Instantons and Monopoles

Lecture 2: Mathematics

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Reducing the (Anti-)Self-Duality Equation

$$* F = F$$

$$F = dA + A \wedge A = F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Self-dual Yang-Mills (Instanton) $\mathbb{R}^4$

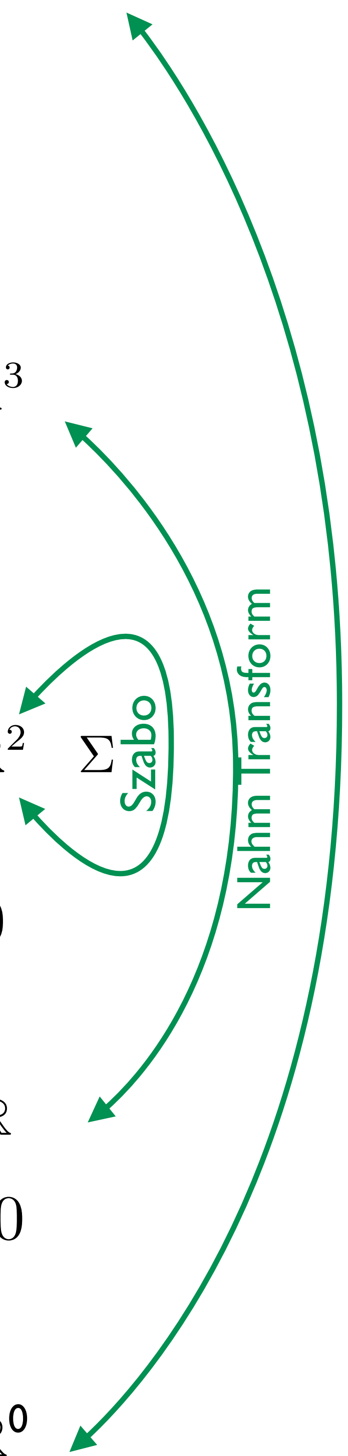
$$\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = \partial_2 A_3 - \partial_3 A_2 + [A_2, A_3]$$
$$\partial_0 A_2 - \partial_2 A_0 + [A_0, A_2] = \partial_3 A_1 - \partial_1 A_3 + [A_3, A_1]$$
$$\partial_0 A_3 - \partial_3 A_0 + [A_0, A_3] = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$$

Bogomolny Equation (Monopole) $\mathbb{R}^3$

$$\partial_0 = 0$$

$$\Phi = -A_0$$

$$F = *_3 D\Phi$$



Hitchin System $\mathbb{R}^2$

$$\partial_2 = 0, \partial_3 = 0$$

$$\Phi = A_3 - iA_2$$

$$\begin{cases} F_{z\bar{z}} = -\frac{i}{4}[\Phi, \Phi^\dagger] \\ \bar{D}\,\Phi = 0 \end{cases}$$

Nahm Equation $\mathbb{R}$

$$\partial_1 = \partial_2 = \partial_3 = 0$$

$$\partial_0 A_1 + [A_0, A_1] = [A_2, A_3]$$
$$\partial_0 A_2 + [A_0, A_2] = [A_3, A_1]$$
$$\partial_0 A_3 + [A_0, A_3] = [A_1, A_2]$$

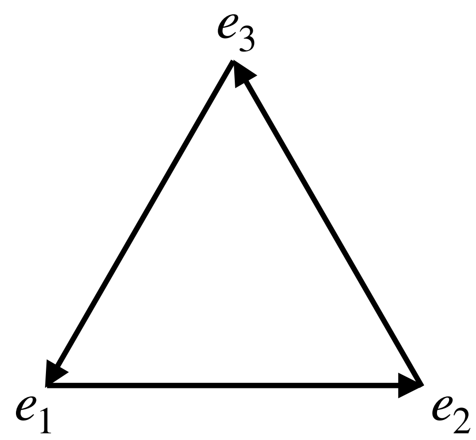
ADHM Equations $\mathbb{R}^0$

These equations are secretly quaternionic!

- Quaternions: $$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1$$

$\mathbb{H} = \mathbb{R}^4$ $$q = 1 \otimes q_0 + e_1 \otimes q_1 + e_2 \otimes q_2 + e_3 \otimes q_3$$

$$q^* = 1 \otimes q_0 - e_1 \otimes q_1 + e_2 \otimes q_2 + e_3 \otimes q_3$$



Admit 2-dim representation S, e.g. $e_j = -i\sigma_j$ with Pauli matrices σ_j.

- Given a connection on a Hermitian bundle $\mathcal{E} \to \mathbb{R}^4$ form the Dirac operator

$$\not{D} = 1 \otimes D_0 + e_1 \otimes D_1 + e_2 \otimes D_2 + e_3 \otimes D_3$$ acting on sections of $S \otimes \mathcal{E}$

its Hermitian conjugate is $\not{D}^\dagger = -1 \otimes D_0 + e_1 \otimes D_1 + e_2 \otimes D_2 + e_3 \otimes D_3$

- Then Self-Duality is equivalent to $\not{D}^\dagger \not{D}$ being real:

$$\not{D}^\dagger \not{D} = -(D_0^2 + D_j D_j) + \sum_{(i,j,k)=cyc(1,2,3)} e_i \otimes \Big([D_j, D_k] - [D_0, D_i] \Big)$$

- Moreover, all connections form an infinite-dimensional hyperkähler space with triholomorphic gauge group action, and the moment of this action is

$$\mu(A) = \mathrm{Im}\, \not{D}^\dagger \not{D}$$

see Hitchin "Dirac Operator" '02

Nahm Transform

Werner Nahm '80

Quadruplet of Hermitian matrix-valued functions on an interval:

$$(T_0(s), T_1(s), T_2(s), T_3(s)).$$

Corresponding to a connection $\nabla = \frac{d}{ds} + T_0$

and three bundle endomorphisms T_1, T_2, T_3

These form an infinite-dimensional hyperkähler affine space

by viewing $\delta\mathbb{T} = i(\delta\nabla + e_1 \otimes \delta T_1 + e_2 \otimes \delta T_2 + e_3 \otimes \delta T_3)$ as a quaternion

with norm $|\delta\mathbb{T}| := -\int \mathrm{tr}\,(\delta T_0^2 + \delta T_1^2 + \delta T_2^2 + \delta T_3^2) ds$

Gauge transformation action
acts isometrically
and commutes with the complex structures

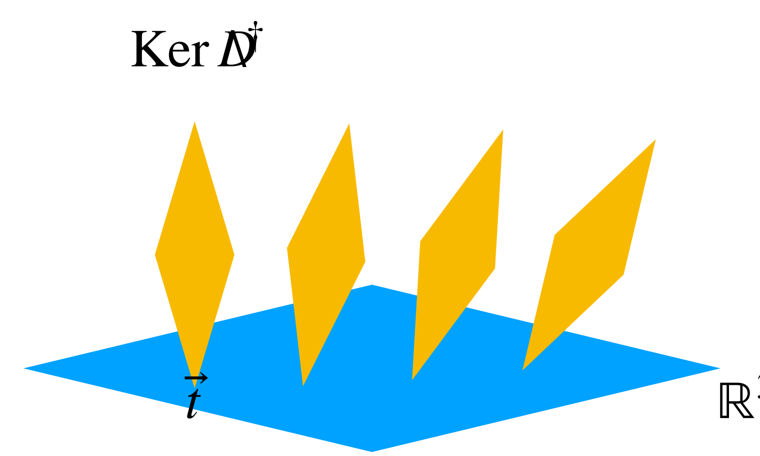
$$g(s) : \begin{pmatrix} T_0(s) \\ T_1(s) \\ T_2(s) \\ T_3(s) \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}T_0 g + i g^{-1}\frac{d}{ds}g \\ g^{-1}T_1 g \\ g^{-1}T_2 g \\ g^{-1}T_3 g \end{pmatrix}$$

- Nahm equations are the moment maps conditions:

  Solve them (with proper boundary conditions),

  rank=monopole number

$$[\nabla, T_1] = [T_2, T_3]$$
$$[\nabla, T_2] = [T_3, T_1]$$
$$[\nabla, T_3] = [T_1, T_2]$$



- Form a family of Dirac (Weyl) operators parameterized by by $\vec{t} \in \mathbb{R}^3$

$$\not{D}^\dagger = \frac{d}{ds} - (\not{T} - \not{t})$$

- Find an orthonormal basis of solutions of

$$\not{D}^\dagger \Psi = 0$$

- Compute the Higgs field

$$\Phi_{ij} = \int ds \Psi_i^\dagger s \Psi_j$$

Endomorphism induced on $\mathrm{Ker}\, D^\dagger$

and the gauge field

$$A_{a,ij} = \int ds \Psi_i^\dagger \frac{\partial}{\partial t^a} \Psi_j$$

Connection induced on $\mathrm{Ker}\, D^\dagger$

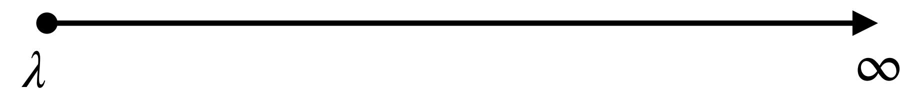
Example

Consider rank one case:

Dirac:

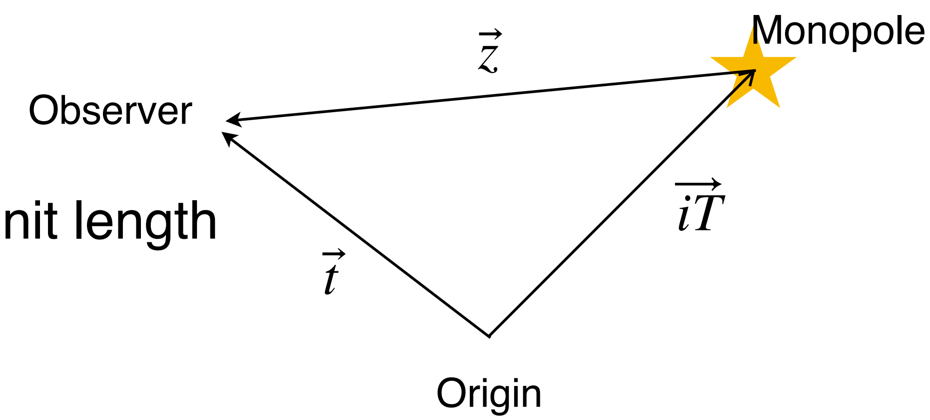
$$\frac{d}{ds}T_1 = \cancel{[T_2, T_3]}^{\,0}$$
$$\frac{d}{ds}T_2 = \cancel{[T_3, T_1]}^{\,0}$$
$$\frac{d}{ds}T_3 = \cancel{[T_1, T_2]}^{\,0}$$

For constant T_j and parameters t_j



Solving the twisted Dirac Eq. $(\frac{d}{ds} - \sigma_j z_j)\Psi = 0$

$$\Psi = \frac{1}{N} e^{\not{z} s}\zeta$$

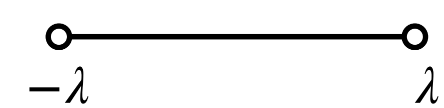


Ensure the solution is L^2 choose negative unit length eigenvector: $\sigma_j z_j \zeta = -z\zeta$

Normalizing factor: $N = \frac{1}{2z} e^{-2\lambda z}$

Resulting in Dirac monopole:

$$\Phi = e^{2\lambda z} 2z \int_\lambda^\infty ds\, s e^{-2sz} = e^{2\lambda z} 2z \left[-\frac{s e^{-2sz}}{2z} - \frac{e^{-2sz}}{(2z)^2} \right]_\lambda^\infty = \lambda + \frac{1}{2z}$$



BPS Monopole:

$$\Psi = \frac{1}{N} e^{(\not{T} - \not{t})s} = \frac{1}{N} e^{\not{z} s} \qquad N^2 = \frac{\sinh 2\lambda z}{2z}$$

Leading to Bogomolny-Prasad-Sommerfield monopole:

$$\Phi(\vec{z}) = \left(\lambda \coth 2\lambda z - \frac{1}{2z} \right) \frac{\not{z}}{z}, \qquad A(\vec{z}) = -\left(\frac{\lambda}{\sinh(2\lambda z)} - \frac{1}{2z} \right) \frac{i[\not{z}, d\not{z}]}{2z}$$

The Problem

- U(n) Instanton A on M with curvature $F_A = dA + A \wedge A$

1. anti-self-dual: $*F = -F$ and
2. finite action: $S_{YM} = -\int \mathrm{tr}\, F \wedge *F < \infty$

(in orientation $dt_1 dt_2 dt_3 d\tau$)

- M can be k-centered Taub-NUT space

$$\tau \sim \tau + 2\pi$$

$S^1 \rightarrow TN_k$:
$\downarrow$
$\mathbb{R}^3$

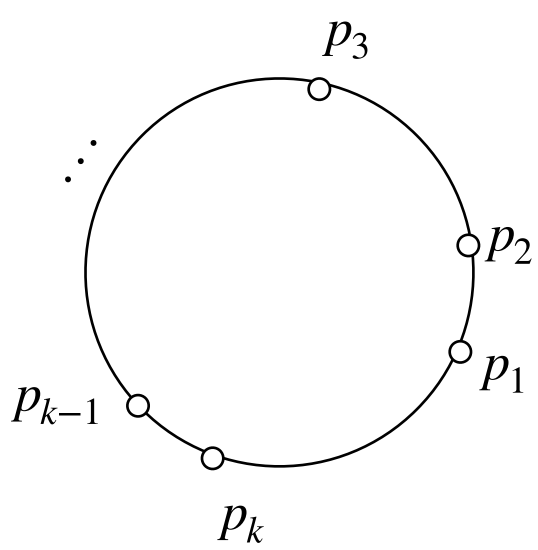
$$ds^2 = V d\vec{t}^{\,2} + \frac{(d\tau + \omega)^2}{V} \quad , \quad V = l + \sum_{\sigma=1}^{k} \frac{1}{2|\vec{t} - \vec{\nu}_\sigma|} \quad , \quad d\omega = *_3 dV$$

Abelian Instantons

- Each NUT ν_σ has associated line bundle $R^{(\sigma)}$ carrying an abelian instanton $a^{(\sigma)} = \frac{1}{2t_\sigma}\frac{d\tau+\omega}{V} - \eta_\sigma$
  - its holonomy around the tau-fiber is trivial and $d\eta_\sigma = *_3 d\frac{1}{2t_\sigma}$
  - its first Chern number is $\frac{1}{2\pi}\int_{C_\rho} da^{(\sigma)} = \delta^\sigma_\rho$
- In addition, there is a family L^s of line bundles carrying an instanton $sa^{(0)} = s\frac{d\tau+\omega}{V}$
  - its holonomy around the tau-fiber is s and
  - its first Chern number is $\frac{1}{2\pi}\int_{C_\rho} da^{(0)} = \frac{s}{l}$
- Importantly, $L^l \otimes \bigotimes_{\sigma=1}^{k} R^{(\sigma)}$ is trivial.

Any k points $p_1, p_2, \ldots, p_k$ on a circle of length l give a family of abelian instantons

$$a_s = sa^{(0)} + \sum_{p_\sigma < s} a^{(\sigma)}$$



Question: Which such family is the best?

U(n) Instantons on TN_k

with Mark Stern and
Andres Larrain-Hubach
arXiv:1608.00018

Def: A U(n) instanton $(\mathscr{E}, A)$ is a Hermitian bundle $\mathscr{E} \to \mathrm{TN}_k$ with s connection, whose curvature F is

a) anti-self-dual: *F=-F and
b) is square integrable $\|F\|^2 = -\int \mathrm{tr}\, F \wedge *F < \infty$

Thm [Uhlenbeck '79]: Instanton curvature on $\mathbb{R}^4$ decays quartically: $|F|(x) < \dfrac{C}{|x|^4}$.

Question: What is the decay rate for instantons on TN_k?

The curvature decay rate is very sensitive to the underlying space volume growth.

For example,

Thm [Mochizuki '14]: For an instanton on T^2x$\mathbb{R}^2$ the curvature norm |F|=$O(1/r^{1+\varepsilon})$.

with Mark Stern and
Andres Larrain-Hubach
arXiv:1608.00018

Thm: Any Hermitian connection with finite action on TN_k that is Yang-Mills, i.e. $D_A{*}F=0$, satisfies $|F|(x) \to 0$, as $d(o,x) \to \infty$.

A technical assumption (**generic asymptotic holonomy)**:
Presume there is a ray in R^3 such that the eigenvalues $\{e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_n}\}$ of the holonomy along the TN fiber have distinct limits at infinity.

Thm: Curvature of an instanton on TN_k which has generic asymptotic holonomy decays quadratically:

$$|F|(x) < \frac{C}{d(0,x)^2}$$

Thm: Generic asymptotic holonomy => limiting holonomy conjugacy class exists and is the case for any ray.

with Mark Stern and
Andres Larrain-Hubach
arXiv:1608.00018

Thm: There is a local trivialization in which the connection one-form A has the form

$$A = -i\,\mathrm{diag}(a_1, a_2, \ldots, a_n) + O(\frac{1}{t^2})$$

with $\quad a_j = (\lambda_j + \frac{m_j}{2t})\frac{d\tau + \omega}{V} - \frac{m_j}{k}\omega\,.$

- The numbers m_j are integers, called magnetic charges.
- We can relabel so that $0 \leq \lambda_1 < \ldots < \lambda_n < l\,.$

Thm: Harmonic spinors decay exponentially fast if no $\lambda_j = 0$,
and quadratically otherwise.

Thm: The index of the associated Dirac operator is

$$\mathrm{ind}_{L^2} D_A^- = \sum_j \left((\{\lambda_j/l\} - \frac{1}{2})(m_j - k\lfloor \lambda_j/l \rfloor) - \frac{k}{2}\{\lambda_j/l\}^2 \right) + \frac{1}{8\pi^2}\int \mathrm{tr}\, F \wedge F$$

Asymptotic term   Bulk term

$\lfloor a \rfloor$ = largest integer not exceeding a, and $\{a\} = a - \lfloor a \rfloor$

Observation: When λ_j crosses 0, the index changed by m_j.

Down Transform
(generalizing ADHM-Nahm transform)

- TN$_k$ is equipped with abelian instantons:

  one associated to each NUT: $a^{(\sigma)} = \frac{1}{2|t-\nu_\sigma|}\frac{d\phi+\eta}{V} - \eta_\sigma \qquad d\eta_\sigma = *d\frac{1}{2|t-\nu_\sigma|}$

  and one more: $a^{(0)} = \frac{d\phi+\eta}{V}$

  These organize into a family parameterized by a bow:

  $a_s := s a^{(0)} + \sum_{p_\sigma < s} a^{(\sigma)}$ instanton connection on a line bundle $e_s = L^s \otimes \bigotimes_{p_\sigma < s} R^{(\sigma)}$

- Given an instanton A on a Hermitian bundle $\mathcal{E}$ over TN$_k$,

  consider a family $A \otimes 1_{e_s} + 1_{\mathcal{E}} \otimes a_s$ on $\mathcal{E} \otimes e_s$.

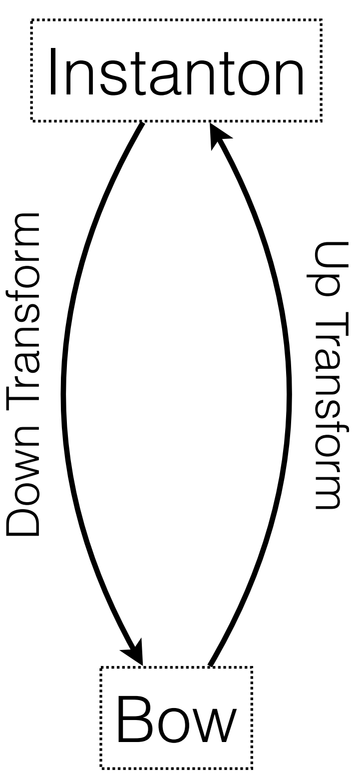
  It has a family of associated Dirac operators D_s

- Eigenvalues of holonomy of A at infinity = $\exp(2\pi i\, \lambda_j)$.

  $\mathrm{Ind}\, D^\dagger_s = R(s)$

  Bow fiber $E_s = \mathrm{Ker}\, D^\dagger_s = \{\Psi \mid D^\dagger_s \Psi = 0\}$

(A;$\mathcal{E}$)

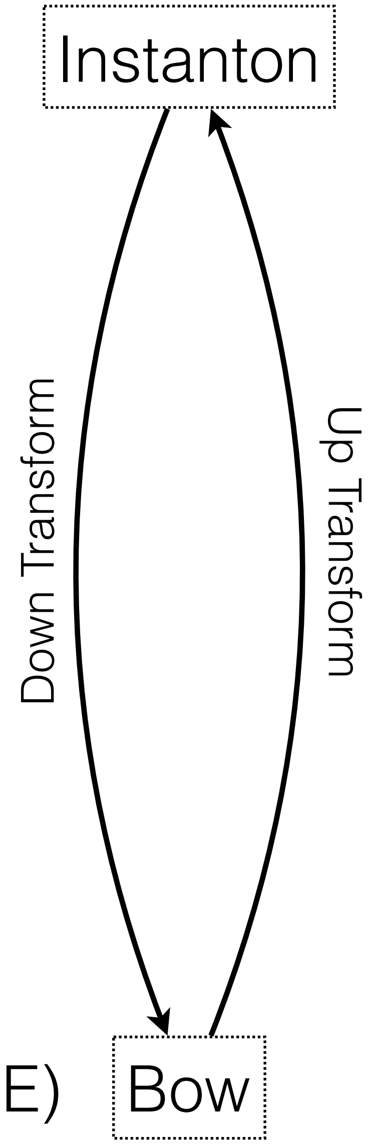


(T,Q,B;E)

Instanton => Bow Representation.

Family of Dirac operators: (A;$\mathcal{E}$)

$$D_s = D_{A\otimes 1_{e_s}+1\otimes a_s}$$



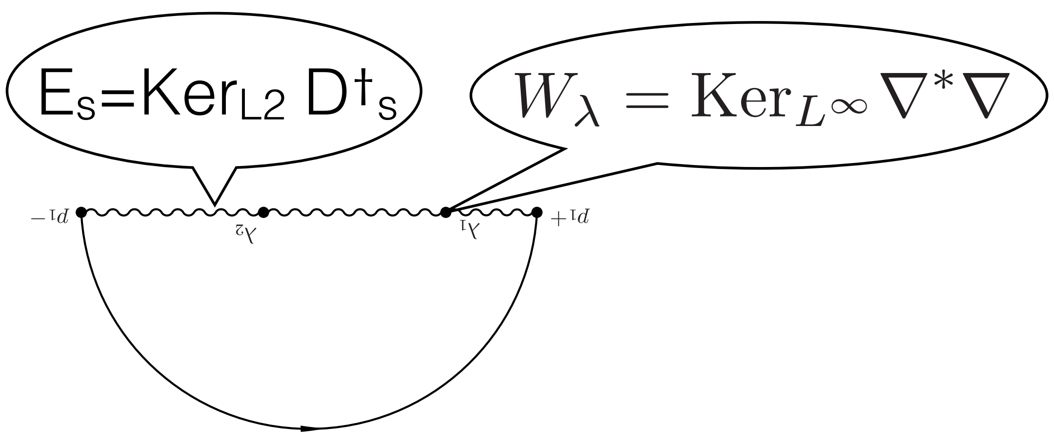
Bundle:

$$\mathcal{E}|_{(t,b)} = \mathrm{Ker}_{L^2}\mathcal{D}_{(t,b)}$$

Instanton:

$$A_{\alpha\beta} = \int_{\mathrm{Bow}} \chi_\alpha^\dagger (d+a_s)\chi_\beta ds.$$

Bow Representation=Index Bundle:

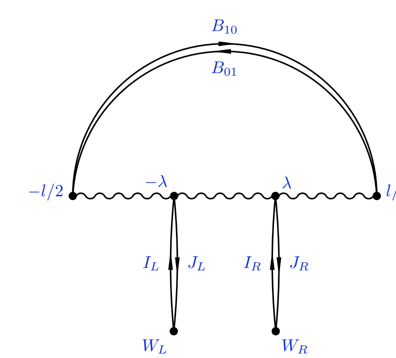


Family of bow Dirac operators:

$$\mathcal{D}_{(t,b)} = \mathcal{D}\otimes 1_e + 1_E \otimes \mathfrak{d}^c \text{ on } S\otimes E\otimes e^*$$

Bow Dirac operator:

$$\mathcal{D} = \begin{pmatrix} i\frac{d}{ds} + T_0 + e_j T_j \\ B_p^\dagger \\ (B_p^c)^\dagger \\ Q^\dagger \end{pmatrix}$$



of a fixed solution (T,Q,B) of a large rep. R.

Bow Solution: (T,Q,B;E)

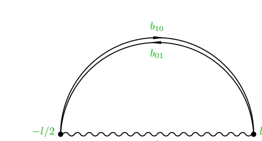
$\{f_\sigma\}$ orthonormal basis of $\mathrm{Ker}_{L^\infty}\nabla^*\nabla$

$\{\Psi_a\}$ orthonormal basis of $\mathrm{Ker}_{L^2}D_s^\dagger$

$$T_{ab}^0 = \int_{\mathrm{TN}} \Psi_a^\dagger i\frac{d}{ds}\Psi_b d\mathrm{Vol},\ T_{ab}^j = \int_{\mathrm{TN}} \Psi_a^\dagger t^j \Psi_b d\mathrm{Vol}$$

$$Q_{a\sigma} = \int_{\mathrm{TN}} \Psi_a^\dagger D_\lambda f_\sigma d\mathrm{Vol},$$

$$B_{ab}^p = \int_{\mathrm{TN}} \Psi_a^\dagger b^p \Psi_b d\mathrm{Vol},$$



Point on a TN = (t,b), solution of a small rep. s.

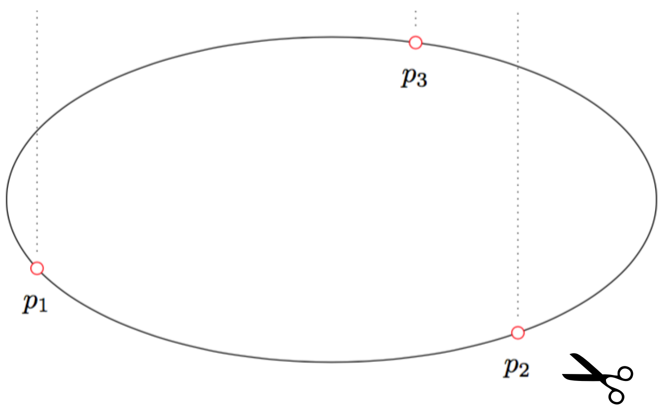
Has corresponding bow Dirac operator $\mathfrak{d}$

Bows

Bow (A_k):

Bow:

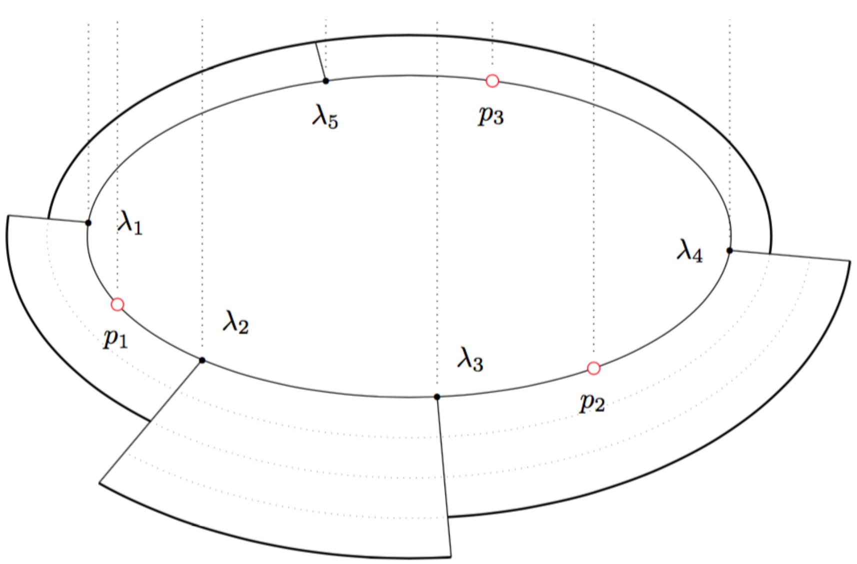
Circle diagram:



Representation *R* **of the bow:**

$$\{\lambda_j\}, R(s)$$

$$\{W_\lambda = \mathbb{C}\}_{R(\lambda-)=R(\lambda+)}$$



Let S be a 2d representation of quaternions,
and e_1, e_2, and e_3 be quaternionic units.

Affine space: Dat(*R*)=B⊕F⊕N is hyperkähler

B: $B^+_\sigma = \begin{pmatrix} B^\dagger_{\sigma,\sigma+1} \\ B_{\sigma+1,\sigma} \end{pmatrix} \in \mathrm{Hom}(E_{p_\sigma -}, S \otimes E_{p_\sigma +})$

F: $Q_\lambda = \begin{pmatrix} J^\dagger_\lambda \\ I_\lambda \end{pmatrix} \in \mathrm{Hom}(W_\lambda, S \otimes E_\lambda)$

N: $D = \frac{d}{ds} + T_0 + e_j T_j \in \mathrm{Con}(S \otimes E)$

Gauge group $\mathcal{G}$ acts triholomorphically on Dat(R)!

$$B^+_\sigma \mapsto g(p_\sigma -) B^+_\sigma g(p_\sigma +),$$

$$Q_\lambda \mapsto g(\lambda) Q_\lambda,$$

$$T_0(s) \mapsto g^{-1}(s) T_0 g(s) + g^{-1}(s) \frac{d}{ds} g(s),$$

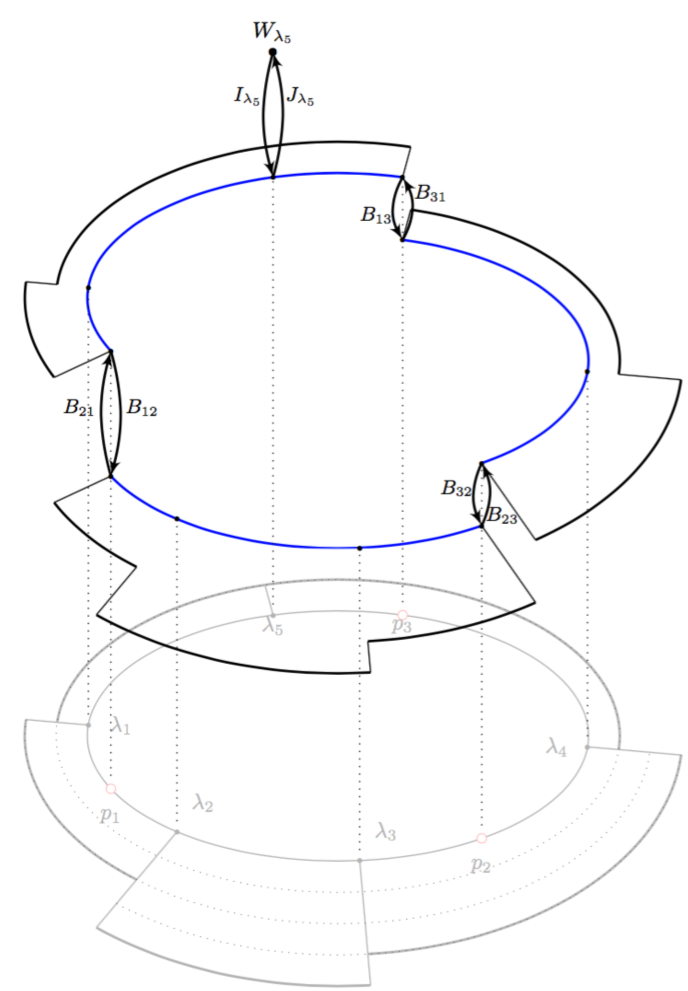
$$T_j(s) \mapsto g^{-1}(s) T_j g(s).$$

Bow rep. moduli space:

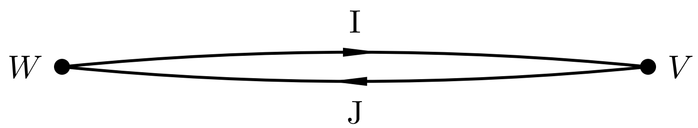
$$\mathcal{M}^{\text{Bow}} = \mathrm{Dat}(R) /\!\!/\!\!/ \mathcal{G} = \mu^{-1}(\nu)/\mathcal{G}$$

* The moment map conditions
$\mu(T, Q, B)$
of the hyperkähler reduction.

* The level ν = NUT positions.

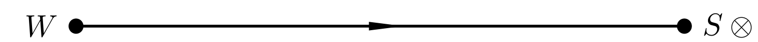


Ingredients 1: Arrows (Fundamental Multiplets)



$V = \mathbb{C}^v$ and $W = \mathbb{C}^w$

$Q = \begin{pmatrix} J^\dagger \\ I \end{pmatrix}$



SU(V) acts isometrically on Hom(W,V)⊕Hom(V,W)

$$g_v : (I, J) \mapsto (g_v^{-1} I, J g_v)$$

Metric: $$ds^2 = \mathrm{tr}_W dQ^\dagger dQ = \mathrm{tr}_W (dJ dJ^\dagger + dI^\dagger dI)$$

Hom(W,V)⊕Hom(V,W) is hyperkähler

With Complex Structures: $e_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ e_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

acting on the space of spinors $S = \mathbb{C}^2$

and Symplectic forms: $\omega_j = g(\cdot, e_j \cdot) \ = \ \mathrm{tr}_W (dQ^\dagger \wedge e_j dQ)$

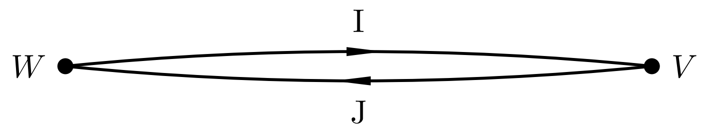
$$\vec{\omega} \equiv \omega_j \sigma_j, \qquad \vec{\omega} = 2i \mathrm{Vec}\, \mathrm{tr}_V dQ \wedge dQ^\dagger$$

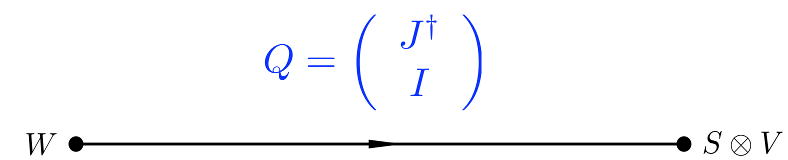
$$\iota_X \omega_j = d\mu_j$$

$$\boxed{\vec{\mu}_V = \mu_V^i \sigma_i = \mathrm{Vec}(Q_V Q_V^\dagger)}$$

$$\mathrm{Vec}(M^0 + M^j \sigma_j) = M^j \sigma_j$$

Ingredients 1: Arrows (Fundamental Multiplets)

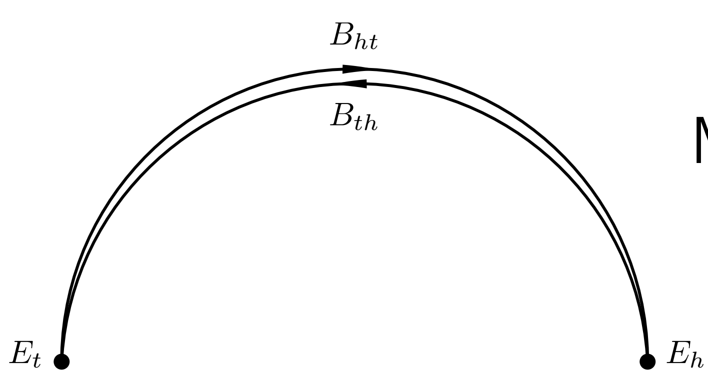




Moment map of U(V) action:

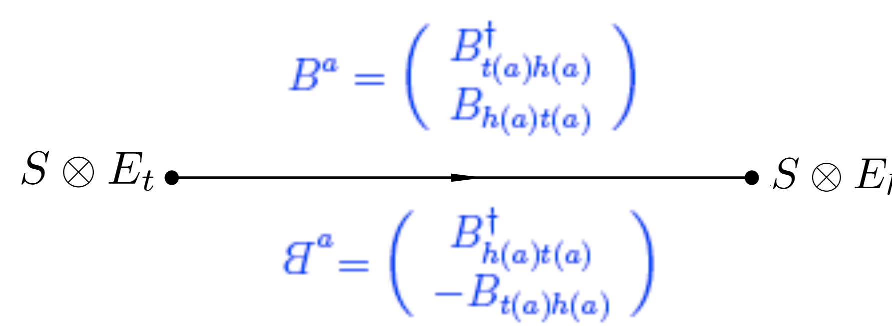
$$\mu_V = \mu_V^i \sigma_i = \mathrm{Vec}(Q_V Q_V^\dagger)$$

Ingredients 1I: Limbs (Bifundamental Multiplets)



Moment map of U(E$_h$) action:

$$\mu_h = \mathrm{Vec}\, B B^\dagger$$



Moment map of U(E$_t$) action:

$\reflectbox{B}^a : E_{h(a)} \to S \otimes E_{t(a)}$, and $B^a : E_{t(a)} \to S \otimes E_{h(a)}$.

$$\mu_t = \mathrm{Vec}\, \reflectbox{B} \reflectbox{B}^\dagger$$

with Mark Stern and
Andres Larrain-Hubach

Thm: Instanton moduli space is isometric
to the moduli space of the corresponding bow representation.

$$M_{\text{Instanon}} = M_{\text{Bow}}$$

Charges and Ambiguities

c.f. Witten 0902.09481
(via brane considerations)

- Distinct holonomy eigenvalues $\{e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_n}\}$
  split the instanton bundle (on a complement of a compact set) $\mathscr{E}_{\mathrm{TN}_k \backslash B} = W_1 \oplus W_2 \oplus \ldots \oplus W_n$

Lemma: The eigenvalues have the form

$$\mu_j(\vec{t}\,) = \frac{\lambda_j}{l} + \frac{\vartheta_j}{2\,|\,\vec{t}\,|} + O(|\,\vec{t}\,|^{-2})$$

comparing to the asymptotic form of our connection $a_j = (\lambda_j + \frac{m_j}{2t})\frac{d\tau + \omega}{V} - \frac{m_j}{k}\omega\,.$

$$l\vartheta_j = m_j + \frac{\lambda_j}{l}k$$

This combination is Independent of any gauge choice!

- Consider the neighborhood of infinity: $\mathrm{TN}_k \setminus B$ contracts to the lens space $S^3/\mathbb{Z}_k$.

Pullback Bundle: W_j^* ———————> W_j

is trivial ↓ ↓

Covering space: S^3 ———————> $S^3/\mathbb{Z}_k$

Thus, what distinguishes different line bundles W_j is $\mathbb{Z}_k$ action on the fiber.
There are only k types of line bundles W_j.

Changing the trivialization of S^3 acts by $e^{i\tilde{\tau}/k} : \begin{pmatrix} \lambda_j \\ m_j \end{pmatrix} \mapsto \begin{pmatrix} \lambda_j + l \\ m_j - k \end{pmatrix}$

- Moral: line bundles W_j are determined by k numbers $\{\hat{m}_j \,|\, \hat{m}_j = m_j \bmod k, 0 \le \hat{m}_j < k\}$

Choice of Twisting Family

- First Chern class: k real numbers $c_1^\sigma = \frac{1}{2\pi i}\int_{C_\sigma} \mathrm{tr}\, F = \sum_j \frac{\lambda_j}{l} - (\mathfrak{c}_\sigma + n\mathfrak{s}_\sigma), \quad \mathfrak{s}_\sigma \in \mathbb{Z}$

$\mathfrak{c}_\sigma \in \{0,1,\ldots,n-1\}$

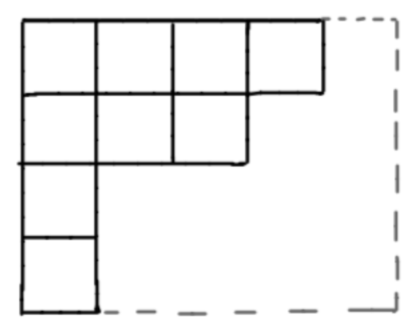
"Stiefel-Whitney classes" (obstructions of PSU(n) to SU(n) lifting)

Relabel the NUTs so that $0 \le \mathfrak{c}_1 \le \mathfrak{c}_2 \le \ldots \mathfrak{c}_k < n$

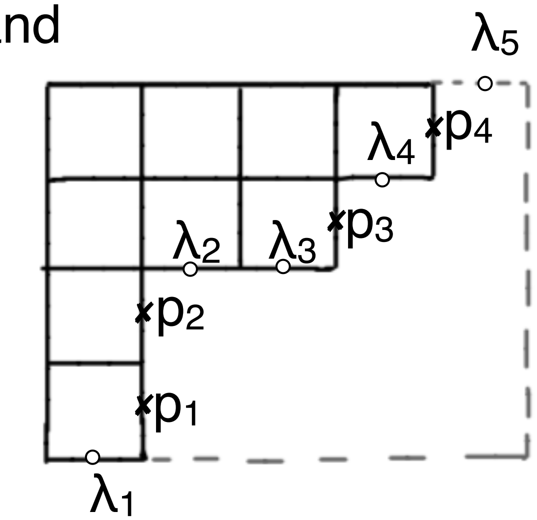
This labels the rows of a Young diagram that fits into an kxn rectangle

For example, for k=4, n=5 and $(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4) = (1,1,3,4)$

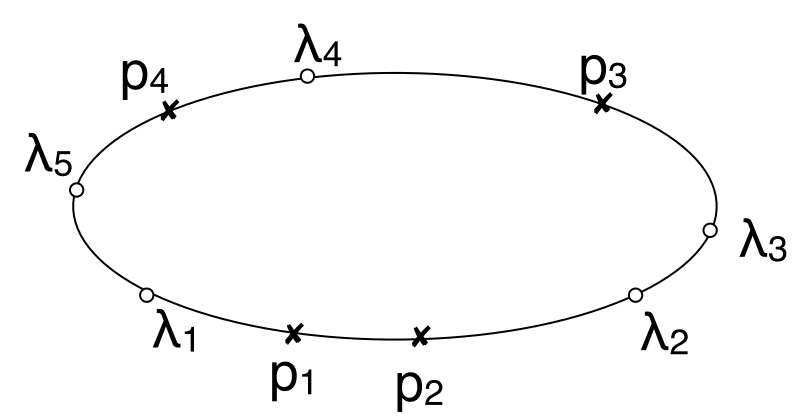
Y_{Bal}=



- View it as a closed path on an kxn torus.
  Mark
  - horizontal segments λ_j and
  - vertical segments p_σ



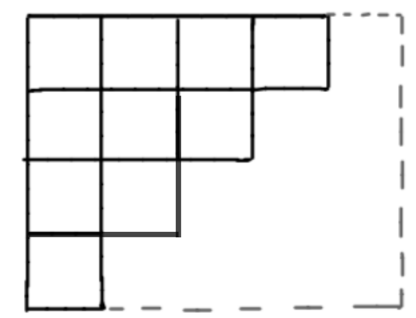
This is the balanced bow representation (the rank is continuous at each p-point):



Note: The set of magnetic charges also gives a Young diagram in nxk rectangle.

$$\left\{ \hat{m}_1, \hat{m}_2, \ldots, \hat{m}_n \,|\, \hat{m}_j \in \{1,2,\ldots,k-1\} \right\}$$

Y_{Cob}=



Interior:
(Chern numbers) Y_{Bal}

Infinity:
(monopole charges) Y_{Cob}

Instanton Number

Since TN_k is not compact, Chern character value $ch_2[E,A]$ does NOT have to be integer.
Need another definition of the instanton number.

Let us focus on a single $TN=TN_1$:

$TN \simeq \mathbb{R}^4$ is contractible, so any bundle over it is trivial,
and connection one-form A is globally defined.

1. On a complement of a compact set, a gauge transformation $g : \mathrm{TN}\backslash B \to U(n)$

   transforms A to the diagonal form

$$A^g = -i\,\mathrm{diag}\begin{pmatrix} \lambda_1 + m_1 l & & & \\ & \lambda_2 + m_2 l & & \\ & & \cdots & \\ & & & \lambda_n + m_n l \end{pmatrix} \frac{d\tau + \omega}{V} + O\!\left(\frac{1}{t^2}\right).$$

   TN\B is contractible to S^3, thus the homotopy class of g is in $\pi_3(U(n)) = \mathbb{Z}$.

   Instanton Number is $m_0 := \deg\,[g] \in \pi_3(U(n))$.

2. Alternatively, holonomy over splits the instanton bundle into orthogonal line bundles, giving a map $S^3_\infty \to U(n)/U(1)^n = N_n$ ← Flag space

   Instanton Number is an element of $\pi_3(U(n)/U(1)^n) = \mathbb{Z}$.

Index Again

Our index theorem expression is not evidently integer.

$$\mathrm{ind}_{L^2} D_A^- = \sum_j \left((\{\lambda_j/l\} - \frac{1}{2})(m_j - k\lfloor \lambda_j/l \rfloor) - \frac{k}{2}\{\lambda_j/l\}^2 \right) + \frac{1}{8\pi^2}\int \mathrm{tr}\, F \wedge F$$

Let us compute it:

$$ch_2[\mathscr{E}, A] = \frac{-1}{8\pi^2}\int_{\mathrm{TN}} \mathrm{tr}\, F \wedge F = \frac{-1}{8\pi^2} \lim_{R\to\infty} \int_{\partial B_R} \mathrm{tr}(A dA + \frac{2}{3}A^3)$$

$$= \deg[g] - \frac{1}{8\pi^2}\lim_{R\to\infty}\int_{\partial B_R} \mathrm{tr}(A^g dA^g + \frac{2}{3}(A^g)^3)$$

$$= \deg[g] + \sum_{j=1}^{n} \frac{1}{2}(m_j - \frac{\lambda_j}{l})^2 .$$

With this our index theorem reads

$$\mathrm{ind}_{L^2} D_A^- = \deg[g] + \sum_{j=1}^{n} \frac{1}{2} \lfloor m_j - \frac{\lambda_j}{l} \rfloor \left(\lfloor m_j - \frac{\lambda_j}{l} \rfloor + 1 \right)$$

Kronheimer Reduction

Kronheimer
MS Thesis '85

An instanton on TN_k that is S^1 invariant has the form

$$\mathscr{A}^g = A + \Phi\frac{d\tau + \omega}{V}$$

in some trivialization.

In this trivialization the S^1 action on the bundle fiber at a fixed point ν_σ with some weights $(n_1, n_2, \ldots, n_n)$.

The anti-self-duality equation $\quad * F_{\mathscr{A}} = - F_{\mathscr{A}}$

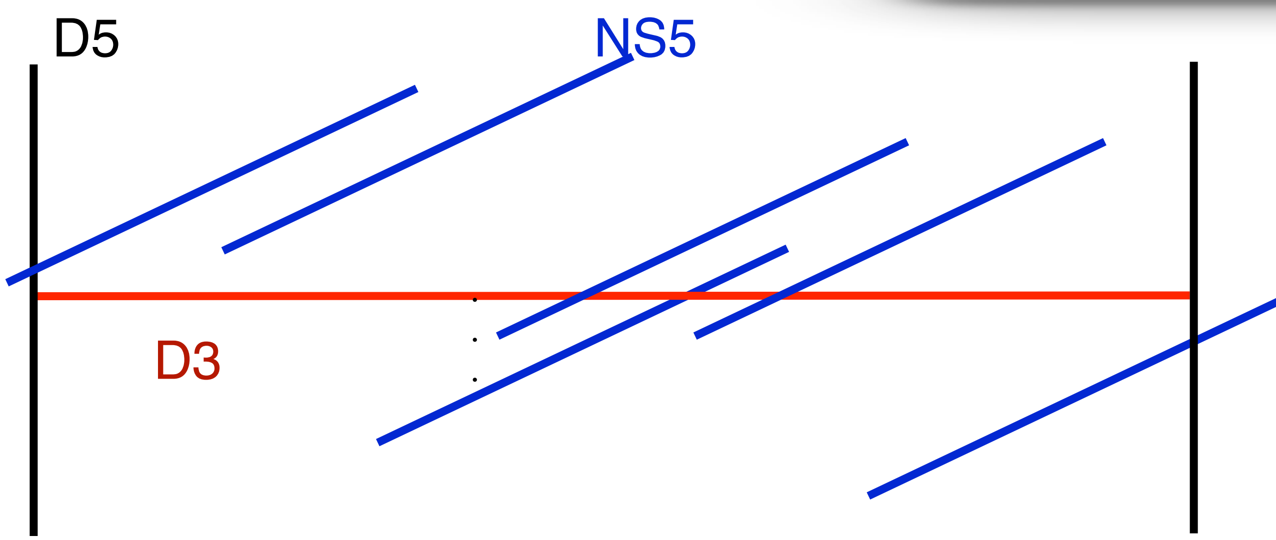
is equivalent to the Bogomolny equation

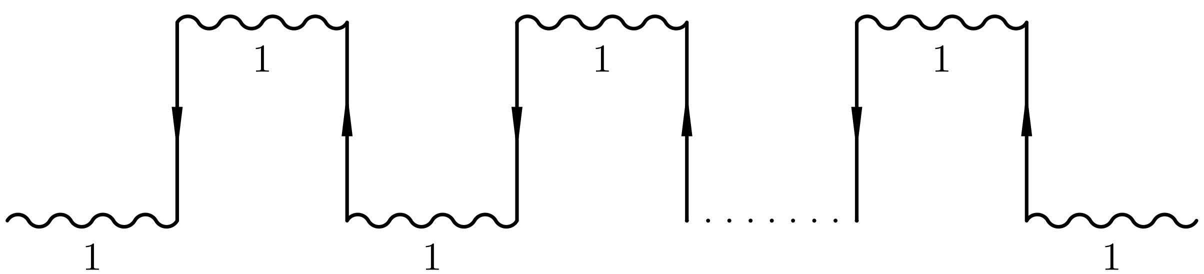
$$* F_A = D_A \Phi$$

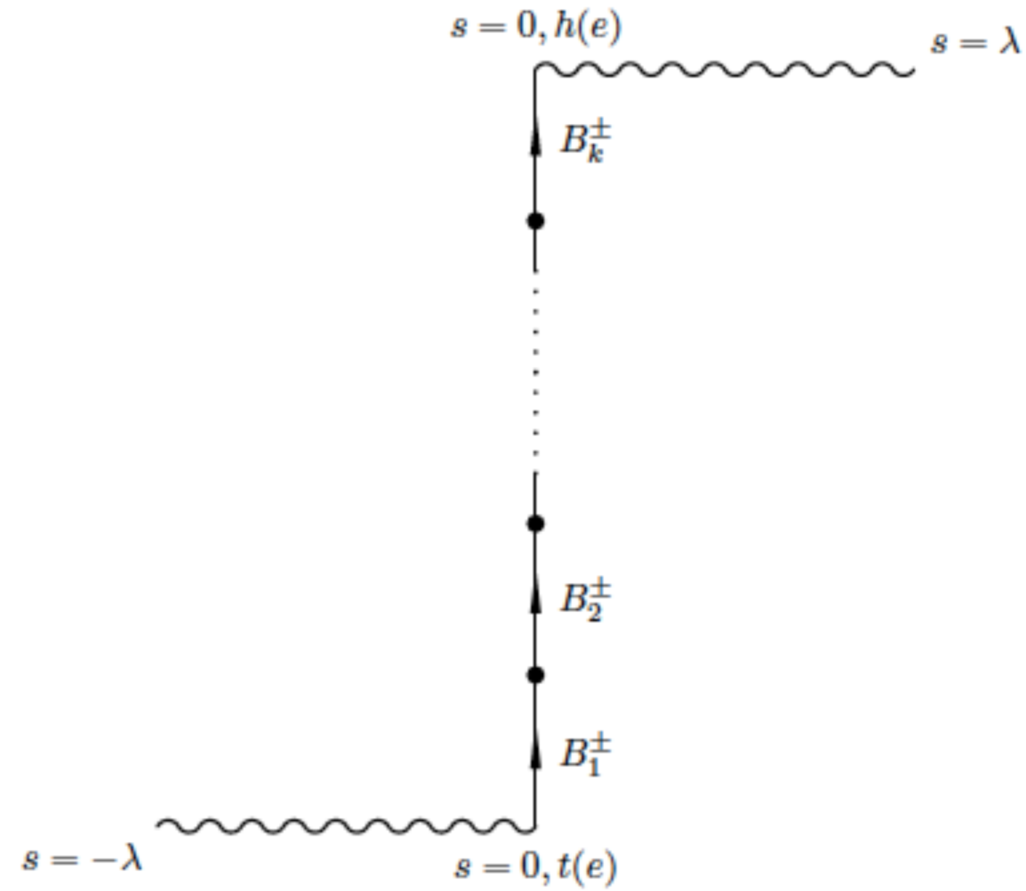
Thus (A, Φ) is a BPS monopole, except with Dirac singularities at the NUTS ν_σ!

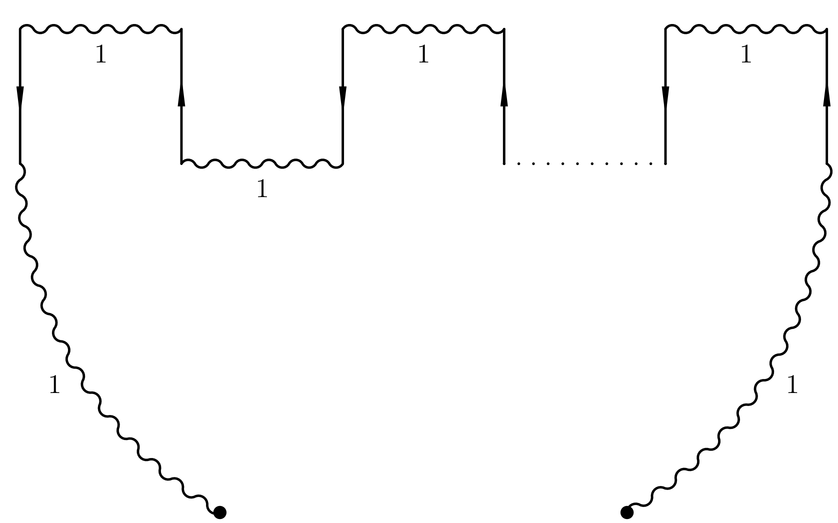
$$\Phi = \frac{\begin{pmatrix} n_1 & 0 & \ldots & 0 \\ 0 & n_2 & & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & n_n \end{pmatrix}}{2|t - \nu_\sigma|} + O(|t - \nu_\sigma|^0)$$

Cheshire Bow



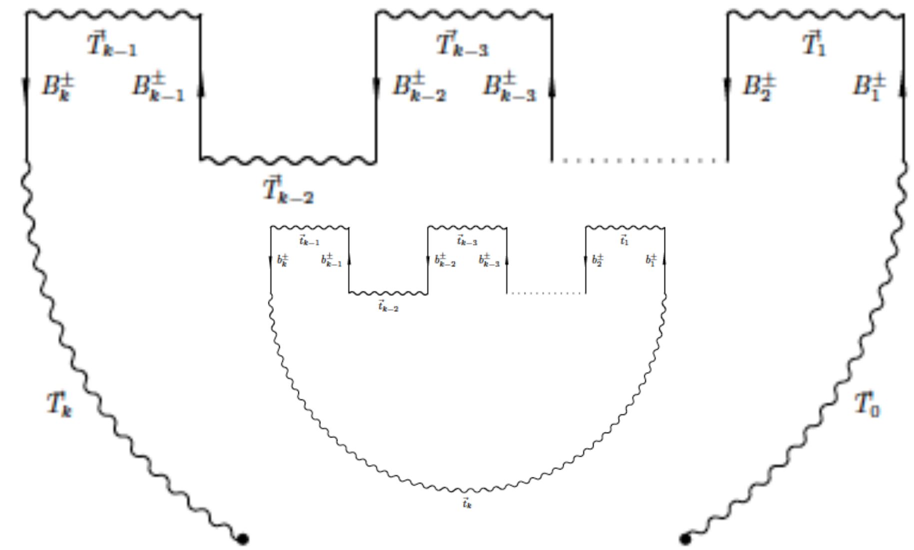






1) Find solutions of the LARGE Cheshire bow and a small mTN bow

2) Form Weyl operators



$$\mathfrak{D}^\dagger = \Big(\frac{d}{ds} - iT_0 - \mathcal{T}\Big) \oplus \bigoplus_\alpha \delta(s-\lambda_\alpha) Q_\alpha \oplus \bigoplus_{a\in\{\text{Arrows}\}} \Big(\delta(s-t(a))\, \tilde{B}^a,\ \delta(s-h(a)) B^a\Big),$$

$$\mathfrak{d}^\dagger = \Big(\frac{d}{ds} - it_0 - \mathfrak{t}\Big) \oplus 0 \oplus \bigoplus_{a\in\{\text{Arrows}\}} \Big(\delta(s-h(a)) b^a \oplus \delta(s-t(a)) \mathrm{d}^a\Big).$$

and the twisted operator

$$\boxed{\mathfrak{D}_t^\dagger = \mathfrak{D}^\dagger \otimes 1 + 1 \otimes \mathfrak{d}^\dagger.}$$

3) Form the orthonormal basis of solutions $\Psi = (\psi(s), \chi_\alpha, \varsigma^a),$

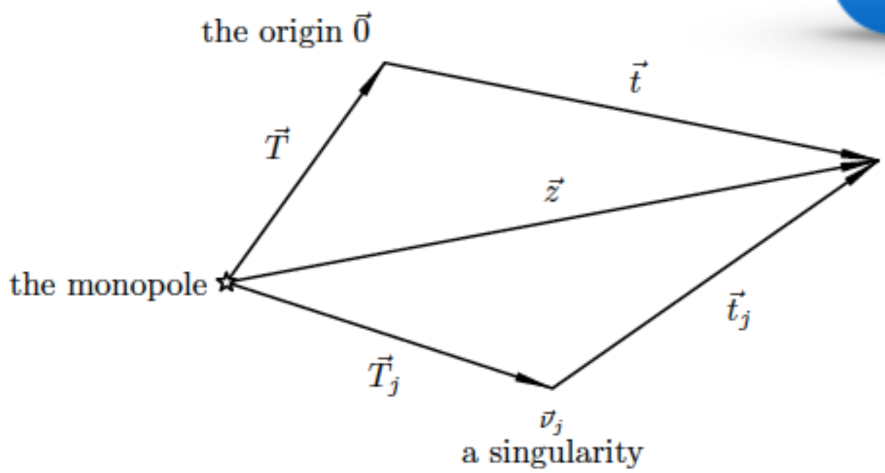
to the Weyl equation $\mathfrak{D}^\dagger \Psi = 0$

4) Find the Higgs field and the gauge field of the singular monopole from

$$\Phi = \left(\Psi, \Big(s + \sum_{j=1}^{\text{int}(s)} \frac{1}{2t_j}\Big)\Psi\right), \qquad A = \left(\Psi, \Big(i\,dt_a \frac{d}{dt_a} + \sum_{j=1}^{\text{int}(s)} \omega_j\Big)\Psi\right)$$

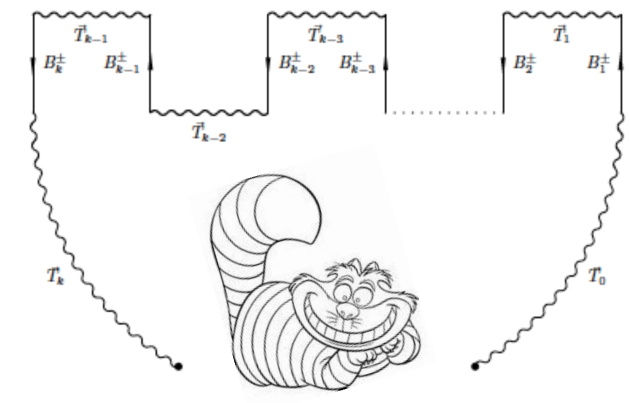
Singular Monopoles

w/ Chris Blair



Define function α by

$$\exp(4\alpha z) = \prod_j \frac{T_j + t_j + z}{T_j + t_j - z}$$



$$\Phi = \left(\left(\lambda + \sum_{j=1}^{k} \frac{1}{2t_j} \right) \coth 2(\lambda + 2\alpha) z - \frac{1}{2z} \right) \frac{\not{z}}{z}$$

$$+ \frac{z}{\sinh 2(\lambda + 2\alpha) z} \sum_{j=1}^{k} \frac{\not{T}_{j\perp}}{2t_j \left(T_j t_j - \vec{T}_j \cdot \vec{t}_j \right)}$$

$$A = \frac{i}{2z} [\not{z}, d\not{z}] \left(- \frac{1}{\sinh 2(\lambda + 2\alpha) z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2(T_j t_j - \vec{T}_j \cdot \vec{t}_j)} \right] + \frac{1}{2z} \right)$$

$$+ \sum_{j=1}^{k} \omega_j \frac{\not{z}}{z} \coth 2(\lambda + 2\alpha) z + \frac{z}{\sinh 2(\lambda + 2\alpha) z} \sum_{j=1}^{k} \frac{i[\not{t}_j, d\not{t}]_\perp}{4t_j \left(T_j t_j - \vec{T}_j \cdot \vec{t}_j \right)}$$

Conclusion

- ADHM transform produces instantons on R^4.
- The Nahm transform produces Monopoles on R^3
- It generalizes to instants on ALF spaces.
- Quadratic curvature decay and index.
- Moduli space isometry.
- Explicit instantons and singular monopoles.