Surface operators, dual quivers and contours

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Quantum fields, Geometry and Representation Theory

This talk is based on . . .

- S. K. Ashok, S. Ballav, M. Billo', E. Dell'Aquila, M. Frau, V. Gupta, R. R. John and A. Lerda, *Surface operators, dual quivers and contours*, [arXiv:1807.06316 [hep-th]]

- S. K. Ashok, M. Billo', E. Dell'Aquila, M. Frau, V. Gupta, R. R. John and A. Lerda, *Surface operators, chiral rings and localization in $\mathcal{N} = 2$ gauge theories*, JHEP **1711**, 137 (2017) [arXiv:1707.08922 [hep-th]].

- S. K. Ashok, M. Billo', E. Dell'Aquila, M. Frau, R. R. John and A. Lerda, Modular and duality properties of surface operators in $\mathcal{N} = 2^{\star}$ gauge theories, JHEP **1707**, 068 (2017) [arXiv:1702.02833 [hep-th]].

- Builds on: Gaiotto '09, Gaiotto-Gukov-Seiberg '13

Motivations

- The list of calculable quantities in quantum field theories is growing:
  - Local operators: $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$
  - Including line operators: $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\ L \rangle$
  - Including surface operators: $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\ S \rangle$
- In supersymmetric gauge theories, it is possible to compute these exactly using localization methods Nekrasov, Pestun, ...
- We focus on the instanton partition function of an $\mathcal{N} = 2$ gauge theory in the presence of a surface operator. In the math literature, this is referred to as the ramified instanton partition function.
- We study pure $\mathcal{N} = 2$ gauge theory in four dimensions with gauge group SU(N). The theory can be studied either via Seiberg-Witten theory or using equivariant localization Nekrasov.

Defects in general $\cdots$

- How the defect affects the bulk theory: involves modifying the path integral by imposing some specified (singular) behaviour at the location of the defect. e.g. 't Hooft loops.
  - Surface operators as monodromy defects.

    Gukov-Witten

- Couple the bulk theory to additional degrees of freedom that are localized on the defect.
  - Surface operators as 2d/4d quiver gauge theories.

    Gukov-Witten, Gaiotto-Gukov-Seiberg

- Realize the bulk and the defect via D-branes in string theory (and lift to M-theory). Frenkel-Gukov-Teschner

Plan of the talk

- Surface operators as monodromy defects: partition function as contour integrals.

- Surface operators as coupled 2d/4d quiver gauge theories: twisted chiral rings and low energy effective actions.

  Dictionary between the two approaches

- 2d Seiberg duality and implications for surface operators.

Surface operators as monodromy defects

Surface operators as monodromy defects

- Surface operators are co-dimension-2 defects in the 4d gauge theory. We always consider $\mathbb{R}^2 \subset \mathbb{R}^4$.
- If $r\, e^{\mathrm{i}\theta}$ is the coordinate of the plane transverse to the defect D, then, as $r \to 0$, the gauge field has the following behaviour: Gukov, Witten '06

$$A \sim \text{diag}\,\Big(\underbrace{\alpha_1,\ldots,\alpha_1}_{n_1},\underbrace{\alpha_2,\ldots,\alpha_2}_{n_2},\ldots\underbrace{\alpha_M,\ldots,\alpha_M}_{n_M}\Big)\,d\theta\,,$$

$$\sum_J n_J = N \qquad \sum_J n_J\,\alpha_J = 0\,.$$

- The gauge field configuration is singular because $F = 2\pi\underline{\alpha}\,\delta_D$.
- In the path integral, one integrates over all gauge field configurations with this prescribed singular boundary condition.
- At the defect, the gauge group is broken to a Levi subgroup

$$SU(N) \longrightarrow S[U(n_1)\times U(n_2)\times\ldots\times U(n_M)]$$

Surface operators as monodromy defects

- For every partition $N = n_1 + \ldots + n_m$, there is a surface operator.
- One can turn on quantized magnetic fluxes for each U(1) factor of the Levi subgroup:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_l)} = m_l\,,$$

such that $\sum_l m_l = 0$.

- Since D is a two dimensional defect, it is also possible to add to the path integral a 2d topological term:

$$2\pi \mathrm{i} \sum_{l=1}^{M} \eta_l \int_D \mathrm{Tr}\, F_{\mathrm{U}(n_l)}$$

- Data of the defect:
  1. The discrete data of the partition $\sum_J n_J = N$.
  2. The $2M$ real parameters (α_I, η_I).

Instanton weight

- The goal is to calculate the instanton partition function in the presence of the defect.
- In the case without a defect, the partition function is weighted by the action of the instanton:

$$S_{\text{inst.}} = -2\pi \mathrm{i}\tau \left(\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F \right) = -2\pi \mathrm{i}\tau\, k\,.$$

$$Z_{\text{inst.}} = \sum_k (e^{2\pi \mathrm{i}\tau})^k\, Z_k\,.$$

- The 2d topological term has to be taken into account.
- One has to sum over the magnetic fluxes.
- In addition, the presence of the defect modifies the weight factor for each sector.

Topological action

- The instanton action receives contributions from the magnetic fluxes.

$$S_{\text{top}}[\vec{n}] = -2\pi\mathrm{i}\tau\left(\frac{1}{8\pi^2}\int_{\mathbb{R}^4\setminus D}\mathrm{Tr}\, F\wedge F\right) - 2\pi\mathrm{i}\sum_{l=1}^{M}\eta_l\left(\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_l)}\right)$$

- From the behaviour of the gauge connection near the surface operator:

$$\boxed{\frac{1}{8\pi^2}\int_{\mathbb{R}^4\setminus D}\mathrm{Tr}\, F\wedge F = k + \sum_{l=1}^{M}\alpha_l\, m_l}\ .$$

- Substituting this into the topological action:

$$S_{\text{top}}[\vec{n}] = -2\pi\mathrm{i}\tau\, k - 2\pi\mathrm{i}\sum_{l=1}^{M}(\eta_l + \tau\,\alpha_l)\, m_l = -2\pi\mathrm{i}\tau\, k - 2\pi\mathrm{i}\,\vec{t}\cdot\vec{m}$$

- The electric and magnetic parameters (η_l, α_l) have combined into a complex parameter:

$$\vec{t} = \{t_l\} = \{\eta_l + \tau\,\alpha_l\}\ .$$

Weight factors and the partition function

- The instanton partition function in the presence of the surface operator:

$$Z_{\text{inst}}[\vec{n}] = \sum_{k,\vec{m}} \left(e^{2\pi\mathrm{i}\tau}\right)^k \mathrm{e}^{2\pi\mathrm{i}\,\vec{t}\cdot\vec{m}} Z_{k,\vec{m}}[\vec{n}] \; .$$

- By a change of variables, this can be rewritten in the form

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_l\}} (q_l)^{d_l} \boxed{Z_{\{d_l\}}} \, ,$$

where $d_l \in \mathbb{N}$ and

$$\begin{aligned} q_1 &= e^{2\pi\mathrm{i}(t_1 - t_M)} \\ q_l &= e^{2\pi\mathrm{i}(t_l - t_{l-1})} \quad \text{for} \quad l = 2, \ldots M-1 \\ q_M &= e^{2\pi\mathrm{i}\tau} e^{-2\pi\mathrm{i}(t_{M-1} - t_M)} \end{aligned}$$

- "Ramified instantons" weighted by q_l: Interpretation in terms of vortex actions to appear later ..
- One sees that $q = e^{2\pi\mathrm{i}\tau} = q_1 \ldots q_M$.

Ramified instanton partition function

- The partition function is calculated using equivariant localization. Kanno-Tachikawa.
- It reduces to a $\mathbb{Z}_M$ orbifold of the case without surface operator:

$$\mathbb{C}_{\epsilon_1} \times (\mathbb{C}_{\epsilon_2} \times \mathbb{C})\,/\mathbb{Z}_M \times \mathbb{C} \times \mathbb{C}$$

  Study $D3$, $D(-1)$ branes on this background.
- The Ω-deformation parameters (ϵ_1, ϵ_2) regulate the volume of $\mathbb{R}^4$ and localize the partition function. Nekrasov.
- We introduce the 4d Coulomb vevs $\{a_u\}$ which also split according to the Levi subgroup:

$$\langle \Phi \rangle = \left\{ a_1, \ldots, a_{r_1} \middle| \ldots \middle| a_{r_{I-1}+1}, \ldots a_{r_I} \middle| \ldots \middle| a_{r_{M-1}+1}, \ldots, a_N \right\} .$$

  We have introduced $r_J = \sum_{I=1}^{J} n_I$.
- The partition function is calculable (order by order in the q_I):

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} (q_I)^{d_I} \boxed{Z_{\{d_I\}}(a_u, \epsilon_1, \epsilon_2)} \,,$$

The twisted chiral superpotential

- In the limit of vanishing ϵ_i:

$$\log\left(1 + Z_{\text{inst}}\right) = -\frac{\mathcal{F}_{\text{inst}}}{\epsilon_1 \epsilon_2} + \frac{\mathcal{W}_{\text{inst}}}{\epsilon_1} + \ldots$$

- The prepotential describes the effective 4d dynamics; irrespective of partition, it depends only on

$$q_{4d} = \prod_{l=1}^{M} q_l \,.$$

- The (twisted) superpotential describes the effective 2d/4d dynamics on the 2d defect.

- We know this because of the ϵ_i are related to the regulated volumes:

$$\frac{1}{\epsilon_1 \epsilon_2} \sim \text{svol}(\mathbb{C}^2_{\epsilon_1,\epsilon_2}) \sim \int d^4x\, d^2\theta_1\, d^2\theta_2$$
$$\frac{1}{\epsilon_1} \sim \text{svol}(\mathbb{C}_{\epsilon_1}) \sim \int d^2x\, d\theta_1\, d\bar{\theta}_1$$

For instance, in the simplest example of the $[1,1]$ surface operator in SU(2),

$$Z_{\text{inst}} = \frac{q_1}{\epsilon_1(2a+\epsilon_1+\epsilon_2)} + \frac{q_2}{\epsilon_1(-2a+\epsilon_1+\epsilon_2)} + \dots$$

leading to

$$\boxed{\mathcal{W}_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \dots}$$

Can we understand $\mathcal{W}_{\text{inst}}(a_u, q, t_l)$ from a more physical perspective?

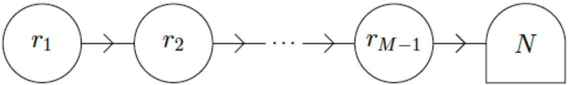
Surface operators as 2d/4d quivers

Surface operators as 2d/4d quivers

- To describe a surface operator of Levi type $\mathbb{L}$ in SU(N) theory, one considers a σ-model with target space: Gukov-Witten, Gadde-Gukov

$$\mathcal{M} = \frac{SU(N)}{\mathbb{L}}$$

- For the Levi subgroup of interest, this is a flag manifold. The defect is 1/2-BPS and preserves $(2, 2)$ supersymmetry in two dimensions.
- Such sigma models have a gauged linear sigma model (GLSM) description Witten '93.



- This is a U(r_1)$\times$ U(r_2) $\times \ldots$ gauge theory in 2d with bi-fundamental matter and SU(N) flavour group.
- The ranks $r_J = n_1 + n_2 + \ldots n_J$. The flavour group or global symmetry group of the 2d theory is identified with the 4d gauge group: flavour defects Gaiotto-Gukov-Seiberg

2d/4d quivers: low energy physics

- The SU(N) flavour symmetry is broken to U$(1)^{N-1}$ by the 4d Coulomb vevs. From the 2d perspective, these play the role of twisted masses.
  Hanany-Hori
- *All* chiral multiplets massive. We integrate them out and write an effective action for the vector multiplets of the 2d theory.
- In 2d, the vector multiplet is completely encoded in a twisted chiral multiplet Σ.
- The $(2,2)$ supersymmetry ensures that the low energy effective action is completely specified by a twisted chiral superpotential $\widetilde{\mathcal{W}}(\Sigma)$.
- The massive vacua are obtained by extremizing $\widetilde{\mathcal{W}}(\Sigma)$:

$$\exp\left(\frac{\partial\widetilde{\mathcal{W}}}{\partial\Sigma}\right) = 1$$

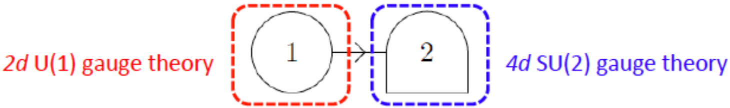
Surface operators as 2d/4d quivers

- There is a one-to-one map between massive vacua $\Sigma_\star$ of the 2d/4d theory and surface operators.

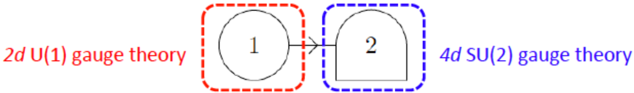
- Evaluate the twisted chiral superpotential on the solution:

  $\widetilde{\mathcal{W}}(\Sigma_\star)$ is identified with $\mathcal{W}_{\text{inst.}}$ calculated using localization.

- Let us see how this works in the simplest example: the $[1, 1]$ surface operator in SU(2) theory: U(1) gauge theory with two chiral multiplets.



2d/4d coupled theory: SU(2)



- For the purely 2d theory with SU(2) flavour symmetry, the effective superpotential is given by

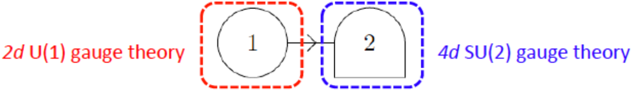
  d'Adda-Davis-di Vecchia-Salomonson, Cecotti-Vafa, Witten

$$\widetilde{\mathcal{W}} = 2\pi \mathrm{i}\, t\, \Sigma - 2 \left(\Sigma \log \frac{\Sigma}{\mu} - \Sigma \right)$$

  Here

  - Here μ is the uv scale
  - $t = \mathrm{i}\zeta + \frac{\theta}{2\pi}$ is the complexified FI coupling
  - The 2 is the number of flavours

2d/4d coupled theory: SU(2)



- With the twisted masses turned on (at the classical level) Hanany-Hori

$$\widetilde{\mathcal{W}} = 2\pi\mathrm{i}\, t\,\Sigma - \mathrm{Tr}\left((\Sigma - \langle\Phi\rangle) \left(\log \frac{(\Sigma - \langle\Phi\rangle)}{\mu} - 1 \right) \right)$$

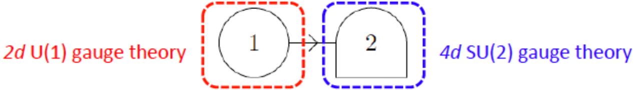
- Quantum-mechanically: Gaiotto-Gukov-Seiberg

$$\widetilde{\mathcal{W}} = 2\pi\mathrm{i}\, t\,\Sigma - \left\langle \mathrm{Tr}\left[(\Sigma - \Phi) \left(\log \frac{(\Sigma - \Phi)}{\mu} - 1 \right) \right] \right\rangle$$

- The function that appears here is the double-integral of the resolvent of the 4d gauge theory, which is the generating function of chiral correlators:

$$\left\langle \mathrm{Tr}\,\frac{1}{\Sigma - \Phi} \right\rangle = \sum_{\ell=1}^{\infty} \frac{1}{\Sigma^{\ell+1}} \left\langle \mathrm{Tr}\,\Phi^{\ell} \right\rangle$$

2d/4d coupled theory: SU(2)



- We begin with this 2d/4d action and we introduce the dynamically generated 2d scale:

$$\widetilde{\mathcal{W}} = 2\pi \mathrm{i}\, t\, \Sigma - \left\langle \mathrm{Tr}\left[(\Sigma - \Phi)\left(\log\frac{(\Sigma-\Phi)}{\mu} - 1\right)\right]\right\rangle$$
$$= -\left\langle \mathrm{Tr}\left[(\Sigma - \Phi)\left(\log\frac{(\Sigma-\Phi)}{\Lambda_1} - 1\right)\right]\right\rangle$$

- The new scale Λ_1 is defined by $t(\Lambda_1) = 0$ or equivalently

$$\Lambda_1^2 = e^{2\pi \mathrm{i} t}\mu^2 \,.$$

- The massive vacua are determined by

$$\exp\left(\frac{\partial \widetilde{\mathcal{W}}}{\partial \Sigma}\right) = 1 \Leftrightarrow \exp\left\langle \mathrm{Tr}\log\frac{(\Sigma-\Phi)}{\Lambda_1}\right\rangle = 1\,.$$

2d/4d coupled theory: SU(2)

- The resolvent of the pure $\mathcal{N}=2$ gauge theory in four dimensions is
  Cachazo-Douglas-Seiberg-Witten

$$\left\langle \mathrm{Tr}\,\log \frac{(z-\Phi)}{\Lambda_1} \right\rangle = \log\left(\frac{P_N(z)+y}{\Lambda_{4d}^N}\right)$$

- y is given by the Seiberg-Witten curve:

$$y^2 = P_N(z)^2 - 4\Lambda_{4d}^{2N}\,.$$

- Using this result, the twisted chiral ring equation for the 2d/4d theory is:

$$\boxed{P_2(\Sigma) = \Sigma^2 - \frac{1}{2}\left\langle \mathrm{Tr}\,\Phi^2 \right\rangle = \Lambda_1^2 + \frac{\Lambda_{4d}^4}{\Lambda_1^2}\,.}$$

- This is solved order by order in the non-perturbative scales:

$$\Sigma_\star = \sqrt{\frac{1}{2}\langle \mathrm{Tr}\,\Phi^2\rangle + \Lambda_1^2 + \frac{\Lambda_{4d}^4}{\Lambda_1^2}} = \left(a + \frac{1}{2a}\left(\Lambda_1^2 + \frac{\Lambda_{4d}^4}{\Lambda_1^2}\right) + \dots\right)$$

2d/4d coupled theory

- Evaluating W on the solution we find

$$\widetilde{\mathcal{W}}(\Sigma_\star)\Big|_{\text{1-inst.}} = \frac{1}{2a}\left(\Lambda_1^2 - \frac{\Lambda_{4d}^4}{\Lambda_1^2}\right)$$

- Compare with the localization result obtained earlier for ramified instantons:

$$\mathcal{W}_{\text{1-inst.}} = \frac{q_1}{2a} - \frac{q_2}{2a},$$

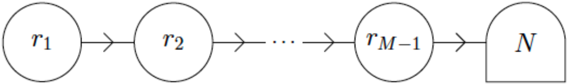
- We find a perfect match provided:

$$\boxed{q_1 = \Lambda_1^2 \qquad q_2 = \frac{\Lambda_{4d}^4}{\Lambda_1^2}\,.}$$

- Note that q_1 is precisely the weight of the "vortex" solution in the pure 2d theory Witten:

$$S_{vortex} = -(2\pi\mathrm{i}\, t(\mu)) = -\log(\Lambda_1^2)\,.$$

General case: $[n_1, n_2, \ldots n_M]$



- The bi-fundamentals are integrated out to obtain an effective twisted chiral superpotential:

$$\widetilde{\mathcal{W}} = 2\pi \mathrm{i} \sum_{l=1}^{M-1} \tau_l \,\mathrm{Tr}\,\Sigma^{(l)} - \sum_{l=1}^{M-2} \sum_{s=1}^{r_l} \sum_{t=1}^{r_{l+1}} \varpi\left(\Sigma_s^{(l)} - \Sigma_t^{(l+1)}\right) - \sum_{s=1}^{r_{M-1}} \left\langle \mathrm{Tr}\,\varpi\left(\Sigma_s^{(M-1)} - \Phi\right)\right\rangle$$

  where

$$\varpi(x) = x\left(\log\frac{x}{\mu} - 1\right),$$

- The twisted chiral ring equations are written for each node:

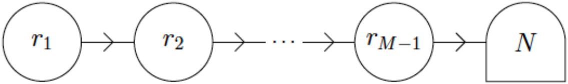
$$\exp\left(\frac{\partial \widetilde{\mathcal{W}}}{\partial \Sigma_s^{(l)}}\right) = 1 \qquad s \in \{1, \ldots r_l\} \quad \text{and} \quad l \in \{1, \ldots M-1\}$$

General case: $[n_1, n_2, \ldots n_M]$

- The solutions are all isolated (massive vacua). But what is the map to surface operators?
- In the localization calculation, the Coulomb vevs split in a particular way:

$$\begin{aligned}\langle \Phi \rangle &= \mathrm{diag}\left\{ a_1, \ldots, a_{r_1} \middle| \ldots \middle| a_{r_{I-1}+1}, \ldots a_{r_I} \middle| \ldots \middle| a_{r_{M-1}+1}, \ldots, a_N \right\} \\ &= \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \mathcal{A}_I \ldots \oplus \mathcal{A}_M \;.\end{aligned}$$

- This gives a natural classical solution to the twisted chiral ring equations of the quiver:



$$\Sigma^{(I)}_{\text{class.}} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_I$$

This is an inclusion:

$$\Sigma^{(1)}_{\text{class.}} \subset \Sigma^{(2)}_{\text{class.}} \subset \ldots \subset \Sigma^{(M-1)}_{\text{class.}} \subset \langle \Phi \rangle \;.$$

General case: $[n_1, n_2, \ldots n_M]$

- Solve the twisted chiral ring equations order by order in the dynamically generated scales $\Lambda_l^{b_l}$ and Λ_{4d} (via the resolvent).

$$b_l = r_{l+1} - r_l = n_l + n_{l+1} \,.$$

- Evaluate the twisted chiral superpotential *on* the solution $\widetilde{\mathcal{W}}(\Sigma_\star)$. This reproduces the instanton expansion on the localization side provided

$$q_1 = \Lambda_1^{b_1} \,, \quad q_2 = \Lambda_2^{b_2} \,, \quad \ldots \quad q_{M-1} = \Lambda_{M-1}^{b_{M-1}} \,, \quad q_M = \frac{\Lambda_{4d}^{2N}}{q_1 q_2 \ldots q_{M-1}} \,.$$

Summary of results so far

Monodromy defect	2d/4d quiver
Partition of N: $[n_1, n_2, \ldots, n_M]$	Ranks of 2d gauge nodes
4d Coulomb v.e.v.'s	2d twisted masses
Partition of Coulomb v.e.v.'s	Choice of classical (massive) vacuum
Instanton counting parameters q_I, q_M	2d/4d strong coupling scales Λ_I, Λ_{4d}
$\mathcal{W}_{\text{inst}}(a, q)$	$\mathcal{W}(\Sigma, a, \Lambda_I, \Lambda_{4d})\|_{\Sigma_\star}$

Which contour?

- The ramified instanton partition function is a multi-dimensional contour integral. For instance, at 1-instanton:

$$-\sum_{I=1}^{M}\frac{q_I}{\epsilon_1}\int\frac{d\chi_I}{2\pi\mathrm{i}}\prod_{s\in\mathcal{N}_I}\frac{1}{\left(a_s-\chi_I+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)}\prod_{t\in\mathcal{N}_{I+1}}\frac{1}{\left(\chi_I-a_t+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)}\,.$$

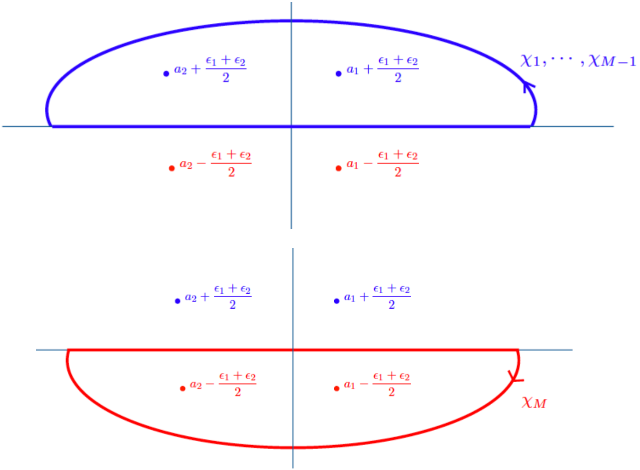
- So far we have not been very precise about the contour of integration on the localization side.
- Which poles contribute to the partition function?
- Assign $\mathrm{Re}(a_u)=0$ and $\mathrm{Im}(\epsilon_1)>>\mathrm{Im}(\epsilon_2)>>0$. Then, contour amounts to closing in the upper or lower half plane for each set of χ_I.

Contour at 1-instanton: $(+, +, \ldots +, -)$

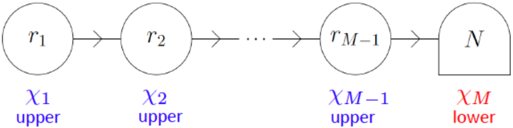
$$\chi_I = a_s + \frac{1}{2}(\epsilon_1 + \epsilon_2) \quad s = 1, 2, \ldots n_I \quad I = 1, \ldots M-1$$

$$\chi_M = a_t - \frac{1}{2}(\epsilon_1 + \epsilon_2) \quad t = 1, 2, \ldots n_1$$

Gorsky et al., SKA et al.



More general formulation



- At higher instantons it is not sufficient to simply say whether the contour is closed in the upper or lower half plane. The order of integration is also important.

Ramified instanton partition function

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}}[\vec{n}] \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M} \left[\frac{(-q_I)^{d_I}}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi \mathrm{i}} \right] z_{\{d_I\}}$$

where

$$z_{\{d_I\}} = \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})}{(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{(\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2)}{(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2)}$$
$$\times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \frac{1}{\prod_{s \in \mathcal{N}_I} \left(a_s - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) \right)} \frac{1}{\prod_{t \in \mathcal{N}_{I+1}} \left(\chi_{I,\sigma} - a_t + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) \right)} .$$

Jeffrey-Kirwan residue prescription

- Alternatively there is an elegant way to specify the contour via Jeffrey-Kirwan reference vector

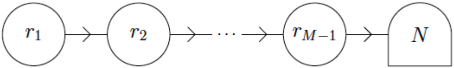
$$\eta = -\zeta_1\chi_1 - \zeta_1\chi_2 - \ldots - \zeta_{M-1}\chi_{M-1} + \zeta_M\chi_M \,.$$

  with $\zeta_1 << \zeta_2 << \ldots \zeta_{M-1} << \zeta_M$.

- With this choice there is a term by term match between the superpotentials calculated via localization and via the twisted chiral ring analysis.

- We identify the ζ_I with the FI parameters of the $M-1$ two dimensional gauge nodes. The hierarchy of ζ_I then corresponds to a hierarchy of scales:

$$\Lambda_1 \quad >> \quad \Lambda_2 \quad >> \quad \ldots \quad \Lambda_{M-1} \quad >> \Lambda_{4d}\,.$$



$$\left|\frac{\Lambda_I}{\mu}\right|^{b_I} \sim e^{-2\pi\,\zeta_I}\,.$$

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4d Coulomb v.e.v.'s	2d twisted masses
Partition of Coulomb v.e.v.'s	Choice of classical (massive) vacuum
Instanton counting parameters q_I, q_M	2d/4d strong coupling scales Λ_I, Λ_{4d}
$\mathcal{W}_{\text{inst}}(a, q)$[+++...+ -]	$\mathcal{W}(\Sigma, a, \Lambda_I, \Lambda_{4d})\vert_{\Sigma_\star}$

A physics question

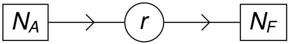
- There is a "duality" or equivalence relation in the two dimensional gauge theory that allows one to write distinct 2d/4d quivers that all have the same infrared behaviour.
- In our case, this means a one-to-one map between the massive vacua.

How do we understand these dual quivers from the localization point of view?

2d Seiberg duality

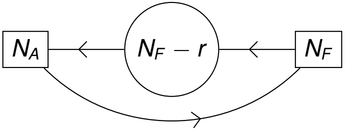
Duality: the basic move

- We start with Theory A and assume $N_F > N_A$:



The classical twisted superpotential $\widetilde{\mathcal{W}}^A = 2\pi\mathrm{i}\,\tau\mathrm{Tr}\,\Sigma$.

- We dualize on the $\mathrm{U}(r)$ node to get Theory B:



The classical twisted superpotential gets modified according to Benini-Park-Zhao

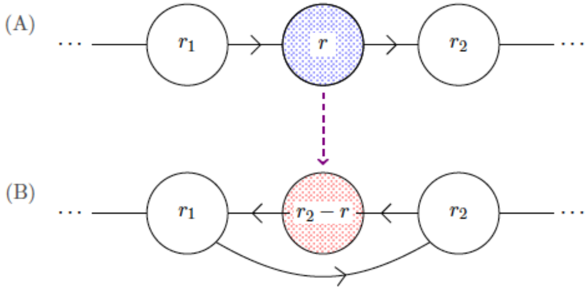
$$\widetilde{\mathcal{W}}^B = -2\pi\mathrm{i}\,\tau\mathrm{Tr}\,\Sigma' + 2\pi\mathrm{i}\tau\sum_{f=1}^{N_F} m_f$$

- There is also an ordinary superpotential induced by the loop in the diagram.

Duality

Case 1:

$$\widetilde{\mathcal{W}}^A = \ldots + 2\pi \mathrm{i}(\tau_1 \mathrm{Tr}\,\Sigma_1 + \tau \mathrm{Tr}\,\Sigma_2 + \tau_2 \mathrm{Tr}\,\Sigma_2) + \ldots$$

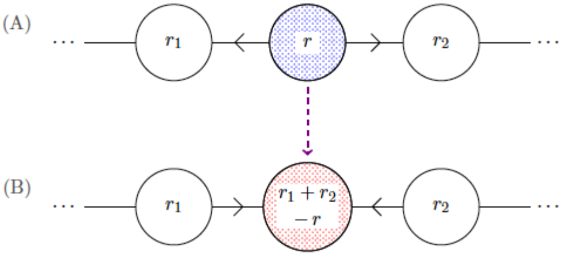


$$\widetilde{\mathcal{W}}^B = \ldots + 2\pi \mathrm{i}(\tau_1 \mathrm{Tr}\,\Sigma_1 - \tau \mathrm{Tr}\,\Sigma_2 + (\tau_2 + \tau) \mathrm{Tr}\,\Sigma_2) + \ldots$$

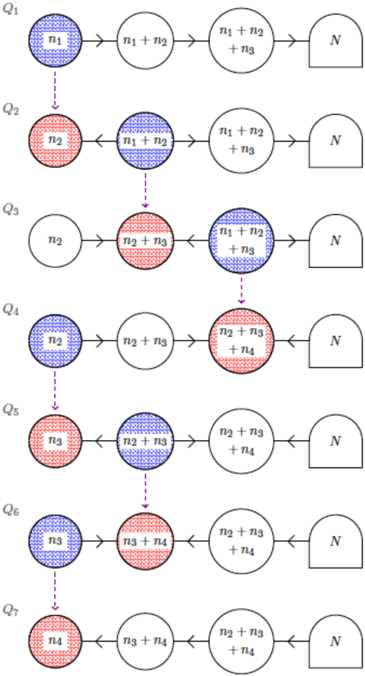
Duality

Case 2:

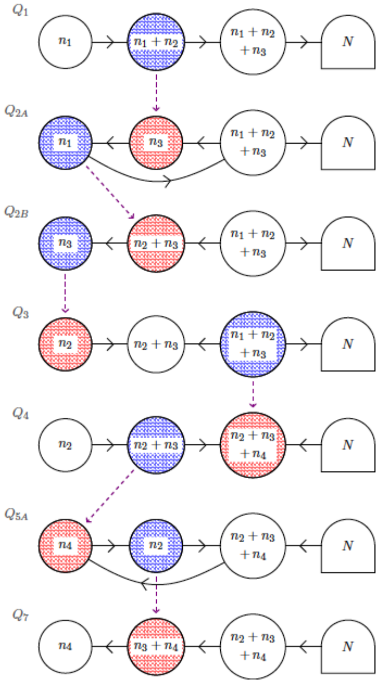
$$\widetilde{\mathcal{W}}^A = \ldots + 2\pi i(\tau_1 \mathrm{Tr}\,\Sigma_1 + \tau \mathrm{Tr}\,\Sigma_2 + \tau_2 \mathrm{Tr}\,\Sigma_2) + \ldots$$



$$\widetilde{\mathcal{W}}^B = \ldots + 2\pi i((\tau_1 + \tau)\mathrm{Tr}\,\Sigma_1 - \tau \mathrm{Tr}\,\Sigma_2 + (\tau_2 + \tau)\mathrm{Tr}\,\Sigma_2) + \ldots$$



Q_1
n_1
$n_1 + n_2$
$n_1 + n_2 + n_3$
N
Q_2
n_2
$n_1 + n_2$
$n_1 + n_2 + n_3$
N
Q_3
n_2
$n_2 + n_3$
$n_1 + n_2 + n_3$
N
Q_4
n_2
$n_2 + n_3$
$n_2 + n_3 + n_4$
N
Q_5
n_3
$n_2 + n_3$
$n_2 + n_3 + n_4$
N
Q_6
n_3
$n_3 + n_4$
$n_2 + n_3 + n_4$
N
Q_7
n_4
$n_3 + n_4$
$n_2 + n_3 + n_4$
N



Q_1
n_1
$n_1 + n_2$
$n_1 + n_2 + n_3$
N
Q_{2A}
n_1
n_3
$n_1 + n_2 + n_3$
N
Q_{2B}
n_3
$n_2 + n_3$
$n_1 + n_2 + n_3$
N
Q_3
n_2
$n_2 + n_3$
$n_1 + n_2 + n_3$
N
Q_4
n_2
$n_2 + n_3$
$n_2 + n_3 + n_4$
N
Q_{5A}
n_4
n_2
$n_2 + n_3 + n_4$
N
Q_7
n_4
$n_3 + n_4$
$n_2 + n_3 + n_4$
N

Comments

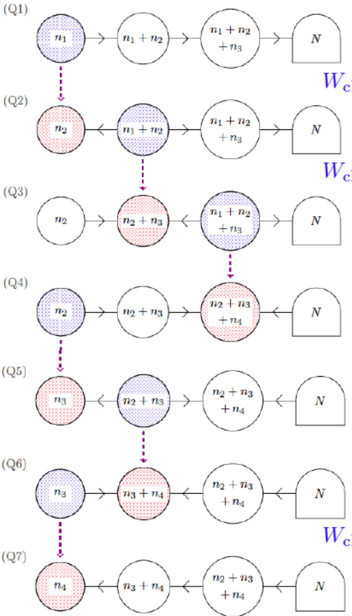
- All these quivers provide different realizations of the same surface operator:
$$SU(N) \longrightarrow S\left[U(n_1) \times \ldots \times U(n_M)\right] .$$
- Seiberg duality is an infrared equivalence. We should expect a one-to-one map between the massive vacua of these distinct quivers.
- We expect that $\widetilde{\mathcal{W}}$ evaluated in these vacua should be equal.
- This suggests the solution to our problem: dual quivers should correspond to different but equivalent choices of contours for the localization integrand.
- The equality of the superpotential is a consequence of residue theorems.
- In order to distinguish the contours, one has to not just focus on the twisted superpotential itself but on the individual residues themselves.

Comments

In principle, one can proceed as we did before for each quiver:

1. Integrate out the matter to write an effective twisted chiral superpotential.
2. Write the twisted chiral ring equations.
3. Find the particular classical vacuum about which one solves the equations order by order in the strong coupling scales.
   Equate classical superpotentials.
4. Evaluate $\widetilde{\mathcal{W}}$ on the particular solution.
5. Find the q-vs-Λ map to match this term by term with the residues chosen by a particular choice of poles on the localization side.
   Follows from action of duality on FI parameters.

Is there an easier way to obtain the quivers vs. contours map?



$$W_{\text{class}} = 2\pi i\big(t_1 \operatorname{Tr}\sigma^1 + t_2 \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

$$W_{\text{class}} = 2\pi i\big(-t_1 \operatorname{Tr}\sigma^1 + (t_1+t_2) \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

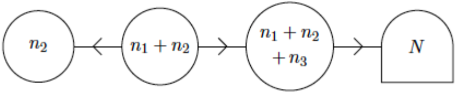
$$\vdots$$

$$W_{\text{class}} = 2\pi i\big(t_3 \operatorname{Tr}\sigma^1 - (t_2+t_3) \operatorname{Tr}\sigma^2 - t_1 \operatorname{Tr}\sigma^3\big)$$

$$W_{\text{class}} = 2\pi i\big(-t_3 \operatorname{Tr}\sigma^1 - t_2 \operatorname{Tr}\sigma^2 - t_1 \operatorname{Tr}\sigma^3\big)$$

The solution

- The classical vevs and the q-vs-Λ map follows from a simple study of the classical superpotentials.
- For instance, for the second quiver in the duality chain:



The classical superpotential is

$$\mathcal{W}_{Q_2} = 2\pi \mathrm{i}(-\tau_1 \mathrm{Tr}\,\Sigma_1 + (\tau_1 + \tau_2)\mathrm{Tr}\,\Sigma_2 + \tau_3 \mathrm{Tr}\,\Sigma_3)$$

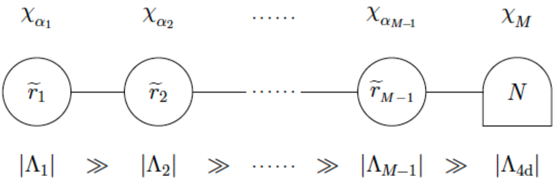
- We read off from here the following:

$$\Sigma_1 = \mathcal{A}_2 \qquad \Sigma_2 = \mathcal{A}_1 \oplus \mathcal{A}_2 \qquad \Sigma_3 = \mathcal{A}_1 + \mathcal{A}_2 \oplus \mathcal{A}_3$$

$$q_1 \sim \Lambda_1^{n_1+n_2} \qquad q_2 \sim \frac{\Lambda_2^{n_1+2n_2+n_3}}{\Lambda_1^{n_1+n_2}} \qquad q_3 \sim \Lambda_3^{n_3+n_4} \qquad q_4 \sim \frac{\Lambda_{4d}^{2N}}{q_1 q_2 q_3} \, .$$

The solution

- From this information it is possible to associate a given 2d node to one of the χ_I-integrals.

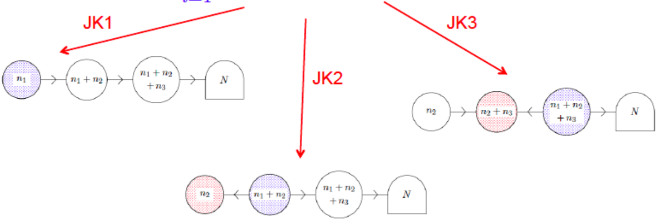


- The sign of the beta-function for a given 2d node determines whether the χ_I is integrated in the upper or lower half plane; this in turn determines the sign of the particular term in the Jeffrey-Kirwan vector.

- We find a term by term match of the twisted chiral superpotential calculated in the two approaches.

- We verified this for each of the quivers in the duality chains.

$$\prod_{i=1}^{M} \oint \frac{d\chi_i}{2\pi i} \frac{q_i^{d_i}}{d_i!} Z_{d_i}(a, \epsilon_1, \epsilon_2)$$



$$\eta_{Q_1} = -\zeta_1\,\chi_1 - \zeta_2\,\chi_2 - \zeta_3\,\chi_3 - \zeta_4\,\chi_4 \quad \text{with} \quad \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$$

$$\eta_{Q_2} = \underline{+\zeta_1\,\chi_1} - \zeta_2\,\chi_2 - \zeta_3\,\chi_3 - \zeta_4\,\chi_4 \quad \text{with} \quad \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$$

$$\eta_{Q_3} = -\zeta_2\,\chi_2 + \underline{\zeta_1\,\chi_1} - \zeta_3\,\chi_3 - \zeta_4\,\chi_4 \quad \text{with} \quad \zeta_2 < \zeta_1 < \zeta_3 < \zeta_4$$

Summary of results

Monodromy defect	2d/4d quiver models
Partition of N: $[n_1, n_2, \ldots, n_M]$	Ranks of 2d gauge nodes
4d Coulomb v.e.v.'s	2d twisted masses
Partition of Coulomb v.e.v.'s	Choice of classical (massive) vacuum
Instanton counting parameters q_I, q_M	2d/4d strong coupling scales Λ_I, Λ_{4d}
$\mathcal{W}_{\text{inst}}(a, q)$	$\mathcal{W}(\Sigma, a, \Lambda_I, \Lambda_{4d})\|_{\Sigma_\star}$
Choice of contour prescription	2d Seiberg duality frame

The duality frame affects all the entries above.

Final remarks

- It would be worthwhile to understand from first principles the appearance of ratios of scales by studying vortex equations for bi-fundamentals. For instance, we found that for Q_2:

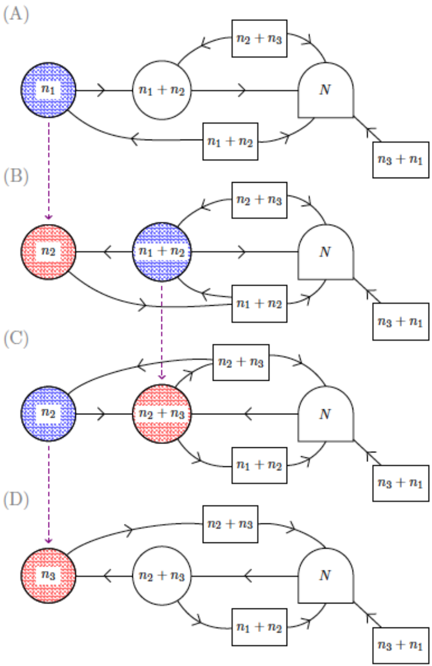
$$q_1 \sim \Lambda_1^{n_1+n_2} \qquad q_2 \sim \frac{\Lambda_2^{n_1+2n_2+n_3}}{\Lambda_1^{n_1+n_2}} \qquad q_3 \sim \Lambda_3^{n_3+n_4} \qquad q_4 \sim \frac{\Lambda_{4d}^{2N}}{q_1 q_2 q_3} .$$

- A string theory analysis should clarify how the "ramified instantons" combine to form a gauge instanton:

$$q_{4d} = \prod_{I=1}^{M} q_M .$$

- For the conformal SQCD case, the map between contours and quivers is manifest: the superpotentials calculated for different contours do not agree with each other but agree with those calculated from the corresponding quivers!

- Work in progress to reconcile the residue at infinity with Seiberg duality.


(A)
n_1
$n_1 + n_2$
$n_2 + n_3$
N
$n_1 + n_2$
$n_3 + n_1$
(B)
n_2
$n_1 + n_2$
$n_2 + n_3$
N
$n_1 + n_2$
$n_3 + n_1$
(C)
n_2
$n_2 + n_3$
$n_2 + n_3$
N
$n_1 + n_2$
$n_3 + n_1$
(D)
n_3
$n_2 + n_3$
$n_2 + n_3$
N
$n_1 + n_2$
$n_3 + n_1$