)^G$ under the G-action on $Hom(X, Y)$ by $u \in G$, $f : X \to Y$ then $(u \bullet f)(x) = u(f(u^{-1}x))$.

Definition 3.5. *An indecomposable G-set X is irreducible if for all indecomposable G-sets $Y, f \in Hom_G(X, Y), f$ is either injective or "trivial".*

An irreducible G-set is the same as a primitive permutation group.

4 Actions on Surfaces and Graphs

Historically, the problem of finite-group-actions on compact Riemann surfaces arose in connections with Riemann's cryptic remark: "the set of complex structures on a compact orientable topological surface of genus $g \geq 2$ has $3g - 3$ complex moduli."

[By Riemann mapping theorem, the sphere (a surface of genus 0) has a unique complex structure, and a torus (a surface of genus 1) has 1- complex parameter family of $\mathbb{C}$ -structures which Gareth described.]

Riemann's remark has been made precise in the following form: the set of $\mathbb{C}$ -structures on a compact, orientable surface of genus $g \geq 2$ can be made into a $3g$-3 $\mathbb{C}$- dimensional normal, quasi-projective complex algebraic variety. Its singular locus consists of $\mathbb{C}$-structures which admit non-trivial automorphisms. The automorphism group of a $\mathbb{C}$-structure on a compact orientable surface of genus $g \geq 2$, is always finite. Conversely, given an orientable surface X, and a finite group G acting continuously on X, there always exists a complex structure on X which is invariant under G.

Except for the last remark, Gareth described this story. But because of the last remark, we can forget about complex structures altogether! One may proceed purely in the topological category.

One may pose the following natural question:

$(*)$**Problem**: Given a compact orientable surface X, understand all actions by finite groups on X, preserving orientation.

In other words, more precisely we are asking for all conjugacy classes of finite subgroups of the group of orientation-preserving homeomorphisms of X.

In a seminal paper, Hurwitz(1892) explained the underlying topological aspects - in particular, the (branched) covering space theory, and obtained the famous upper bound for the order of a finite group G action on a compact, orientable surface of genus $g \geq 2$:

$$\mid G \mid \leq 84(g-1).$$

Later Wiman(1897) solved $(*)$ for $g = 2$. For $g = 3$, see Weaver (1995), Broughton( ). The lists grow rapidly with g, but do not show any pattern. Also it is easy to make errors. Computers can help! It is desirable to develop some GAP-package to develop these lists.

Also$(*)$ asks: given X, what are the G's? Now, knowing more about finite groups, it is better to ask

$(**)$ Given G, what are the X's on which it acts, and what are the actions?

Theorem 4.1. *Given G, there exists a natural number N such that*

1) If G acts on a compact, orientable surface of genus g, then $g \equiv 1 \mod (N)$.

2) If $g \equiv 1 \mod (N)$, then G does act on a compact orientable surface of genus g, except for finitely many exceptions.

Here, the natural number N is computable in terms of the Sylow p-group structure of G, and the finitely many exceptions involves solving certain equations in groups. There are only finitely many finite simple groups which can act on $g's$ so that $g-1$ has a given 2-adic part. This is an analogue of the Brauer-Suzuki theorem that there are only finitely many finite simple groups in which the order of the centraliser of an involution is bounded.

The question of finite group-actions on graphs is closely related to the question of finite group-actions on surfaces.

Here is an analogue of Hurwitz's "84(g-1)-theorem".

Theorem 4.2. *Let Γ be a finite connected graph without vertices of valence 1, and cycle-rank $r \geq 2$. Let G be a group acting effectively on Γ such that no $g \in G - \{e\}$ fixes any edge pointwise. Then*

1)$\mid G \mid \leq 6(r-1)$.

2)If $\mid G \mid = 6(r-1)$,then Γ is a trivalent graph, and G is isomorphic to a quotient of $\mathbf{PSL_2}(\mathbb{Z})$ *by a torsion-free normal subgroup of finite index.*

There is a lot of work which remains to be done in this area. There is a deep connection with the area of integral representations of finite groups.

5 Linear representations

Let $\mathbb{F}$ be a field , and V= a vector space over $\mathbb{F}$, dim $V = n < \infty$. V may be regarded as a set X with a certain structure σ. So the first basic issue:

Problem Understand

$$\mathcal{S}_{(X,\sigma)} \diagdown \mathcal{M}_{(X,\sigma)} = \mathrm{GL}(V) \diagdown \mathrm{End}(V).$$

This is the concern of the theory of "rational canonical form", which is a high point of Linear Algebra. Closely related to this chapter is the Frobenius's theory of centralizers of operators. At present, these two chapters of Linear Algebra hang somewhat separately. A "dynamical perspective" brings them together more closely.

Let $T : V \to V$ be a linear operator. (Equivalently, it is a representation of the semigroup $\mathbb{W}$, of whole numbers in End V.)

Definition 5.1. *V is T-indecomposable if V is not a direct sum of proper T-invariant subspaces.*

Definition 5.2. *V is T-irreducible if there does not exist a proper T-invariant subspace.*

Definition 5.3. *V is T-cyclic if there exist a $v \in V$ such that V = span $\{T^k(v) \mid k \in \mathbb{W}\}$*

Definition 5.4. *V is completely reducible with respect to T if V is a direct sum of T-irreducible subspaces.*

The theory of rational canonical forms attaches to the dynamical system (V, T) the following polynomial invariants:
$\chi_T(x) =$ the characteristic polynomial of T,
$m_T(x) =$ the minimal polynomial of T,
the invariant factors, or elementary divisors of T.

Interestingly,
1) V is T-indecomposable if and only if $\chi_T(x) = m_T(x) = p(x)^d$ where $p(x)$ is a monic irreducible polynomial.($p(x)^d$ is the only elementary divisor of T.)
2)V is T-irreducible if and only if $\chi_T(x)$ is irreducible (as a polynomial).
3)V is T-cyclic if and only if $\chi_T(x) = m_T(x)$.
4)V is completely reducible if and only if $m_T(x)$ is a product of distinct, monic, irreducible polynomials.

Moreover, the highpoint:

(∗) Invariant factors (or elementary divisors) determine T up to similarity.

Let
$F[T]$= the polynomials in T
$Z[T] = \{S \in \text{End } V \mid ST = TS\}$
$Z^*(T) = Z[T] \bigcap \text{GL}(V)$.

In the usual approach, we derive (∗) regarding V as a module over $F[T]$. It is better to regard V as a module over $Z[T]$(or $Z^*(T)$). This brings together the two chapters of rational canonical forms, and Frobenius's theory of centralizers of operators.

First, let

$$m_T(x) = \prod_{i=1}^{k} p_i(x)^{d_i},$$

$p_i(x)$ monic, irreducible, and for $i \neq j, p_i(x) \neq p_j(x)$.
Let $V_i = kerp_i(T)^{d_i}$. By the Chinese Remainder Theorem,

$$V = \bigoplus V_i$$

(internal direct sum). This decomposition is invariant under $Z[T]$(not just $F[T]$).

Theorem 5.5. *Assume $m_T(x) = p(x)^d$ where $p(x)$ is monic, irreducible.Then*

1) $Z(T)^*$- has finitely many orbits in its actions on V.

2) Each $Z(T)^*$-orbit is of the form $W_2 - W_1$ where W_1, W_2 are subspaces of V.

3) There is a canonical, maximal, $Z(T)$-invariant flag of subspaces

$$0 = W_0 \subset W_1 \subset W_2 \subset ... \subset W_r = V$$

such that

i) $W_{i+1} - W_i$ is a $Z(T)^*$-orbit, $i = 0, 1, , 2, ..., r - s$,

ii) Each W_i is irreducible with respect to $Z[T]$ (or $Z(T)^*$).

An important remark: Let $ch : \mathbb{W} \to F$

$$k \to \text{trace } T^k$$

Theorem 5.6. *Suppose V is completely reducible with respect to T. Then ch determines T up to similarity.*

Let $\mathcal{M}$ be an abstract magma, and $\alpha : \mathcal{M} \to \text{End } V$ a representation.
Let $ch\ \alpha : \mathcal{M} \to F$

$$u \to \text{trace } \alpha(u).$$

Theorem 5.7. *Suppose V is completely irreducible with respect to α. Then ch determines α upto equivalence.*

cf. Van der Waerden: Modern Algebra.II, Ch.14, Theorem 14.4.

(Usually, in the texts on representation theory of finite groups, one proves "character determines the representation" under the hypothesis characteristic of F does not divide the order of G and invoking Maschke's theorem. However this is a general fact regarding completely reducible representations of magmas.)

For further details, and the dynamic viewpoint, on the results mentioned in this section see: "Dynamics of Linear and Affine Maps", Asian J. of Math (2008)

This paper was a sequel to an earlier paper :

Dynamical Types and Centralizers ..... Ramanuman Math Soc. (2007).

I would end my talk by briefly drawing your attention to this paper. The main theorem 2.1 of this paper shows how the notions like centralizer, normalizer, "Weyl group",... arise naturally in a dynamic set-up.

Let a group G act on a set X. Then

(*) $\qquad X = \bigcup_{x \in X} G(x)$

is a partition of X.

There is actually a second partition of X, which is coarser than (*), but which carries additional structure.

For $x, y \in X, x \sim_o y$ if the orbits $G(x), G(y)$ have the same type, i.e., the corresponding two conjugacy classes of subgroups are the same.

If x, y are in the same orbit then, of course, $x \sim_o y$. But not (necessarily) conversely.

Let

$$R(x) = \{y \in X \mid y \sim_o x\}.$$

Then

(**)

$$X = \bigcup R(x)$$

is the second partition of X.

The interesting point is that $R(x)$ has two structures of a (set-theoretic) bundle!

Let

F_x = the fixed point set of G_x

$F_x^{'}$ = the fixed points y of G_x such that $G_x = G_x$.

Intuitively, $F_x^{'}$ is the set of "generic" fixed points of G_x.

Let N_x = the normalizer of G_x $W_x = N_x/G_x :=$ the "Weyl group" at x.

Then W_x has two canonical <u>free</u> actions on both G/G_x and $F_x^{'}$. This leads to the diagonal action of W_x on $G/G_x \times F_x^{'}$.

Theorem 5.8. *There exist a canonical bijection*

$$W_x \diagdown (G/G_x \times F_x^{'}) \longrightarrow R(x).$$

The two fibrations of $R(x)$ arise from i) projecting $R(x)$ onto G/N_x, and ii) projecting $R(x)$ onto "a fundamental domain of the W_x-action on $F_x^{'}$", by using the above bijection of $R(x)$ with $W_x \diagdown (G/G_x \times F_x^{'})$.

Apply this to any action you like. I promise you, (most of the times) you will be happy !

♡ ♡ ♡