Knot Polynomials from Chern-Simons theory & their string theoretic interpretation

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- Current status and ongoing work

Just like Periodic Table of chemical elements

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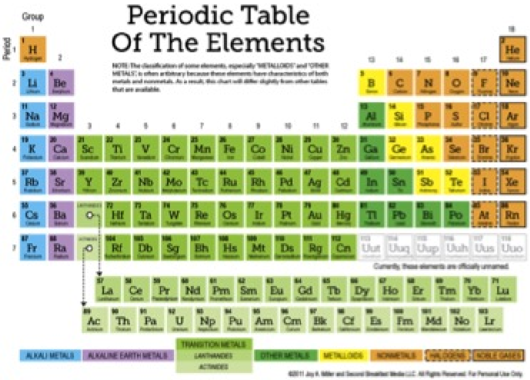
Atomic number and atomic weight distinguishes different chemical



elements!

Periodic table of Knots

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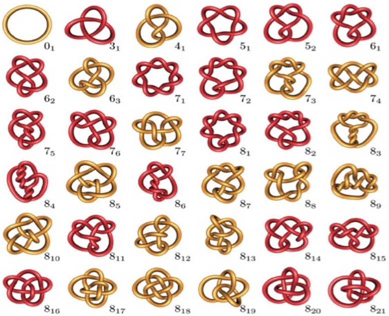
In the context of knots, what quantity will distinguish them!



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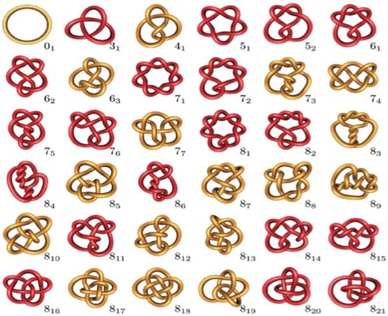

crossing number is a weak invariant

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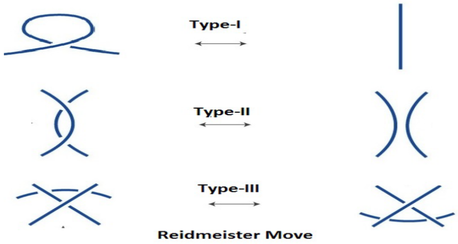

Knot Invariant through recursive method

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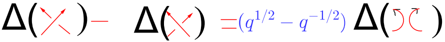
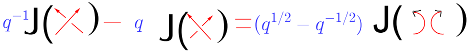
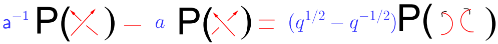
Alexander Polynomial (1923) Δ(K; q) :



Jones Polynomial(1984) J(K; q):



HOMFLY-PT Polynomial $\mathbf{P(K; a = q^N, q)}$:



Jones Polynomial

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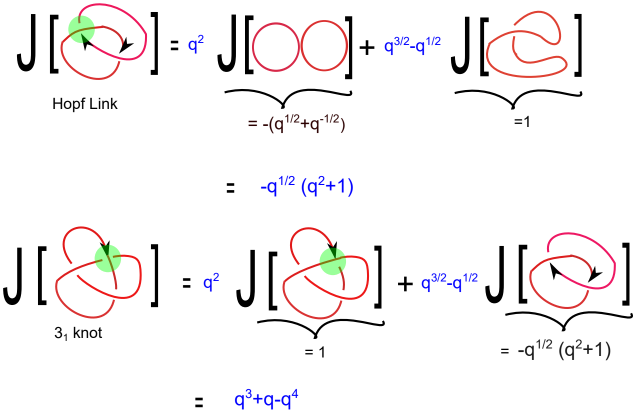
For Hopf link and trefoil knot



Mutant Knots

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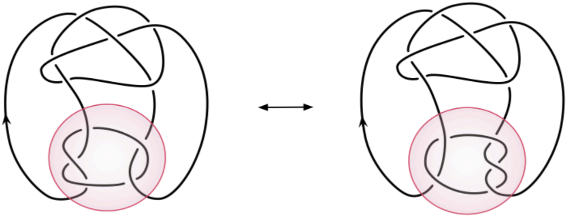

Chern-Simons Theory

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$$S = \frac{k}{4\pi} \int_{M^3} \left(A \wedge DA + \frac{2}{3} A \wedge A \wedge A \right)$$

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$$S = \frac{k}{4\pi} \int_{M^3} \left(A \wedge DA + \frac{2}{3} A \wedge A \wedge A \right)$$

- Knot invariants are given by expectation value of Wilson loop operators:

$$\tilde{P}_R^G[K] = \langle W_R^G[K] \rangle = \frac{1}{Z[M^3]} \int \mathcal{D}A \, \mathrm{Tr}_R \left(\exp \oint_K A \right) \, e^{iS}$$

where K is the knot and R is the representation of gauge group G.

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- Why these well-known polynomials cannot distinguish **mutant pairs**?
- We have more generalised knot invariants for arbitrary R and G- Can they distinguish mutant pairs?- small step towards classification!

Knot Invariants from Chern-Simons

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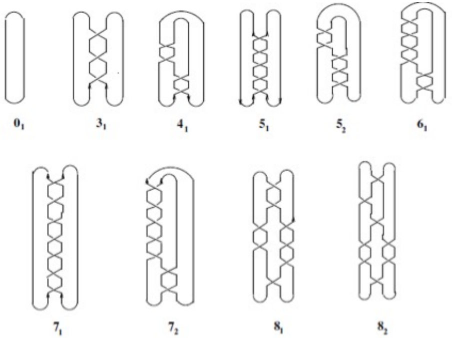
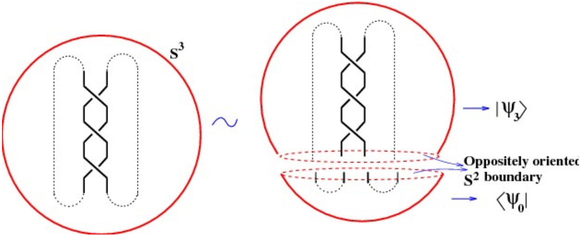
- Relation between Chern-Simons theory to G_k Wess-Zumino conformal field theory (WZNW) (Witten 1989)
- Any knot can be obtained as a **closure/plat/quasiplat of braid** **(Alexander, Birman)**



Example: Trefoil invariant

Basically, the trefoil T in S^3 is viewed as gluing two three-balls with oppositely oriented S^2 boundaries.



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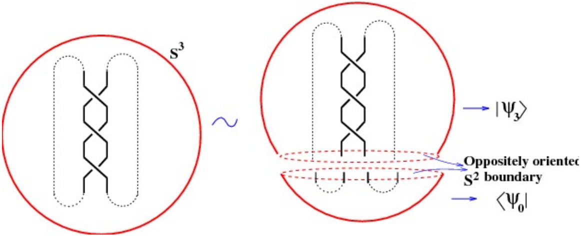

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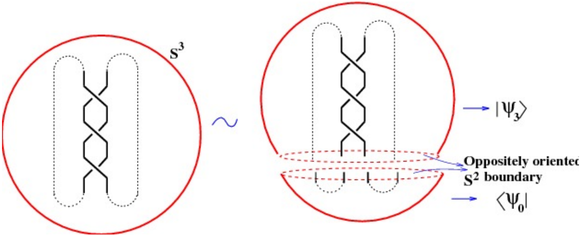

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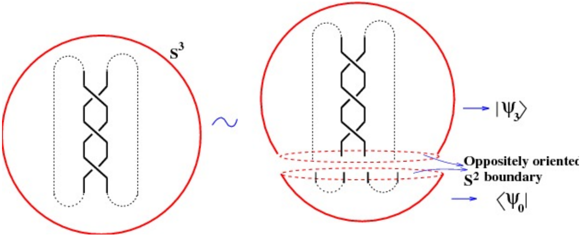

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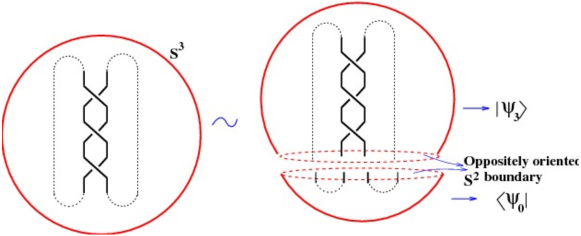

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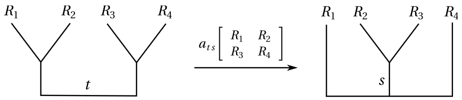
Braiding operator $\mathcal{B}$ eigenbasis will determine the polynomial form in variable q

Eigenbasis of Braiding operator B

For the four-punctured S^2 boundary, the conformal block bases are:



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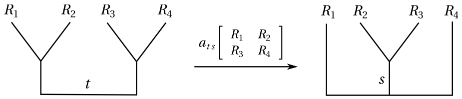

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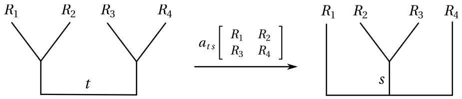

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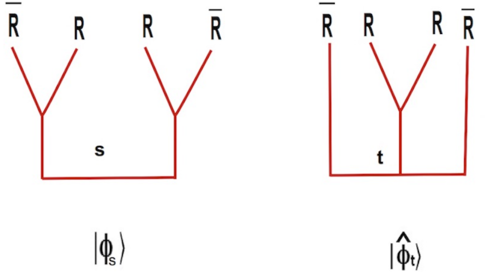
For knots, two of the R_i's will be R and the other two will be conjugate $\bar{R}$ depending on the orientation.

Polynomial invariant of trefoil



In the braid diagram for trefoil, middle two strands are parallely oriented and they are braided.

$$|\Psi_0\rangle \;=\; \sum_{s \in R \otimes R} \mu_s \; |\hat{\Phi}_t(\bar{R}, R, R, \bar{R})\rangle$$

where $\mu_s \;=\; \sqrt{S_{0s}/S_{00}} \equiv \sqrt{dim_q s}$ (**unknot** normalisation)

Trefoil evaluation continued

$$\tilde{P}_R[3_1] \;=\; \langle \Psi_0 | \mathcal{B}^3 | \Psi_0 \rangle \;=\; \sum_s dim_q s \; (\lambda_s(R,R))^3$$

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$$\lambda_s^{(+)} \equiv \lambda_s(R,R) \ = \ (-1)^{\epsilon_s} q^{2C_R - C_s/2}, \ q = e^{\frac{2\pi i}{k+C_v}} \text{ where } \epsilon_s = \pm 1$$

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$$J[3_1] = \tilde{P}_{\square}[3_1] / \tilde{P}_{\square}[U]$$

Figure 8 knot invariant

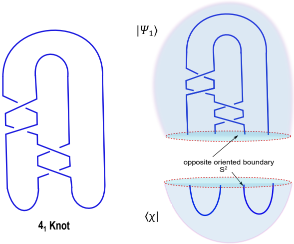
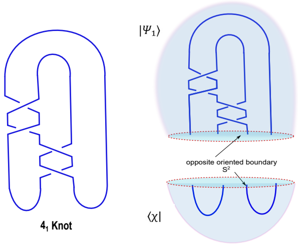


Figure 8 knot invariant



Involves antiparallel braidings in middle as well as side two-strands.

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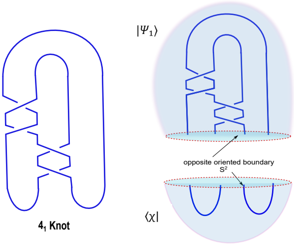

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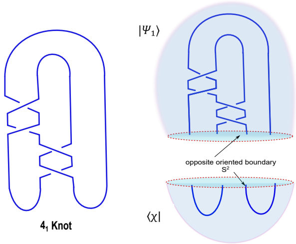
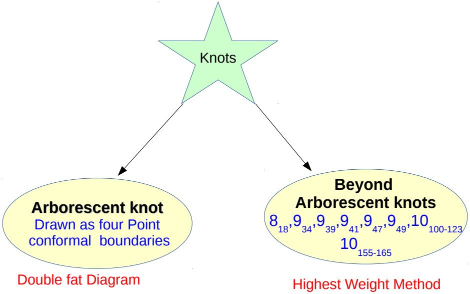


Involves antiparallel braidings in middle as well as side two-strands. Duality matrix required to go from middle to side-strand basis! The antiparallel braiding eigenvalue will be $\lambda_s^{(-)} \equiv \lambda_s(R,\bar{R}) \;=\; (-1)^{\epsilon_s} q^{C_s/2}$

$$\tilde{P}_R[4_1] \;=\; \sum_{t,s \in R\otimes\bar{R}} \sqrt{dim_q t\; dim_q s}\; a_{ts} \begin{bmatrix} \bar{R} & R \\ \bar{R} & R \end{bmatrix} \{\lambda_t^{(-)}\}^2 \{\lambda_s^{(-)}\}^{-2}$$

The method is straightforward to write invariants for knots from *n*-strand quasi-plat.

Broad classification of knots



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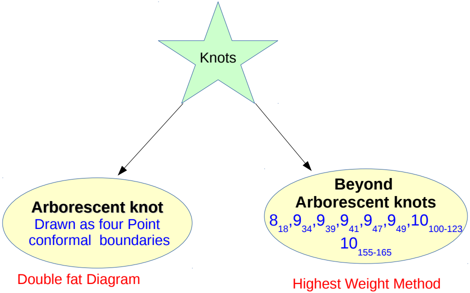
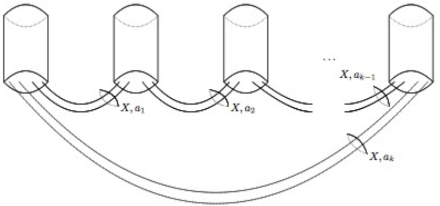


We will now discuss arborescent knots and their invariants

Arborescent Knots

- We had illustrated trefoil from 4- plat diagram and figure-eight from quasi-plat diagram.
- The knots with more than four-strands which can be drawn as



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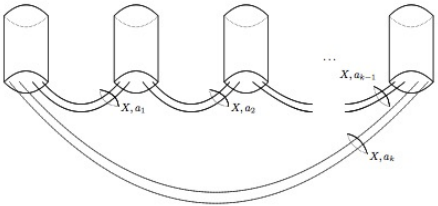

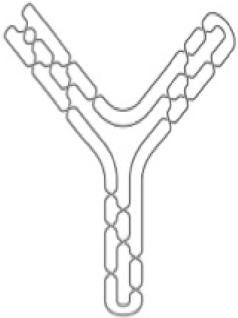

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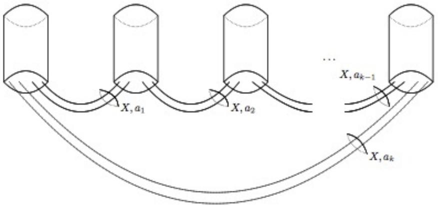

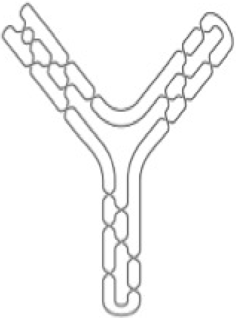

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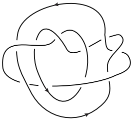
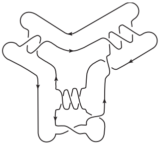
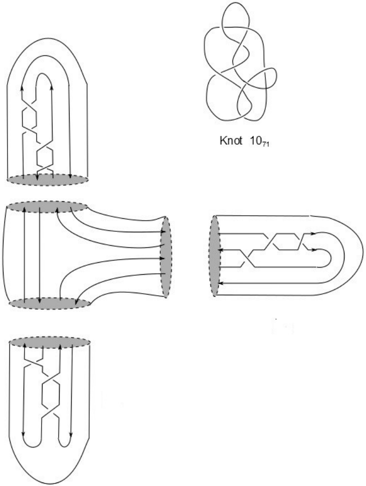
• These knots in S^3 are obtained from gluing three-balls where some three-balls have two or more four-punctured S^2 boundaries

10_{152} and 10_{71} arborescent knots

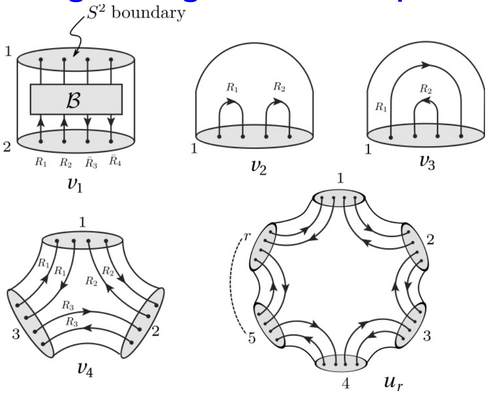






Building blocks

Requires the following building blocks to compute knot polynomials



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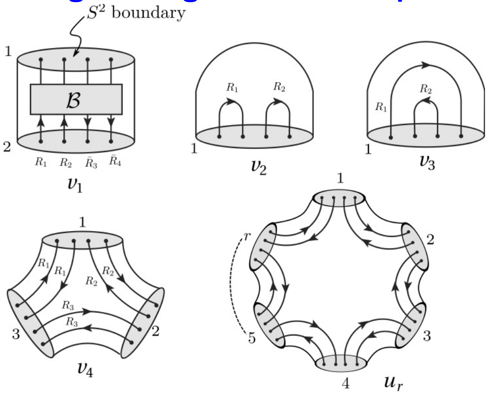

$$\nu_r = \sum_t (dim_q t)^{(1-r/2)} |\phi_t^{(1)}\rangle \ldots |\phi_t^{(r)}\rangle$$

Equivalent Building Blocks

- To write states of some diagrams, equivalent diagrams are shown:

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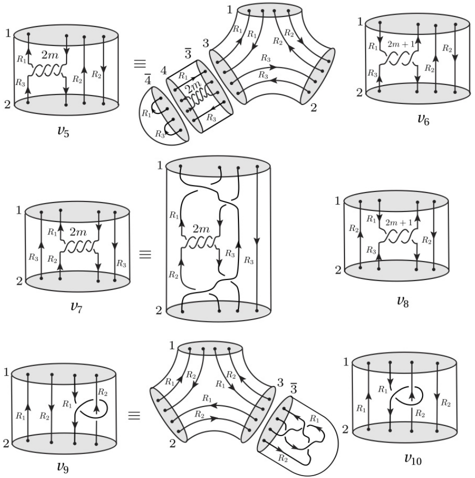
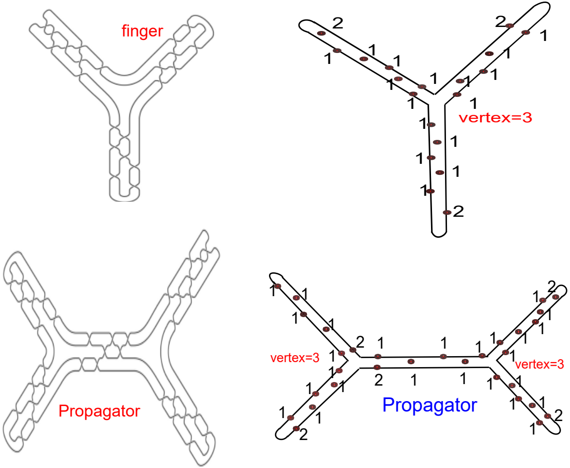


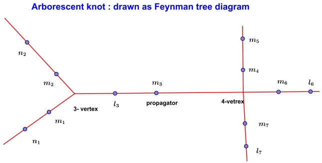
Arborescent knot- Feynman diagram analogy



Arborescent knots (Feynman tree diagram)

Family Approach: Arborescent knots

one universal invariant as a function of parameters- choice of parameters gives different knot invariants!



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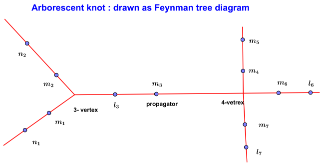

The best parametric family (for describing upto 10-crossing knots) in this class (of 4-point Feynman trees with up to 7 parameters)

A.Mironov, A. Morozov, An. Morozov, V.Singh, A. Sleptsov, PR (2016)

$d_R \sum_{X,\bar{Y}} F_{ap}(X) F_{pap}(X) T_X^n \bar{P}_{X\bar{Y}} F_{apa}(\bar{Y}) F_{aa}(\bar{Y})$

$9_{32-33}, 10_{45}, 10_{57}, 10_{62}, 10_{64}, 10_{66}, 10_{79-85}, 10_{87-91}, 10_{94}, 10_{98}, 10_{99}, 10_{139}, 10_{141}, 10_{143}, 10_{148-154}$- list not contained!

Arborescent knot invariants

- arborescent knot invariants will involve braiding eigenvalues and two types of duality matrices $a_{ts} \begin{bmatrix} \bar{R} & R \\ \bar{R} & R \end{bmatrix}$ and or $a_{ts_1} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix}$

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- However, other duality matrices are needed for **non-arborescent knots** invariants!

Do we know duality matrix elements

- **Duality matrices proportional to quantum Wigner 6j** (completely known for $SU(2)$ (Kirillov, Reshetikhin)

$$a_{j_{12}\, j_{23}} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{[2j_{12}+1][2j_{23}+1]} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \,,$$

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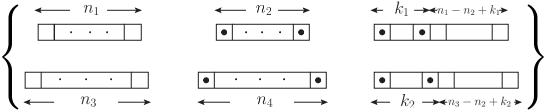

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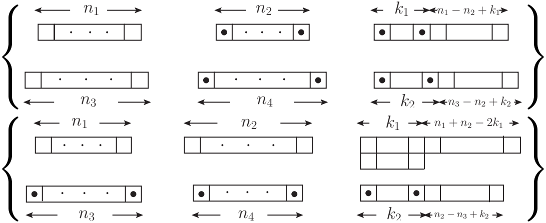

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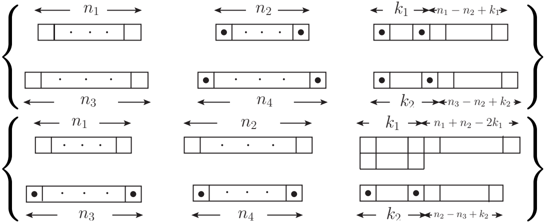

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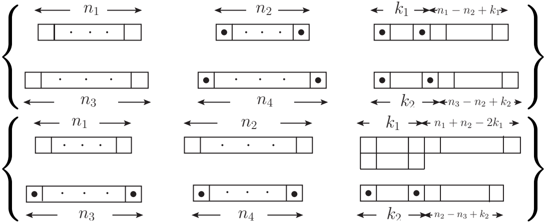
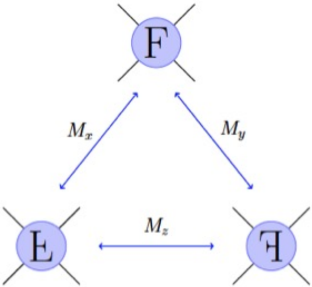


- So we can obtain colored HOMFLY-PT of any arborescent knot in variables $q, a = q^N$ for symmetric and antisymmetric colors.

Detection of Mutation

- On any two tangle, mutation refers to π rotation about x or y axis (M_x, M_y)



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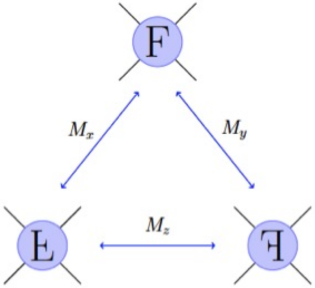

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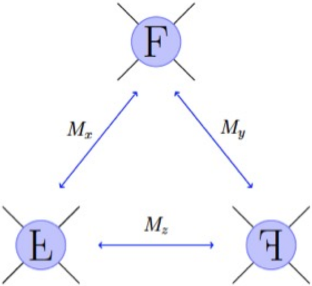

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- need to go beyond symmetric representation.

[2,1] colored HOMFLY-PT

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- We indicate some of the features of mixed representation leading to mutation detection

Additional information in mixed representation

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- Crucial input in the context of mixed representation: *multiplicity*

$$
\begin{aligned}
(21;0)\otimes(21;0) \quad = \quad & (42;0)_0 \oplus (2^3;0)_0 \oplus (31^3;0)_0 \oplus (321;0)_0 \\
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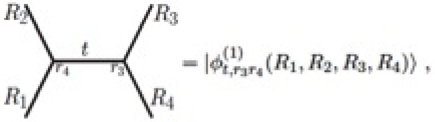
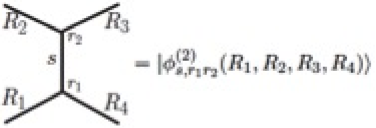
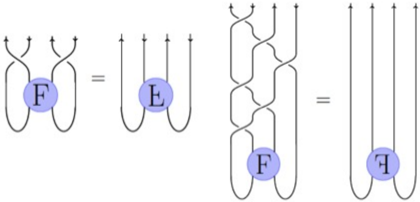
- Hence the states in the four-point conformal blocks involve multiplicity index r_i : $|\phi_{s,r_1,r_2}\rangle$





Mutation operation on two-tangles



$$
\begin{aligned}
|\text{Ⅎ}\rangle &= b_1^{(-)}[b_3^{(-)}]^{-1}|\mathbf{F}\rangle \\
&= \sum_{t,r_1,r_2} \{R,\bar{R},\bar{t},r_1\}\{R,\bar{R},\bar{t},r_2\}|\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})\rangle\langle\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})|\mathbf{F}\rangle
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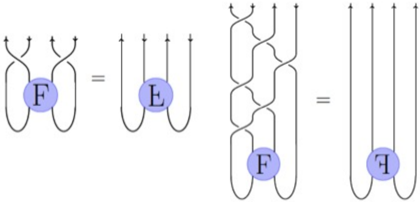

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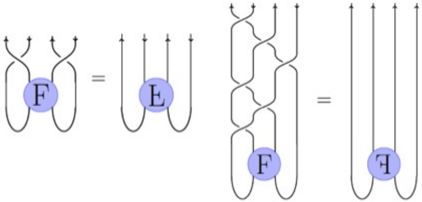

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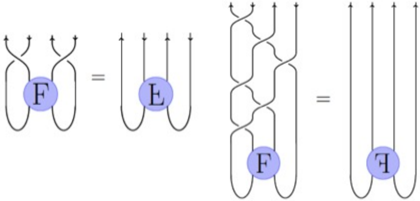
parenthesis denotes signs ± 1.

Mutation operation on two-tangles



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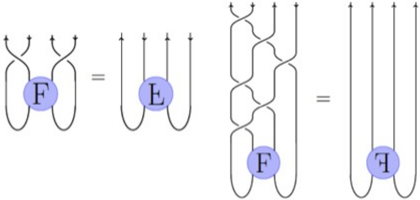

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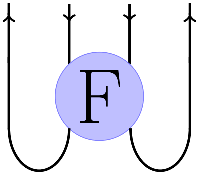
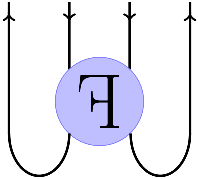
parenthesis denotes signs ± 1.Notice the amplitudes of mutant tangles are related by sign when $r_1 \neq r_2$ (occurs only for irreps with multiplicity)

Tangle and its M_y mutation

• The mutation operation (M_y) on $|\mathbf{F}\rangle$ which gives $|\reflectbox{F}\rangle$ whose state can also be obtained.



$$= \sum_{s,r_1,r_2} f_{s,r_1,r_2} |\phi^{(1)}_{s,r_1,r_2}(R,\bar{R},\bar{R},R)\rangle ,$$

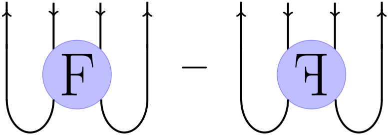


$$= \sum_{s,r_1,r_2} \tilde{f}_{s,r_1,r_2} |\phi^{(1)}_{s,r_1,r_2}(R,\bar{R},\bar{R},R)\rangle$$

• The coefficients are related by mutation operation :

$$\tilde{f}_{s,r_1,r_2} = (-1)^{r_1+r_2} f_{s,r_2,r_1} .$$

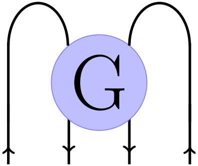
Difference between tangle F and mutant tangle of F



$$|\mathbf{F}\rangle - |\reflectbox{$\mathbf{F}$}\rangle = \left(f_{(1;1),0,1} + f_{(1;1),1,0}\right) \sum_{r_1 \neq r_2} |\phi^{(1)}_{(1;1),r_1,r_2}(R, \bar{R}, \bar{R}, R)\rangle \ .$$

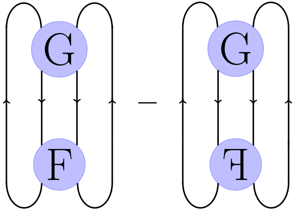
Knot and its mutant

Let us cap each of these tangles with a tangle $\langle \mathbf{G}|$, which we write



$$= \sum_{s,r_1,r_2} g_{s,r_1,r_2} \langle \phi^{(1)}_{s,r_1,r_2}(R, \bar{R}, \bar{R}, R)| \ .$$

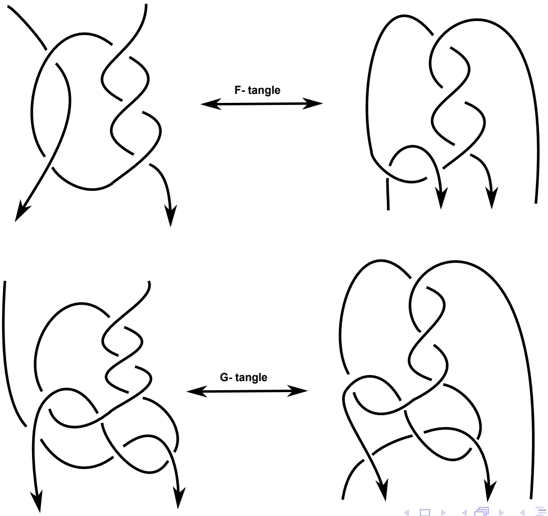
Then, the difference between the invariants of the mutant pairs arising from these 2-tangles will be



$$= (f_{(1;1),0,1} + f_{(1;1),1,0})(g_{(1;1),0,1} + g_{(1;1),1,0})$$

Kinoshita-Terasaka and Conway mutants

- This mutant pair is made of the following F and G-tangle



Knot invariant for the mutant pair

The explicit expression for the coefficient for tangle G turns out to be

$$g_{t,r_{10},r_{11}} = \dim_q R \sum \Omega(i,r_1,r_2,r_3)\Omega(j,r_6,r_7,r_8)\ {\lambda^{+}_{l;r_5}}^{2} a^{*0;0,0}_{l;r_5,r_5}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}$$
$$a^{*i;r_2,r_3}_{l;r_5,r_5}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix} {\lambda^{+}_{k;r_4}}^{-3} a^{0;0,0}_{k;r_4,r_4}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix} a^{i;r_1,r_2}_{k;r_4,r_4}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}(\lambda^{-}_{s;r_9})^2$$
$$a^{*0;0,0}_{s;r_9,r_9}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix} a^{*j;r_7,r_6}_{s;r_9,r_9}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix} a^{t;r_{10},r_{11}}_{j;r_8,r_9}(\lambda^{-}_{t;r_{10}})^{-1}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}$$
$$a^{i;r_1,r_3}_{j;r_8,r_6}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}$$

Similarly, the coefficients in the tangle F state is

$$f_{t,r_{10},r_{11}} = \sum_{w,u}\sum_{r_{14},r_{13},r_{12}} \Omega(t,r_{10},r_{11},r_{12})\ (\lambda^{+}_{w;r_{14}})^3 a^{*0;0,0}_{w;r_{14},r_{14}}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}$$
$$a^{t;r_{11},r_{12}}_{w;r_{14},r_{14}}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}(\lambda^{-}_{u;r_{13}})^{-2} a^{0;0,0}_{u;r_{13},r_{13}}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix} a^{*t;r_{12},r_{10}}_{u;r_{13},r_{13}}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}$$

Non-Arborescent Knots

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Knot Polynomials

- All the colored HOMFLY-PT polynomials are Laurent series

$$P_R^K(a, q) = \sum_{i,j} c_{ij} a^i q^j$$

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- Interestingly, the coefficients c_{ij} are integers
- Needs interpretation or reasoning for integer coefficients

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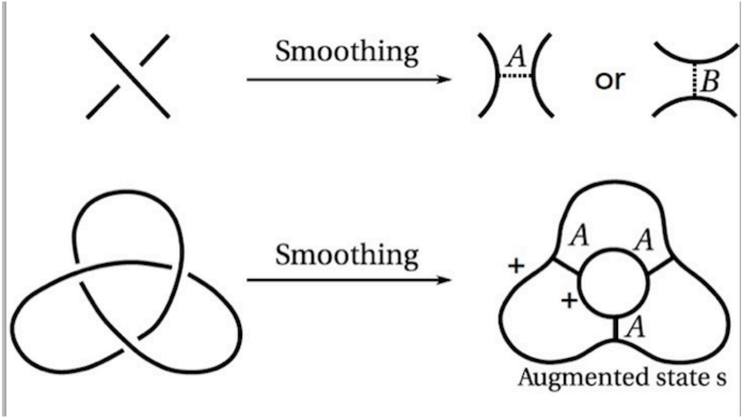
chain of dualities connecting Witten's construction with that of Ooguri-Vafa and their M-theory description

Khovanov Homology



Define $n(s) = n_B$, $j(s) = n_B + n_+ - n_-$ whose values for the above state is $n(s) = 0, j(s) = 2$

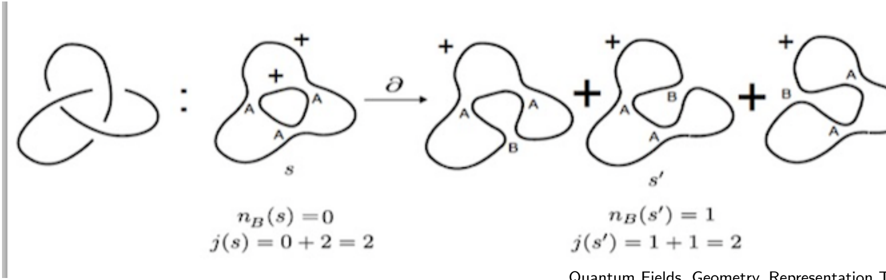
Chain Complex

- C_{nj} is the vector space with basis as states with $n(s) = n$ and $j(s) = j$. Then Jones polynomial is

$$J[K] = \sum_{n,j} (-1)^n q^j \dim(C_{nj})$$

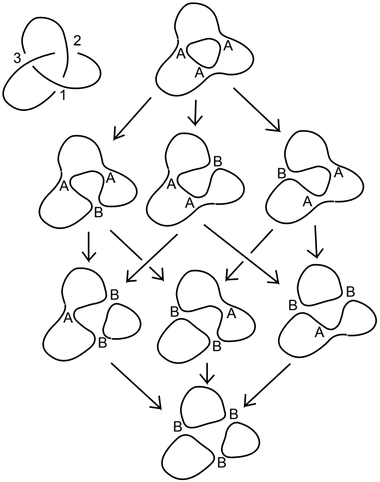
where the homology chain

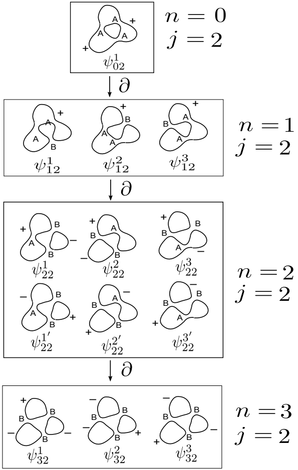
$$\partial : C_{n,j} \longrightarrow C_{n+1,j}, \quad \partial^2 = 0$$



• The vector space

$$H_n(C_{*j}) = \frac{\ker(\partial : C_{n,j} \longrightarrow C_{n+1}, j)}{\mathrm{Image}(\partial : C_{n-1,j} \longrightarrow C_{n,j})}$$





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- Taking $t = -1$ gives the Jones polynomial $J[K; q]$
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Take the partition function of $SU(N)$ Chern-Simons theory on S^3. It has the following expansion in the large N limit ($\lambda = \frac{2\pi}{k+N}, t = i\lambda N$):

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$\mathcal{F}_g$ computes certain low-energy effective theory terms in type IIA string theory compactified on Calabi-Yau. In terms of **D-branes in *IIA*** *string theory, $\mathcal{F}_g$ is calculable.(Gopakumar-Vafa)*

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Open string version in the presence of N-D branes wrapped on S^3 gives rise to Chern-Simons gauge theory on S^3.

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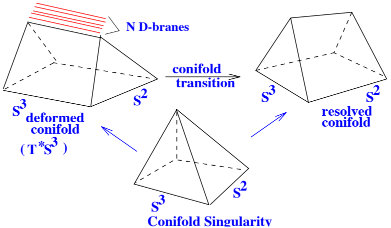

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$$\langle Z(U,V)\rangle = \sum_{n=1}^{\infty}\sum_{R} f_R(q^n,\lambda^n)\,\frac{1}{n}\,Tr_R V^n$$

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where $\mathcal{N}_{R,Q,s}$ are integers giving the number of $D2$-branes carrying bulk charge Q and spin s ending on $\mathcal{C}$ (in the presence of M $D4$-branes wrapping $\mathcal{C}$ and the uncompactified R^2 spacetime) and transforming in the representation R of $SU(M)$ group.

$\mathcal{N}$ integers from knot polynomials

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Note: Obtaining these integer coefficients $\mathcal{N}$ from full string theory calculation looks to be a difficult task.
The robust techniques of direct computation of knot invariants *(ramadevi-govindarajan-kaul)* in Chern-Simons theory indeed determines the **integer coefficients.**

• **Chern-Simons field theoretic knot polynomials gives D-brane interpretation for the** *integer coefficients*- **yet to be obtained from string theory!**

VERIFICATION USING KNOT INVARIANTS

Using group-theoretic methods, we can rewrite $f_R(t, \lambda)$ in terms of knot invariants $P_{R'}$.

For example,

$$f_{\yng(2)}(q, \lambda) = P_{\yng(2)}(q, \lambda) - \frac{1}{2} P_{\yng(1)}(q, \lambda)^2 - \frac{1}{2} P_{\yng(1)}(q^2, \lambda^2)$$

$$f_{\yng(1,1)}(q, \lambda) = P_{\yng(1,1)}(q, \lambda) - \frac{1}{2} P_{\yng(1)}(q, \lambda)^2 + \frac{1}{2} P_{\yng(1)}(q^2, \lambda^2)$$

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For trefoil $\mathcal{K}$

$$f_{\square\square} = \frac{1}{(q^{1/2}-q^{-1/2})}\left[\lambda q^{-\frac{1}{2}}\left(1+q^2\right)\left(-1+\lambda\right)^2\left(q-\lambda-q^2\lambda+q\lambda^2\right)\right]$$

$$f_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{(q^{1/2}-q^{-1/2})}\left[\lambda q^{-\frac{3}{2}}\left(1+q^2\right)\left(-1+\lambda\right)^2\left(-q+\lambda+q^2\lambda-q\lambda^2\right)\right]$$

Verified for many knots (work with T. Sarkar (2000)

Extension to Framed Knots and Links

- The topological scalar operator:

$$Z(\{U_\alpha\},\{V_\alpha\}) = exp\left[\sum_{\alpha=1}^{r}\sum_{n=1}^{\infty}\frac{1}{n}TrU_\alpha^n\ TrV_\alpha^n\right] ,$$

$$\langle Z(\{U_\alpha\},\{V_\alpha\})\rangle_A = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\{R_\alpha\}} f_{R_1,\ldots R_r}(q^n,\lambda^n)\prod_{\alpha=1}^{r} Tr_{R_\alpha} V_\alpha^n\right]$$

where

$$f_{R_1,\ldots R_r}(q,\lambda) = \lambda^{\frac{1}{2}\ell_\alpha p_\alpha}\sum_{Q,s}\frac{1}{(q^{1/2}-q^{-1/2})}N_{(R_1,R_2\ldots R_r),Q,s}q^s\lambda^Q$$

$N_{(R_1,\ldots R_r),Q,s}$ are integers only for $U(N)$ invariants *with a specific choice of the $U(1)$ charge*

That is

$$\langle\prod_\alpha Tr_{R_\alpha}(U_\alpha)\rangle = (-1)^{\sum_\alpha \ell_\alpha p_\alpha}\ V^{SU(N)}_{R_1\ldots R_r}[L]\ V^{\{U(1)\}}_{\frac{\ell_1}{\sqrt{N}},\ldots\frac{\ell_r}{\sqrt{N}}}[L]$$

Can we write $\ln Z[M]$ as closed string expansion?

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• Using the three-manifold invariants from surgery of links and the input from topological string duality conjectures, we did achieve!

Can we write $\ln Z[M]$ as closed string expansion?

• Using the three-manifold invariants from surgery of links and the input from topological string duality conjectures, we did achieve!
The Chern-Simons partition function $Z[M]$ from these $U(N)$ link invariants

$$\frac{Z[M]}{S_{00}} = \frac{Z_0[M]}{S_{00}} + \mathcal{F}[M, z, \{p_i\}, \{\ell k_{ij}\}]$$

Choosing $N = M$ and the v.e.v

$$\langle Tr_{R_\alpha} V_\alpha \rangle = (-1)^{\sum_\alpha \ell_\alpha p_\alpha} \frac{S_{0R_\alpha}}{S_{00}} \ ,$$

we get

$$\alpha^{\sigma[L]} \frac{Z_0[M]}{(S_{00})^{r+1}} = \langle Z(\{U_\alpha\}, \{V_\alpha\}) \rangle_{A,\tilde{A}}$$

$$\ln\left(\alpha^{\sigma[L]} \frac{Z_0[M]}{(S_{00})^{r+1}}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\{R_\alpha\}} f_{R_1,\ldots R_r}(q^n, \lambda^n) \prod_{\alpha=1}^{r} \langle Tr_{R_\alpha} V_\alpha^n \rangle$$

RHS can be simplified to resemble A-model topological string partition function

$\ln Z(M)$ contd

$$\ln\left(\alpha^{\sigma[L]}\frac{Z_0[M]}{(S_{00})^{r+1}}\right)=\sum_{n=1}^{\infty}\sum_{g}\frac{1}{n}\left(\sinh\frac{dg_s}{2}\right)^{2g-2}\times$$

$$\{\sum_{Q}\sum_{\{\ell_\alpha\}}\sum_{\{s_\alpha\}}\hat{N}_{(R_{\ell_1,s_1},\ldots R_{\ell_r,s_r}),g,Q}\quad(-1)^{\sum_\alpha s_\alpha}(-1)^{n\sum_\alpha \ell_\alpha p_\alpha}\lambda^{\frac{1}{2}n\sum_\alpha \ell_\alpha p_\alpha}$$

$$\left(\lambda^{n\{Q+\sum_\alpha(\frac{-\ell_\alpha}{2}+s_\alpha)\}}-\lambda^{n\{Q+1+\sum_\alpha(\frac{-\ell_\alpha}{2}+s_\alpha)\}}\right)\}$$

$$=\sum_{g,n,m}\frac{1}{n}(2\sinh\frac{ng_s}{2})^{2g-2}n_{g,m}e^{-dmt}$$

where $\hat{N}$'s are refined integer invariants obtained from $f_{R_1,\ldots R_r}(q,\lambda)$ and $n_{g,m}$ are the Gopakumar-Vafa invariants.

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In the large k limit

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whereas we find

$$\ln\left(\sum_c Z_c[M]\right) = \text{Closed String partition function}$$

Hence we cannot predict duality between Chern-Simons gauge theory on M with the A-model string theory with the $n_{g,m}$'s we have determined

Generalisation of the duality to *SO* gauge groups

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A-model closed strings on an orientifold of the resolved conifold is dual to *SO*/*Sp* Chern-Simons theory (Sinha and Vafa)

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- Incorporating Wilson loop observables

$$\ln\langle Z(\{U_\alpha\},\{V_\alpha\})\rangle = \mathcal{F}_{\mathcal{G}}(V) = \frac{1}{2}\mathcal{F}^{(or)}_{\mathcal{R}}(V) + \mathcal{F}^{(unor)}(V)$$

Not clear how to seperate, we showed LHS (Pravina Borhade and PR)

$$\langle Z(\{U_\alpha\},\{V_\alpha\})\rangle = \exp\left[\sum_{n=1}^{\infty}\sum_{\{R_\alpha\}} g_{R_1,R_2,\ldots R_r}(q^n,\lambda^n)\ \frac{1}{n}\prod_{\alpha=1}^{r}\ Tr_{R_\alpha}V_\alpha^n\right]$$

where $g_{R_1,\ldots R_r}(q,\lambda) = \sum_{Q,s}\frac{1}{(q^{1/2}-q^{-1/2})}N_{(R_1,R_2\ldots R_r),Q,s}q^s\lambda^Q$

$N_{(R_1,\ldots R_r),Q,s}$ are integers-how to find oriented contribution?

Oriented contribution

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•Composite representation invariants to extract oriented contribution (Marino)

$$\mathcal{F}_{\mathcal{R}}^{(or)}(V) = \sum_{R,S} \langle W_{(R,S)}[C] \rangle (Tr_R V)(Tr_S V) = \sum_t \mathcal{R}_t[C] Tr_t V$$

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• Obtained invariants of torus knots & links for some composite representations and verified the form of $h_R(q, \lambda)$(2010, C. Paul, P. Borhade,PR)

- Using [r]-colored Kauffman polynomials for figure-eight knot from the study of structural properties of colored Kauffman homologies of knots (Nawata,Zodin,PR,2013) and Morton's result for $(\Box,\Box)$ composite invariant for figure-eight, we verified the integrality structure for $h_{\Box\Box}$ and $g_{\Box\Box}$.
- Results on adjoint polynomials for arborescent knots (Mironov et al), enabled us to verify integrality properties for all arborescent knots upto for $\Box\Box$, $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$ (*Work with Mironov, Morozov, Sleptsov, Vivek*)

Witten's Intersecting brane Construction

Directions	0	1	2	3	4	5	6	7	8	9
NS5	$\surd$	$\surd$	$\surd$	$\surd$	$*$	$*$	$*$	$*$	$\surd$	$\surd$
D3	$\surd$	$\surd$	$\surd$	$*$	$*$	$*$	$\surd$	$*$	$*$	$*$

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- Topological fied theory on Euclidean W using R-symmetry twist to $\mathcal{N} = 4$ SYM on V where $\vec{X} \equiv \phi_\mu$ leading to 3-d CS action on W involving complex $\mathcal{A} = A + \omega\phi$ where $\omega = f(\tau)$

Witten's intersecting brane construction(contd)

- Localization equations of topologically twisted theory are BHN (Bogomolnyi, Hitchin, Nahm) equations
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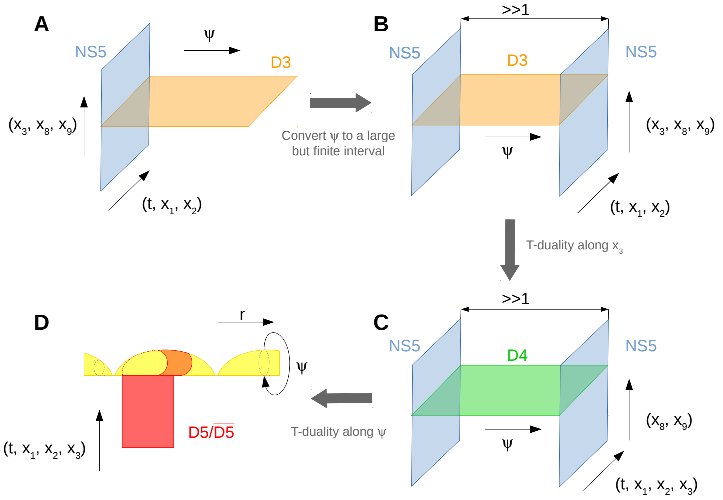
- The equations develop sources due to the presence of surface operators

$$\begin{aligned} F - \phi \wedge \phi &= 2\pi\alpha\delta_K \\ d_A * \phi &= 2\pi\beta dl \wedge \delta_K \\ d_A \phi &= 2\pi\gamma\delta_K \end{aligned}$$

where dl is the line element along knot K, (α, β, γ) are parameters that characterise K and δ_K is the delta function 2-form.

M-Theory description of Witten's model



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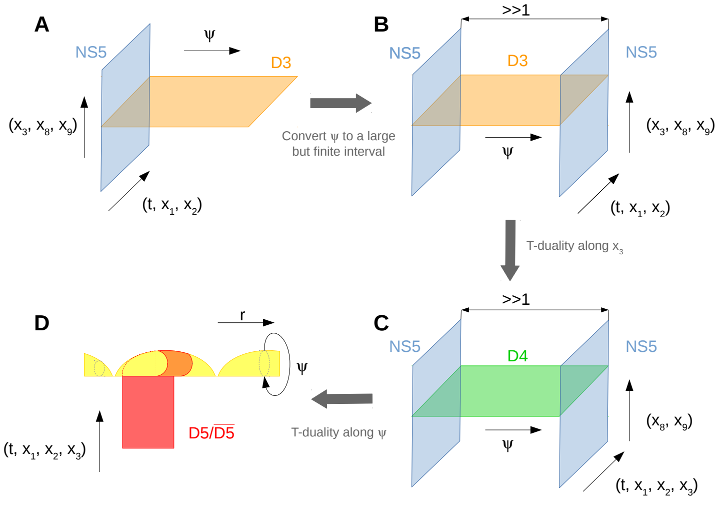

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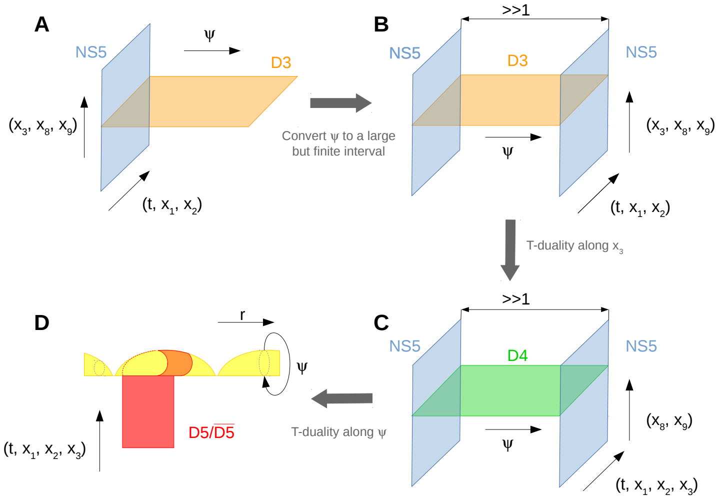

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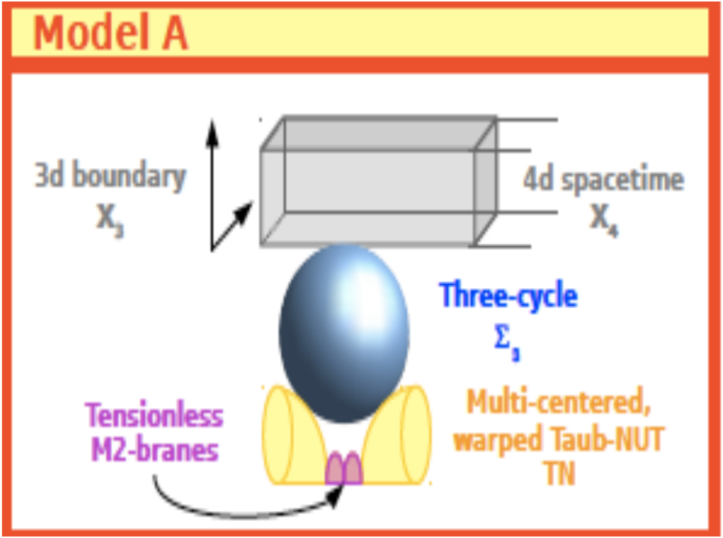
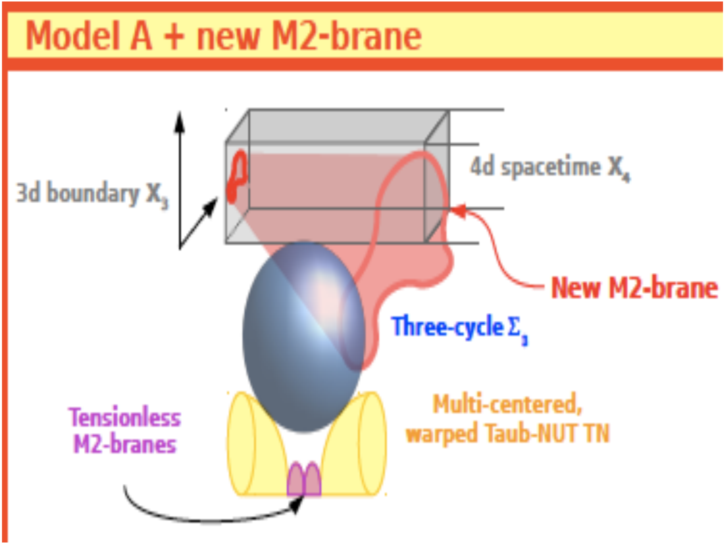
We get the full topological action of Witten !

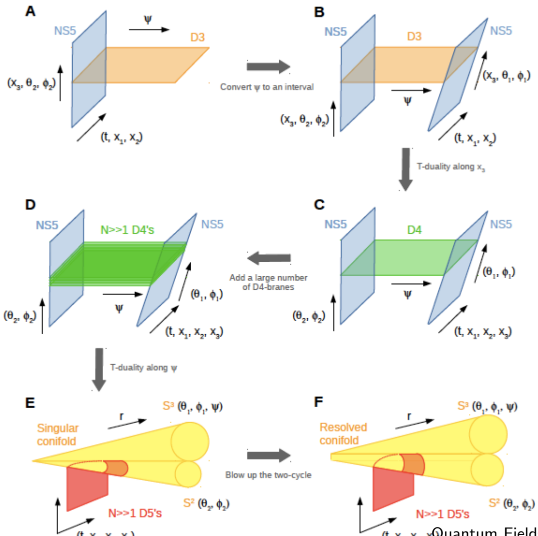
Model A: Witten model



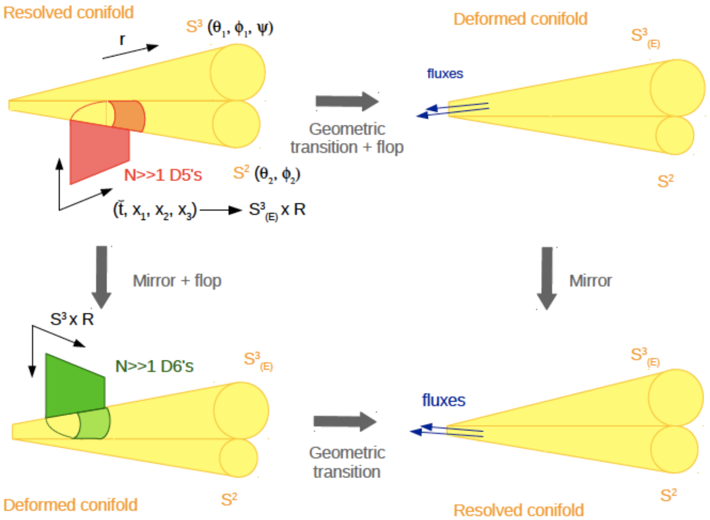
Model A: Witten model with knot



Two NS5-branes with relative orientation from Witten model



Relation to Ooguri-Vafa model



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Thank You