Unusual scaling in long-ranged quantum walks

Parongama Sen

Department of Physics
University of Calcutta, Kolkata, India

Discrete classical and quantum walks

Discrete classical walk in one dimension

$$x_{t+1} = x_t + \xi_t$$

ξ's are independent random variables.

Probability that the walker is at a position x at time t

$$G(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right).$$

Simple random walk (SRW): $\xi = \pm 1$ with equal probability.
As if a translation to the right or left (R/L) following the toss of a coin.
$\implies$ Internal (coin) state dictates the direction (R/L)
Position coordinates only degree of freedom

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Quantum walk: Additional coin degree of freedom (chirality)

Simply stated: Instead of measuring the coin state after a toss, a superposition of states with head and tail ('left' and 'right' chirality) is allowed.

Flipping of a coin replaced by a unitary transformation on the coin state: the coin state then dictates the translation.

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Classical walk: Scaling behaviour (unbiased):
$\langle x \rangle = 0; \quad \langle x^2 \rangle - \langle x \rangle^2 \propto t.$
Quantum walk:
$\langle x^2 \rangle \propto t^2$

Quantum walk: some details

Discrete quantum walk in one dimension: the state of the walker is expressed in the $|x\rangle \otimes |d\rangle$ basis, where $|x\rangle$ is the position (in real space) eigenstate and $|d\rangle$ is the chirality eigenstate (either left ($|L\rangle$) or right ($|R\rangle$)).
The state of the particle: $\psi(x,t)$

$$\psi(x,t) = \left[\begin{array}{c} \psi_L(x,t) \\ \psi_R(x,t) \end{array} \right]$$

Hadamard coin used for the rotation:

$$H = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]$$

Rotation:

$$\boxed{\begin{array}{l} H|R\rangle = \frac{1}{\sqrt{2}}[|R\rangle + |L\rangle] \\ H|L\rangle = \frac{1}{\sqrt{2}}[|R\rangle - |L\rangle] \end{array}}$$

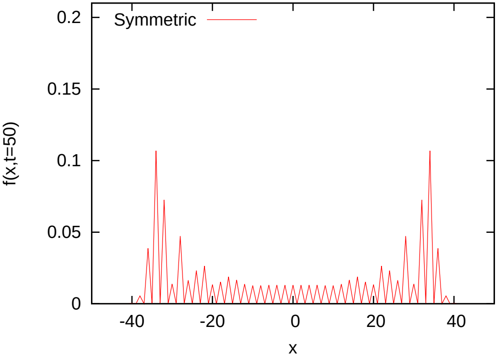
Translation:

$$\boxed{\begin{array}{l} T|x\rangle|L\rangle \rightarrow |x-\ell\rangle|L\rangle \\ T|x\rangle|R\rangle \rightarrow |x+\ell\rangle|R\rangle \end{array} .}$$

ℓ is a constant in the conventional quantum walk

The occupation probability of site x at time t is given by
$f(x,t) = |\psi_L(x,t)|^2 + |\psi_R(x,t)|^2$
Sum of these probabilities over all x is 1 at each time step.

Probability profile in a quantum walk



- Localized initial states: $|0\rangle \otimes (\alpha|R\rangle + \beta|L\rangle)$
- Ballistic peaks at $x \approx \pm t/\sqrt{2}$
- Range R: $R/t \approx \sqrt{2}$

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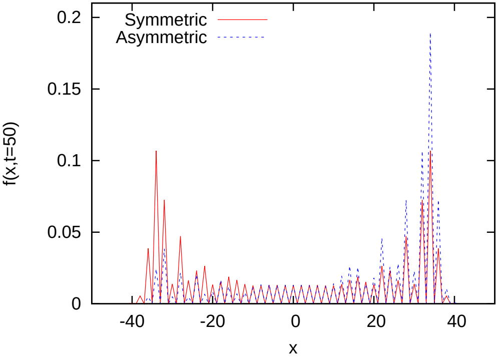

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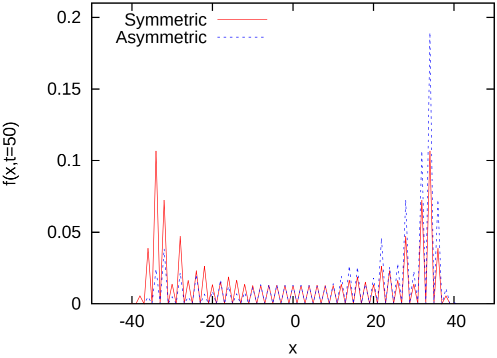

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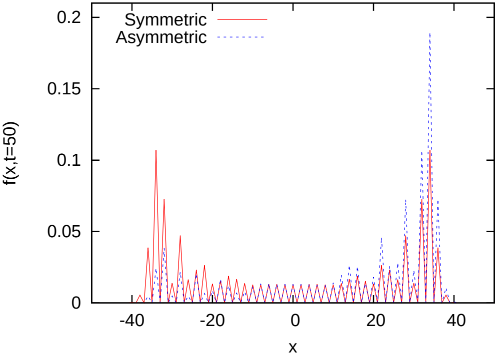
Constructive/destructive interferences can occur at every point making the walk distinct from the classical walk.



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Quantum walk: disorder induces localization/diffusive behaviour.

In earlier works, such disorder incorporated through the coin operator in most cases.

Random long range steps in quantum walk

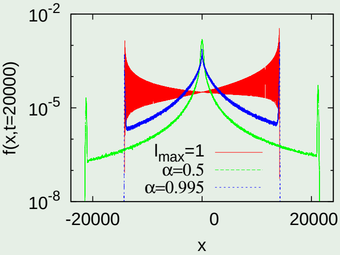
Consider translations in general by a length $\ell \geq 1$ chosen randomly at each step.

PS 2019 Physica A **514** 266; published online 2018 Oct

Binary choice

$$P(\ell) = \alpha\delta(\ell - 1) + (1 - \alpha)\delta(\ell - 2^n)$$

$\alpha = 0, 1$ - usual results (trivial scale factor). $\ell_{max} = 2^n$.



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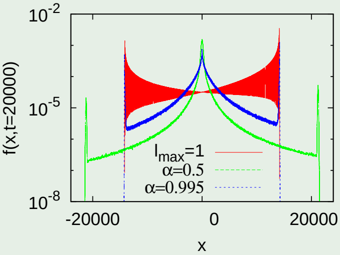
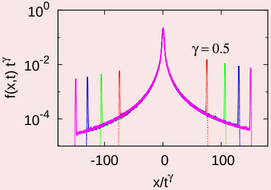
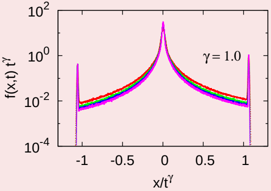


Scaling: $\alpha = 0.5$





Even for α very close to 1 (or 0), the same scaling behaviour found.

Moments

Approximate form of distribution:

$$f(x,t) = a_1 \frac{1}{t}\delta(x - ct)$$

$$+a_2 \frac{1}{t}\delta(x + ct) + a_3 \frac{1}{\sqrt{t}}\delta(x).$$

such that

$$\langle x \rangle = t/(b_1 + b_2\sqrt{t})$$

and

$$\langle x^2 \rangle = t^2/(b_3 + b_4\sqrt{t}),$$

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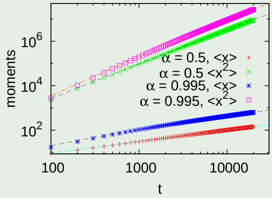
The above forms of $\langle x \rangle$ and $\langle x^2 \rangle$ are consistent with the numerical results.

Asymptotically
$\langle x \rangle \propto t$, $\langle x^2 \rangle \propto t^{3/2}$

Moments with $\ell = 1$ or 2



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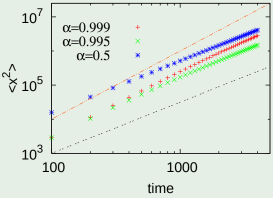
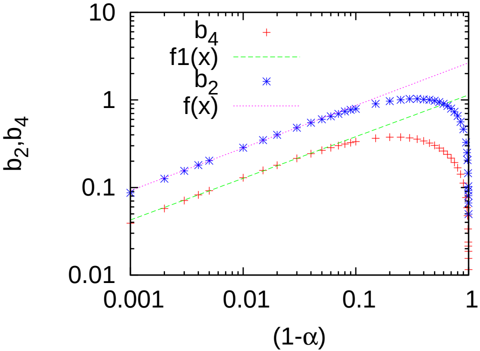


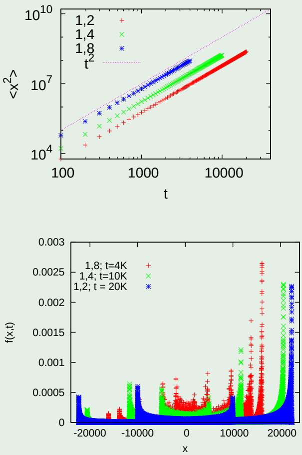
Decoherence parameters: b_2 and b_4



The decoherence parameters vanish in a power law form near $\alpha = 1$ and 0 with an exponent ≈ 0.5.

Non-random long range steps

What if step lengths are not unique but do not vary in a random manner?
Let step lengths be periodic: 1, 2, 1, 2.... etc.



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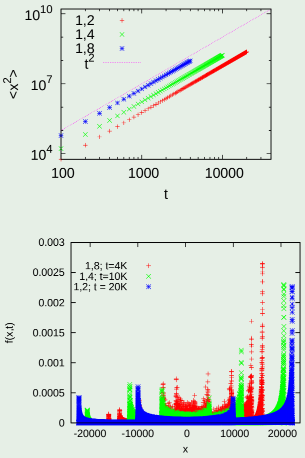

- Scaling behaviour same as in the quantum walk without disorder: $\langle x^2 \rangle \propto t^2$

- Probability density differs: as if a super-imposition of two walks with two different step lengths.

Conclusion: Randomness plays the key role in altering the scaling behaviour

Universal behaviour for random long ranged quantum walks:

S Das et al (arXiv:1806.04024): random step lengths chosen from Poissonian and other exponential distributions (truncated) for integer step lengths 1, 2, ...
Similar scaling: $\langle x^2 \rangle \propto t^{1.5}$ independent of distribution.

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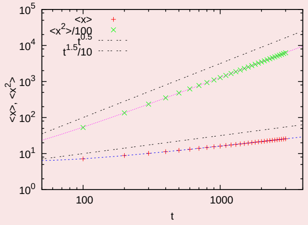
Quantum Levy walk (truncated)

PS 2019 ongoing work

Step lengths chosen from a fat tailed distribution

$$\boxed{P(\ell) = A\ell^{-1-\delta}}$$

$1 \leq \ell \leq \ell_{max}$



Scaling behaviour: $\langle x^2 \rangle \propto t^{3/2}$ asymptotically once again!
This result is independent of ℓ_{max} and δ.

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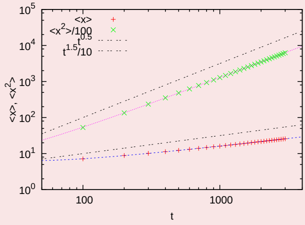

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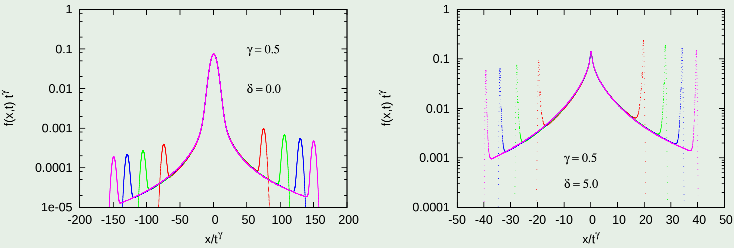
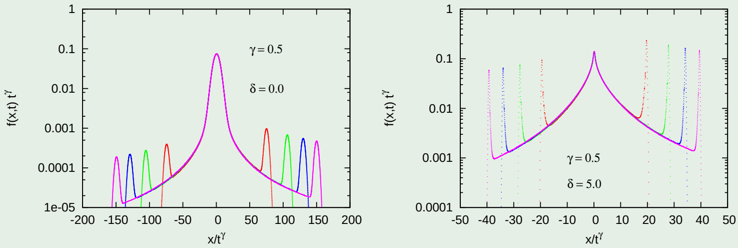
Why such values of the exponents?
Di Molfetta et al (2018 Phys. Rev. A 97, 062112) found in a quantum non Markovian walk with variable length step lengths the same exponent 3/2 for $\langle x^2 \rangle$.
Is memory irrelevant??
$\langle x \rangle \propto t^{1/2}$ as the ballistic peaks do not contribute (empirical).



Scaling function $g(z)$; $\quad z = x/t^{1/2}$

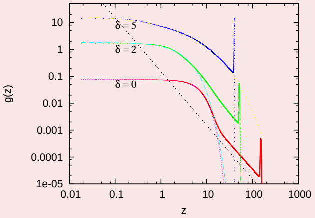
Truncated quantum Levy walk: Scaling function



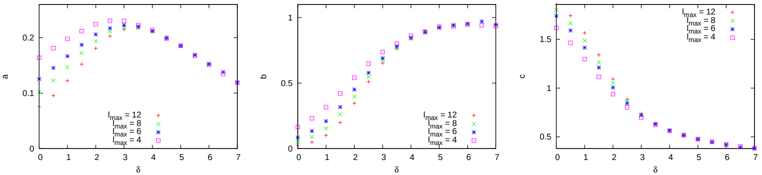
Scaling function $g(z)$; $\quad z = x/t^{1/2}$

$$g(z) = a\exp(-bz^c), \quad z < z^*$$
$$= \text{const} \quad z^{-2}, \quad z^* < z < z_{max}$$

For values of δ greater than $\delta^* \simeq 4$, z^* coincides with z_{max} such that the power law region is absent $\implies$ A crossover behaviour



a, b, c become independent of ℓ_{max} above $\delta^* \implies$ scaling function is universal above δ^*



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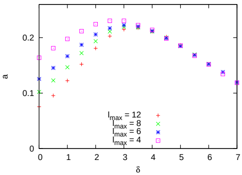
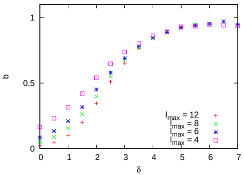
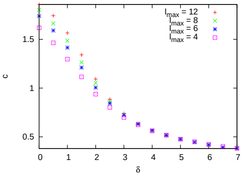
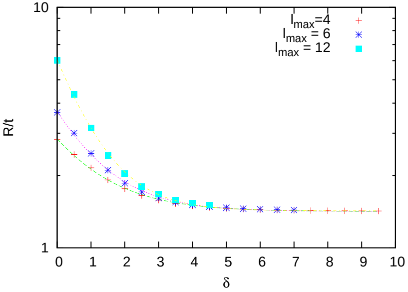




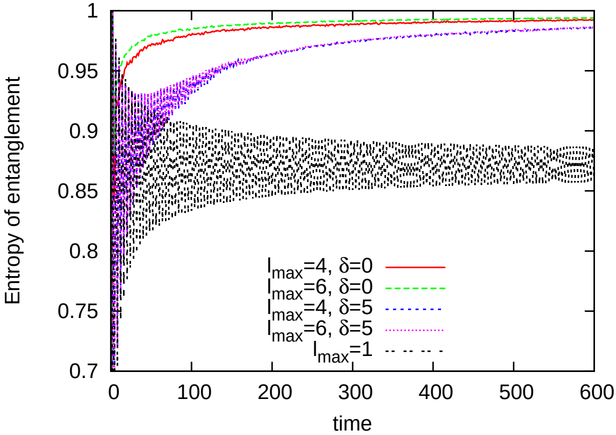


The range R scaled by t, has the same feature.
R/t approaches $\sqrt{2}$ as δ is increased.
However, it has a larger value for $\delta < \delta^*$, increasing with ℓ_{max}
$\implies$
Possible to search remote targets although the walk slows down



Entropy of entanglement

Initially, the state of the walker expressed as product state and the entanglement is zero. With time, the coin and position states become entangled.



Results show that the asymptotic values are larger than the usual quantum walker. For larger randomness, it is larger.

Conclusions

• Sub-ballistic but super-diffusive scaling for quantum walks when step lengths are randomly chosen - universal scaling

• Quantum Levy walk: crossover behaviour observed

• Remote searching possible for fat tailed distribution

• Entanglement enhancement; slower convergence and less amplitude of oscillations.

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Charles Bennett for suggestions

Thank you for your kind attention!