Instantons and Monopoles

Lecture 1: Physics

Sergey Cherkis

(University of Arizona)

Quantum Fields, Geometry and Representation Theory

ICTS-TIFR, Bengaluru

July 16-27, 2018

Outline

- Yang-Mills Instantons
- Significance:
  - Hanany-Witten brane dynamics
  - N=4, D=3 Quantum Gauge Theories
  - Geometric Langlands for surfaces
  - Singular monopoles
- Bows from Branes:
  - Taub-NUT as a Bow Moduli Space
  - Abelian Instantons on Taub-NUT
- Bow moduli Spaces

Yang-Mills Instanton

Yang-Mills gauge field (connection 1-form) A,
with field strength (curvature 2-form) F=dA+A∧A
is governed by the action functional

$$S[A] = \int_{M^4} \mathrm{tr}\ F \wedge *F$$

Hodge star

Its extrema satisfy the Yang-Mills equation (Euler-Lagrange equation for S[A]):

$$D_A * F = 0$$

In quantum theory one is interested in computing path integrals of the form

$$< \mathcal{O}(A) > = \int e^{i\frac{S[A]}{\hbar}} \mathcal{O}(A) \mathcal{D}A$$

Its Euclidean version is

$$< \mathcal{O}(A) > = \int e^{-\frac{S[A]}{\hbar}} \mathcal{O}(A) \mathcal{D}A$$

it is dominated by the minima of S[A], which satisfy the SD or ASD equations:

$$*F = \pm F$$

Belavin-Polyakov-Schwartz-Tyupkin (Bogomolny) Trick

Since the norm of F+*F is non-negative

$$0 \leq \int \mathrm{tr}\,(F + {*F}) \wedge * \,(F + {*F}) = 2S[A] + 2\int \mathrm{tr}\, F \wedge F$$

The action is bounded below:

$$\int \mathrm{tr}\, F \wedge {*F} \geq -\int \mathrm{tr}\, F \wedge F = 8\pi^2 ch_2[M^4]$$ Chern character

with the minimum achieved only if A is anti-self-dual

$$*\,F = -\,F$$

Note: Similar inequality is saturated by self-dual connections.

Taub-NUT Space

A_0 ALE: Euclidean $\mathbb{R}^4$ metric in “radial coordinates” $\mathbb{R}^4 = \mathbb{H} \ni q = a e^{e_1 \tau}$ with a pure imaginary

(t,τ) are “polar coordinates” on $\mathbb{R}^4$ $\quad t = t_1 e_1 + t_2 e_2 + t_3 e_3 = q e_1 \bar{q} = a e_1 \bar{a}$

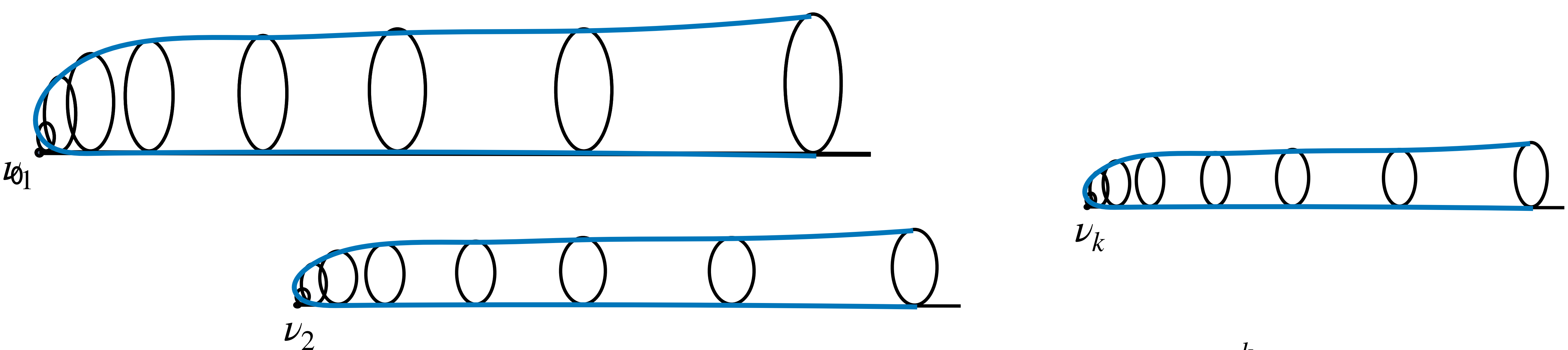
$$ds^2 = dq d\bar{q} = \frac{1}{2|t|} d\vec{t}^{\,2} + \frac{(d\tau + \omega)^2}{\frac{1}{2|t|}}$$

$$d\omega = *_3 dV$$

A_0 ALF: Taub-NUT space (TN)

$$ds^2 = V d\vec{t}^{\,2} + \frac{(d\tau + \omega)^2}{V}$$

$$V(t) = l + \frac{1}{2|t|}$$



A_{k-1} ALF: multi-Taub-NUT space TN_k in Gibbons-Hawking form $\quad V(t) = l + \sum_{\sigma=1}^{k} \frac{1}{2|t - \nu_\sigma|}$

This is the simplest nontrivial Calabi-Yau space:
respects 1/2 SUSY, has 2 covariantly constant spinors, is hyperkähler.

Significance

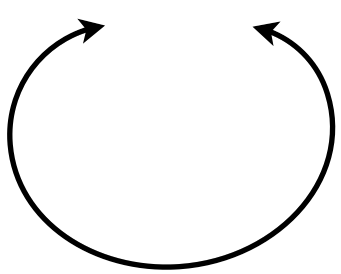
Chalmers-Hanany-Witten
Brane Dynamics
(NS5+D5+D3)

Geometric Langlands
for Complex Surfaces

Instanton on
Taub-NUT

M theory compactification
with background induced
M-brane charges

Quantum Gauge Theory:
N=4, D=3 or
N=2, D=4 with Impurity Walls



Gauge Theory
Mirror Symmetry
(as Isometry of Moduli Spaces)

Generalizations of Quivers:
Bows, Slings, Monowalls…

- A number of ways of realizing instantons:
  Dp-branes inside D(p+4)-branes wrapped on M.

- If M has an isometry or a mirror W, then
  D(p+k)-brane intersecting D(p+4-k)-brane on W.

p=2: Relation to Bows

M	0	1	2	3	4	5	6	7	8	9	10
				Taub-NUT							
								Taub-NUT			
M2	x	x	x								

$S^1_M = S^1_{10}$ ↑

IIA	0	1	2	3	4	5	6	7	8	9
				Taub-NUT						
D6	x	x	x	x	x	x	x			
D2	x	x	x							

T_6 ↓

IIB	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x					x	x	x
D5	x	x	x	x	x	x				
D3	x	x	x				x			

Warning: $TN_k \times TN_k$ in NOT a solution of M theory (even though it is Calabi-Yau)

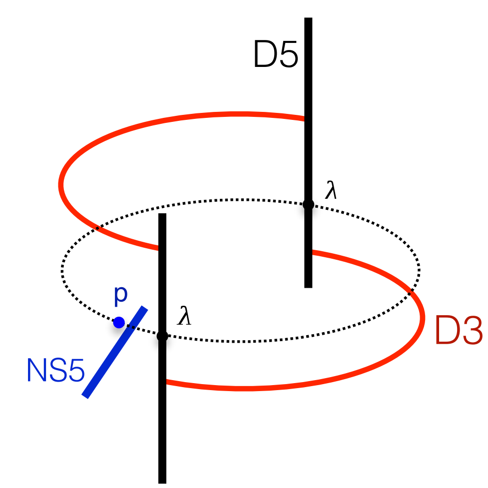
$$S_M^{\text{Eff}} = \ldots + \int C^{(3)} \wedge R^4$$

R^4 source gives rise to C^3 field => fractional M2-brane charge

Hanany-Witten effect in M theory

w/ Ruben Minasian

Chalmers-Hanany-Witten configuration: analogue of Higgs spectral curve

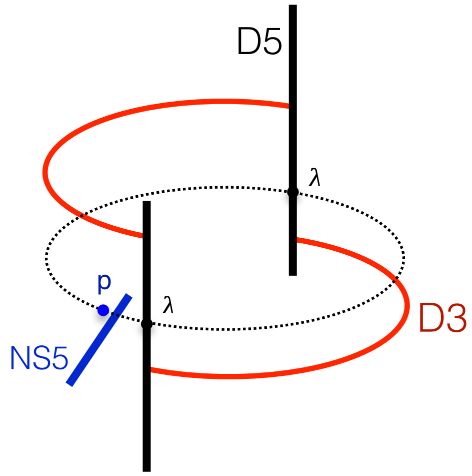


D=4, N=2 Supersymmetric Yang-Mills on D3-brane

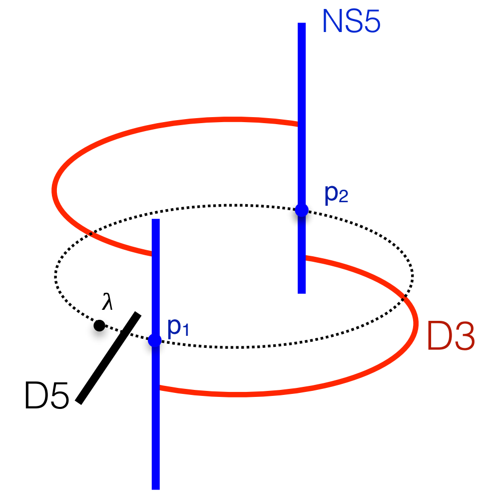
Analyzing its vacua gives rise to a Bow!

What we learn from this brane configuration:

Gaiotto-Witten,
SCh-Saemann-O'Hara



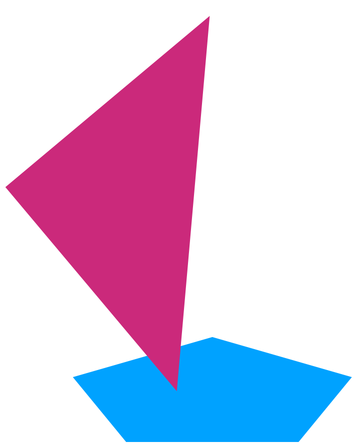
S duality



Gauge Theory
Mirror Symmetry

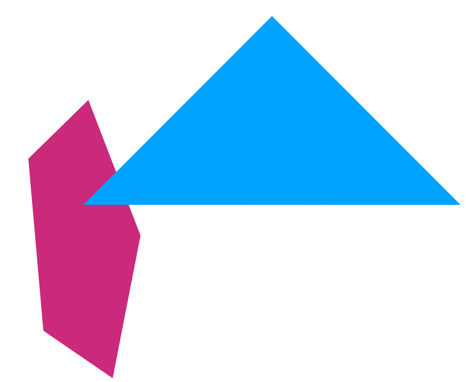
M: N=2 U(n) super-Yang-Mills
in 4D with some defects

W: N=2 U(n) super-Yang-Mills
in 4D with other defects



Coulomb branch of M $\overset{\text{isometry}}{=\!=\!=}$ Higgs branch of W

Higgs branch of M $\overset{\text{isometry}}{=\!=\!=}$ Coulomb branch of W



Note: Gauge theory mirror symmetry is usually stated for D=3, N=4 gauge theories.
These are not isometric, but have same underlying complex varieties.

p=0: Geometric Langlands for Surfaces

Dijkgraaf, Hollands, Sulkowski, Vafa ’07
M-C Tan ’08
Witten ’09

M	0	1	2	3	4	5	(6)	7	8	9	(10)
				Taub-NUT$_k$							
M5	x			x	x	x	x				x
KK	x										x

$S^1_M = S^1_{10}$ ↙ ↘ $S^1_M = S^1_6$

m U(n) instantons on TN_k

	IIA	0	1	2	3	4	5	(6)	7	8	9
					Taub-NUT$_k$						
n	D4	x			x	x	x	x			
m	D0	x									

	IIA	0	1	2	3	4	5	7	8	9	(10)
k	D6	x	x	x				x	x	x	x
n	D4	x			x	x	x				x
m	F1	x						x			

T_6 ↓ ↓ T_{10}

IIB	0	1	2	3	4	5	(6)	7	8	9
NS5	x	x	x					x	x	x
D3	x			x	x	x				
D1	x						x			

S →

IIB	0	1	2	3	4	5	7	8	9	(10)
D5	x	x	x				x	x	x	
D3	x			x	x	x				
F1	x						x			

What we learn from this brane configuration:

M	0	1	2	3	4	5	(6)	7	8	9	(10)
				Taub-NUT$_k$							
M5	x			x	x	x	x				x
KK	x										x

m U(n) instantons on TN$_k$

	IIA	0	1	2	3	4	5	(6)	7	8	9
					Taub-NUT$_k$						
n	D4	x			x	x	x	x			
m	D0	x									

	IIA	0	1	2	3	4	5	7	8	9	(10)
k	D6	x	x	x				x	x	x	x
n	D4	x			x	x	x				x
m	F1	x						x			

D0: Quantum mechanics with configuration space moduli space of instantons on TN

F1: Twisted holomorphic WZW at D4-D6 intersection

BPS States are

$$\bigoplus_m H^*_{L^2\text{harm}}(\mathcal{M}^{\text{inst}}_{m_G}) = \mathcal{H} = \bigotimes_{\text{TN centers}} \mathcal{W}_j^{{}^L G}$$

M-C Tan '08
Witten '09

Meng-Chwan Tan gives explicit formulas for the L^2 Betti numbers of moduli spaces of instantons in TN$_k$ in 0807.1107.

p=1: Unexplored Relation

M	0	1	2	3	4	5	(6)	7	8	9	(10)
M5	x	x	x					x	x	x	
M5	x	x		x	x	x					x
M2	x	x					x				

$S^1_M = S^1_{10}$

IIA	0	1	2	3	4	5	(6)	7	8	9
NS5	x	x	x					x	x	x
D4	x	x		x	x	x				
D2	x	x					x			

T_6

m U(n) instantons on TN_k

	IIB	0	1	2	3	4	5	(6)	7	8	9
k		Taub-NUT$_k$									
n	D5	x	x		x	x	x	x			
m	D1	x	x								

$S^1_M = S^1_6$

IIA	0	1	2	3	4	5	7	8	9	(10)
NS5	x	x	x				x	x	x	
NS5	x	x		x	x	x				x
F1	x	x								

T_{10}

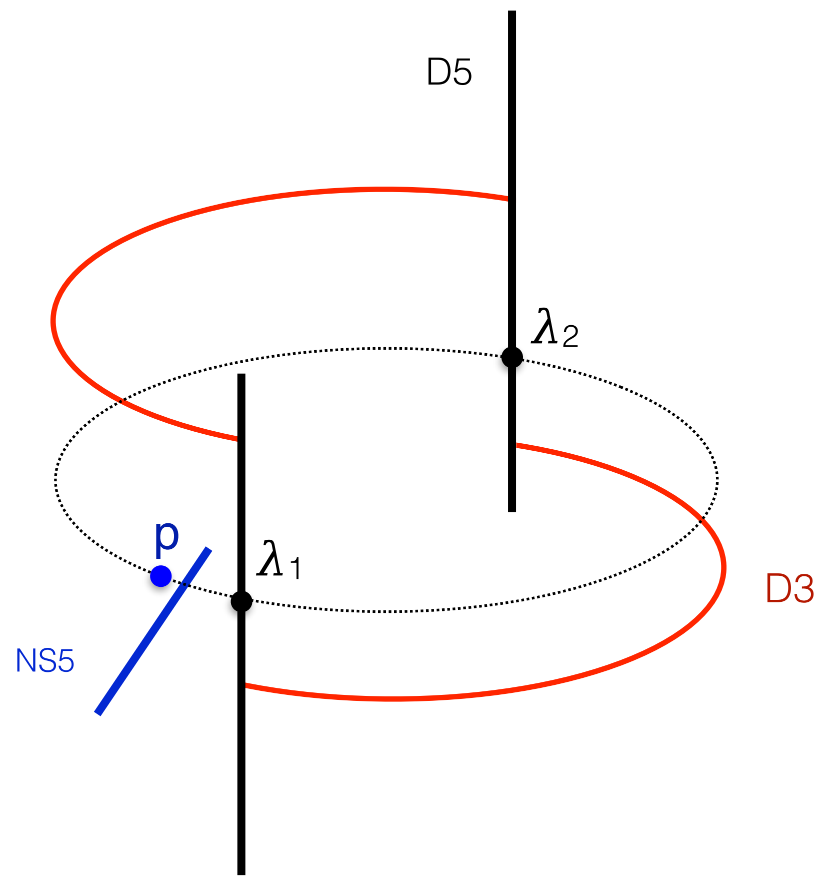
S

IIB	0	1	2	3	4	5	7	8	9	(10)
	Taub-NUT$_k$									
NS5	x	x		x	x	x				x
F1	x	x								

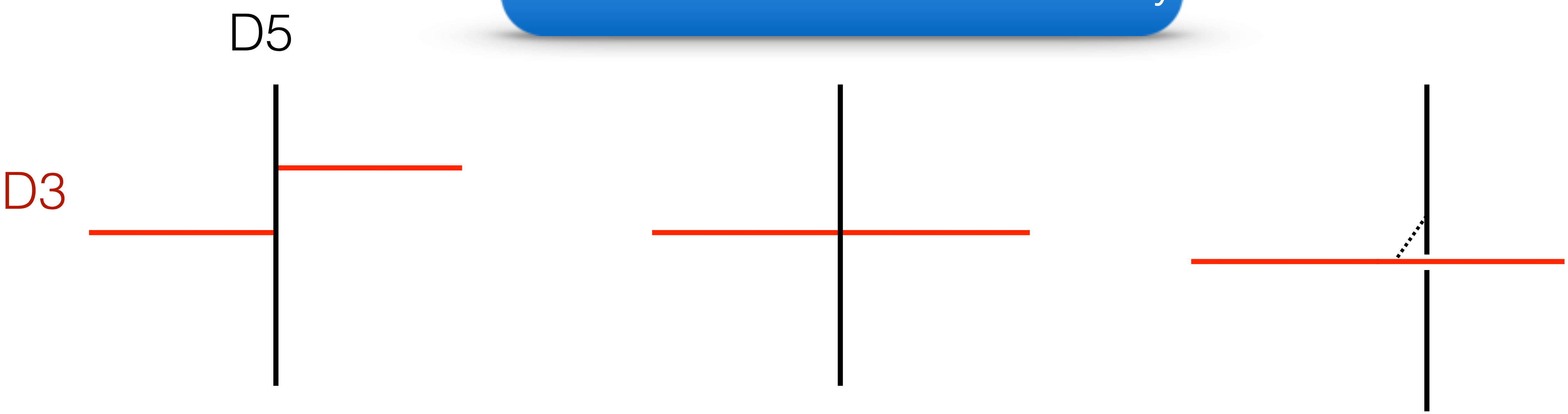
Open Question: what do we learn from this realization?

Back to Hanany-Witten configuration:
What gauge theory lives on the D3-brane?

(Study of defects)

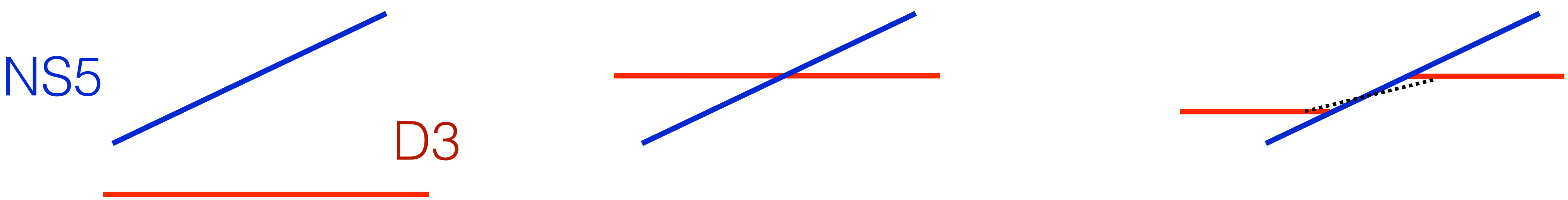


D3-brane world-volume theory



Massless fundamental hypermultiplet: Q

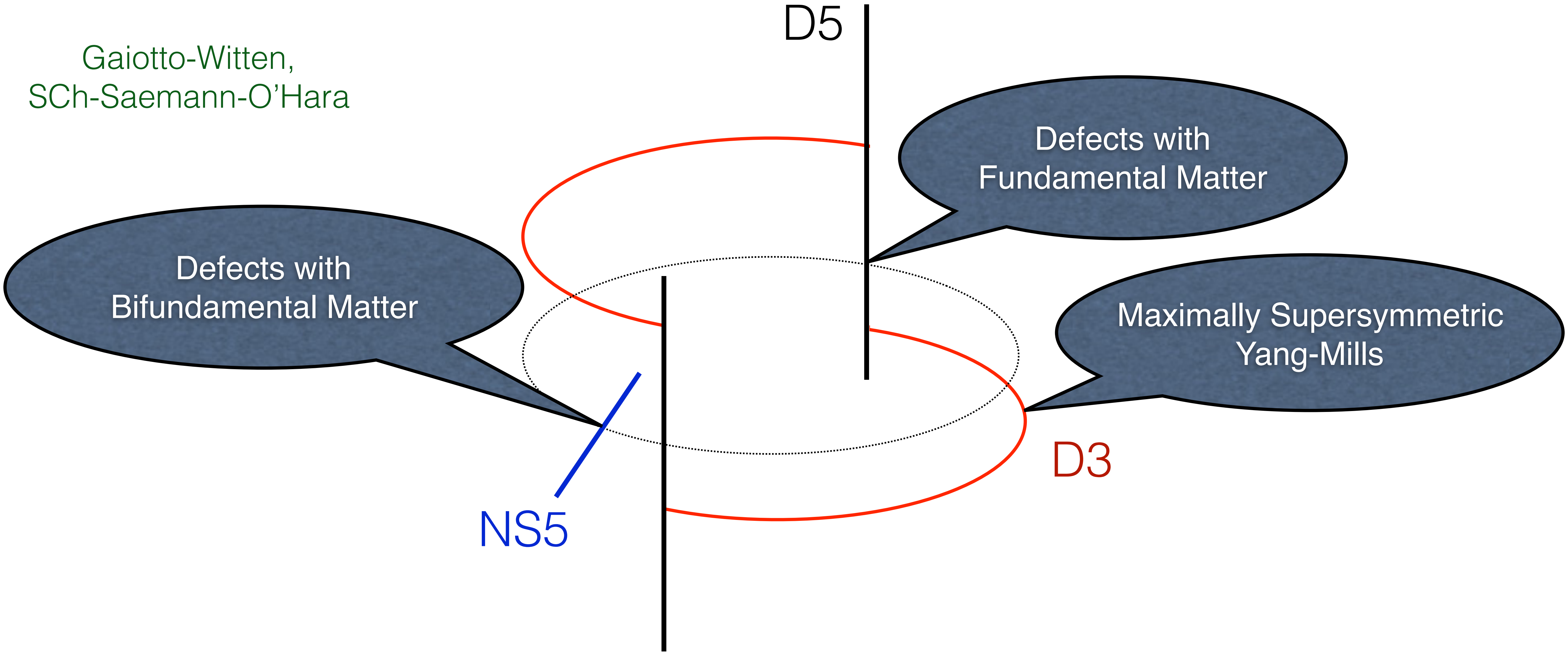
Massive fundamental hypermultiplet from D3-D5 open string mode

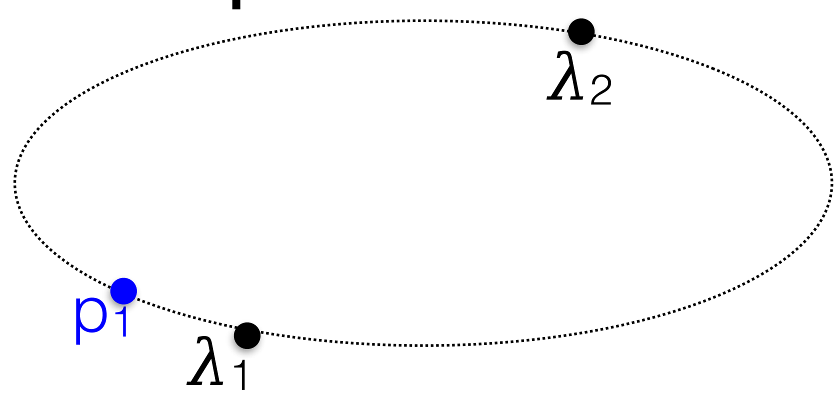


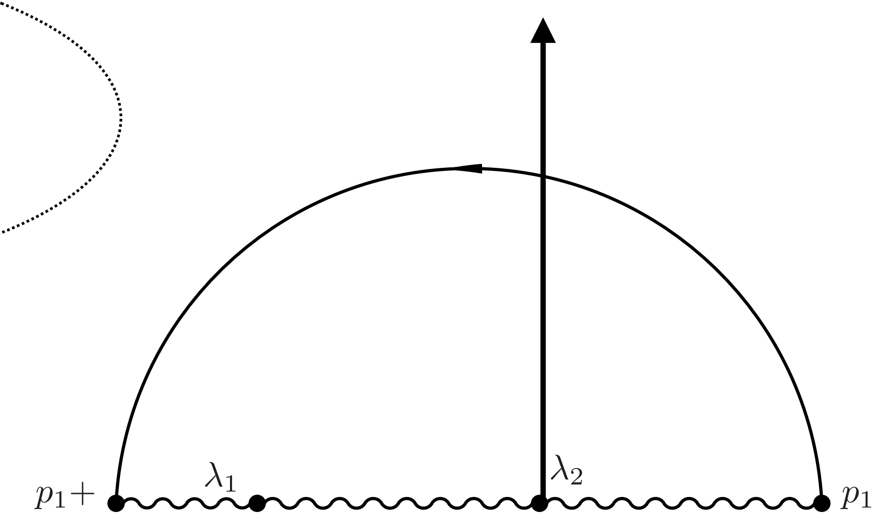
Massless bifundamental hypemultiplet: B

Massive bifundamental hypemultiplet from D3-D3 open string mode

Gaiotto-Witten,
SCh-Saemann-O'Hara





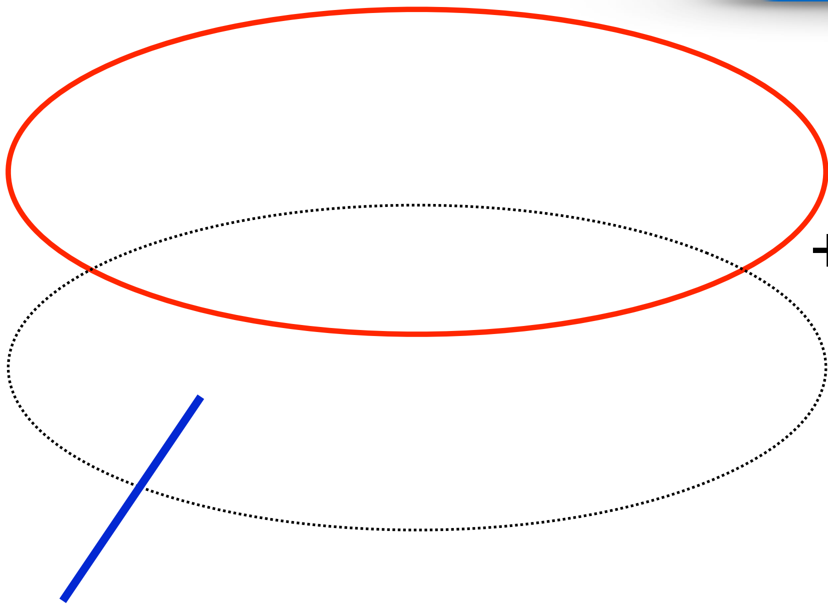


IIA	0	1	2	3	4	5	⑥	7	8	9
				Taub-NUT						
D6	x	x	x	x	x	x	x			
D2	x	x	x							

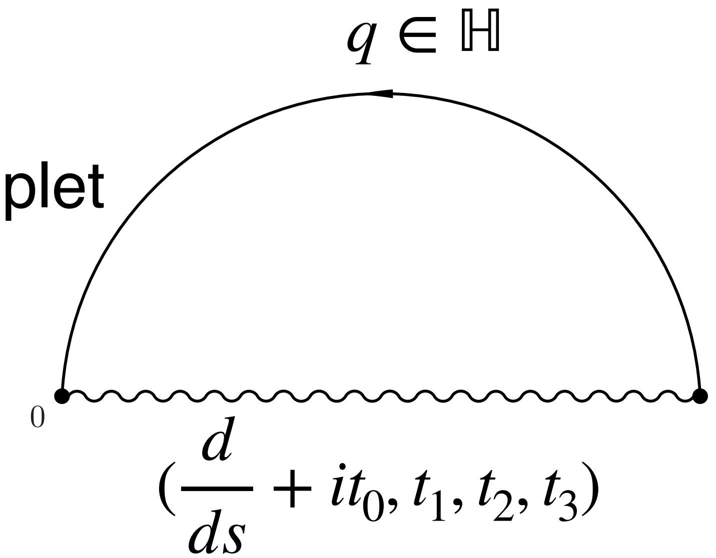
$T_6 \downarrow$

IIB	0	1	2	3	4	5	⑥	7	8	9
NS5	x	x	x					x	x	x
D5	x	x	x	x	x	x				
D3	x	x	x				x			

Basic Example:
Taub-NUT as Bow Moduli Space



U(1) N=4 SYM on D3
+ bifundamental supermultiplet



D- & F-flatness conditions in the interior: $i\frac{d}{ds}t_1 = [t_2, t_3]\ \ (=0)$ Nahm's Equations
(Hyperkähler moments)

Moral: t_1, t_2, t_3 are constant
t_0 can be gauge to constant
by gauge transformations vanishing at the end-points

$(t_0, t_1, t_2, t_3) \in S^1 \times \mathbb{R}^3$

$t_0 \quad t_1, t_2, t_3 \quad q = ae^{e_3\frac{\theta}{2}}$

- The resulting space is the hyperkähler quotient: $S^1 \times \mathbb{R}^3 \times \mathbb{H}///U(1)$

D-flatness conditions at left end:
(Hyperkähler moments)

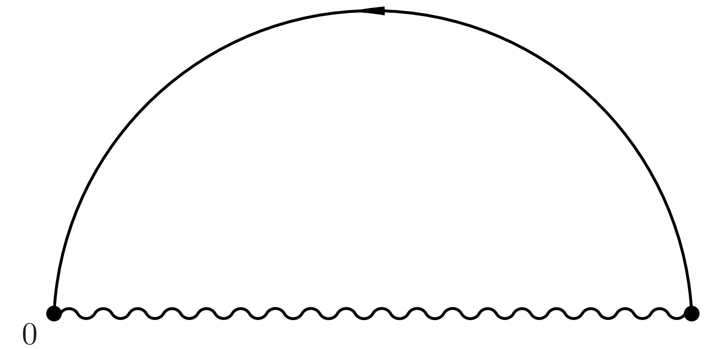
$$\mu = \underbrace{\bar{q}Kq}_{\vec{r}} - (t_1e_1 + t_2e_2 + t_3e_3) \quad (=\nu)$$

this is the Level Set $\mu^{-1}(\vec{\nu})$

divide by remaining gauge action:

$$(t_0, \theta) \mapsto (t_0 + \epsilon, \theta + \epsilon)$$

$$ds^2 = l(dt_0^2 + d\vec{t}^2) + \frac{d\vec{r}^2}{2r} + 2r(d\theta + \omega)^2$$



$$ds^2 = l(dt_0^2 + d\vec{t}^{\,2}) + \left(\frac{d\vec{r}^{\,2}}{2r} + 2r(d\theta + \omega)^2\right)$$

remaining gauge action: $(t_0, \theta) \mapsto (t_0 + \epsilon, \theta + \epsilon)$

Invariant: $\tau = \theta - t_0$

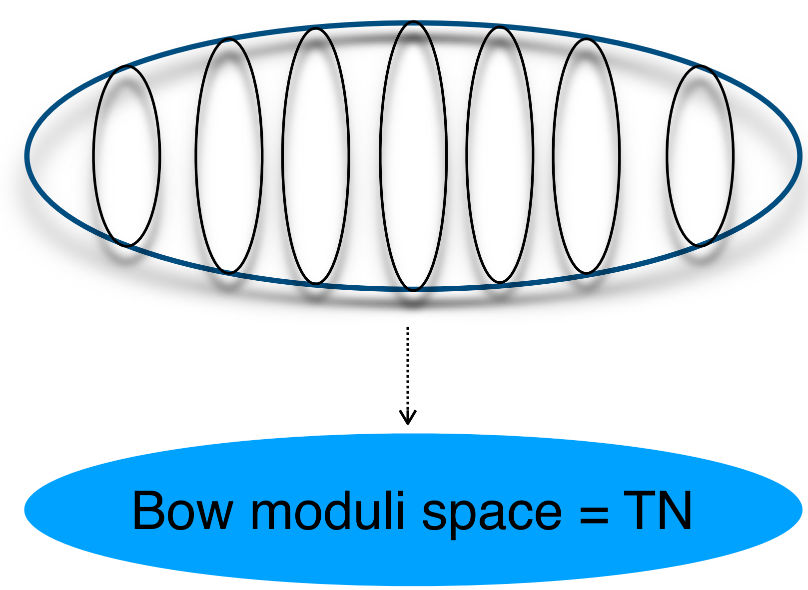
Completing the square:

$$ds^2 = (l + \frac{1}{2t})d\vec{t}^{\,2} + \frac{1}{l + \frac{1}{2t}}(d\tau + \omega)^2 + l\left[dt_0 + \frac{d\tau + \omega}{l + \frac{1}{2t}}\right]^2$$

Abelian anti-self-dual connection

$$a^{(0)} = \frac{d\tau + \omega}{V}$$

Level Set:



Moduli space metrics can be computed explicitly when the bow representation ranks do not exceed 2.

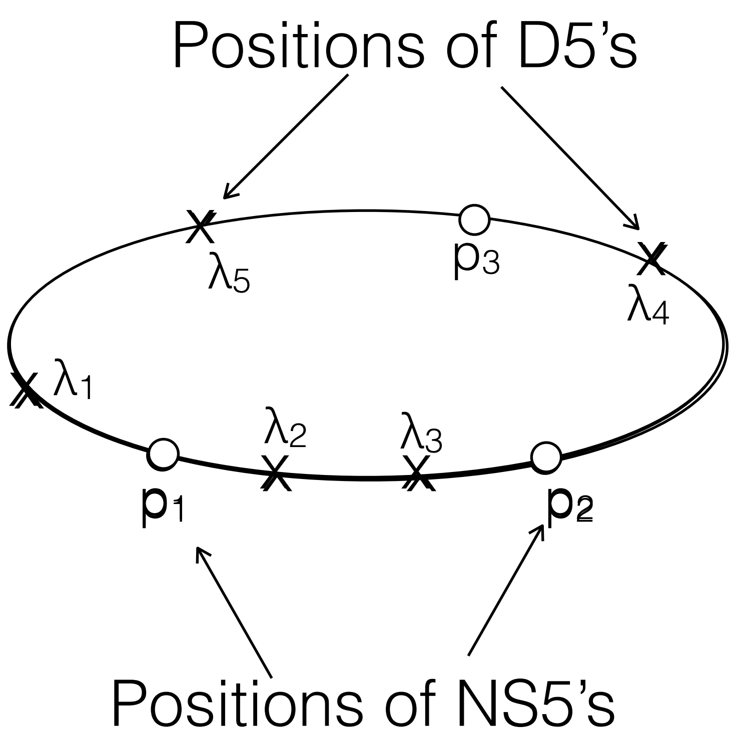
What is the expression for a general case metric?

Exact Metric on Moduli Space (Bow Integrabe System)

w/ Roger Bielawski

U(n) YM instanton on A_k ALF space => two Dynkin diagrams: $\tilde{A}_{k-1}$ and $\tilde{A}_{n-1}$

$\tilde{A}_{k-1}$



A bow representation solution carries above each λ interval the Nahm system:

$$\begin{cases} \frac{d}{ds}T_1 + [T_0, T_1] = [T_2, T_3] \\ \frac{d}{ds}T_2 + [T_0, T_2] = [T_3, T_1] \\ \frac{d}{ds}T_3 + [T_0, T_3] = [T_1, T_2] \end{cases}$$

with Lax pair:

$$M = \frac{d}{ds} + T_0 - iT_3 - \zeta(T_1 - iT_2),$$

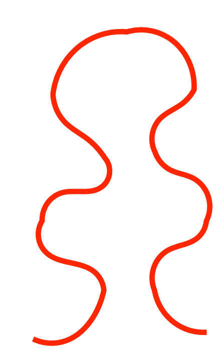
$$L = T_1 + iT_2 + 2\zeta iT_3 - \zeta^2(T_1 - iT_2),$$

$$\left[\frac{d}{ds} + M, L\right] = 0$$

and spectral curve

$$T\mathbb{P}^1 \ni S_{I_j} := \left\{\eta\frac{d}{d\zeta} \in T\mathbb{P}^1 \middle| \det(L - \eta)\right\} = 0$$

$\tilde{A}_{n-1}$

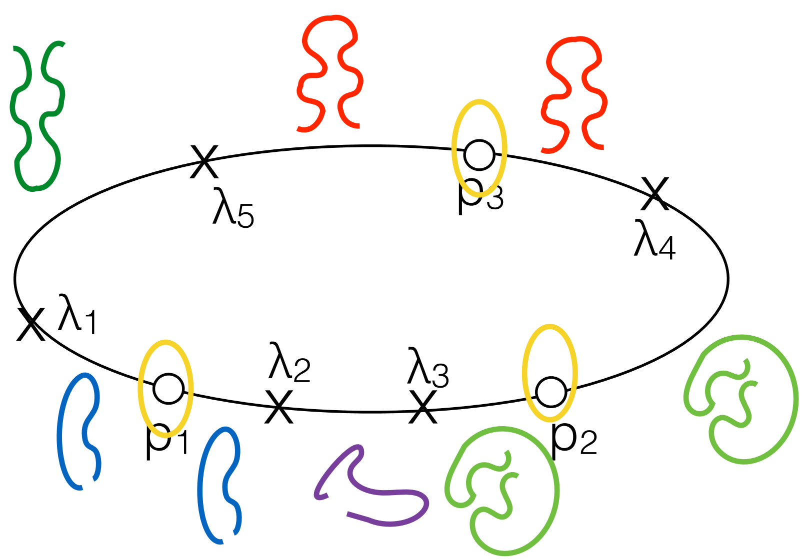


Spectral Curve changes across a λ-point, but remains the same across a p-point:

moment map conditions at a p-point read

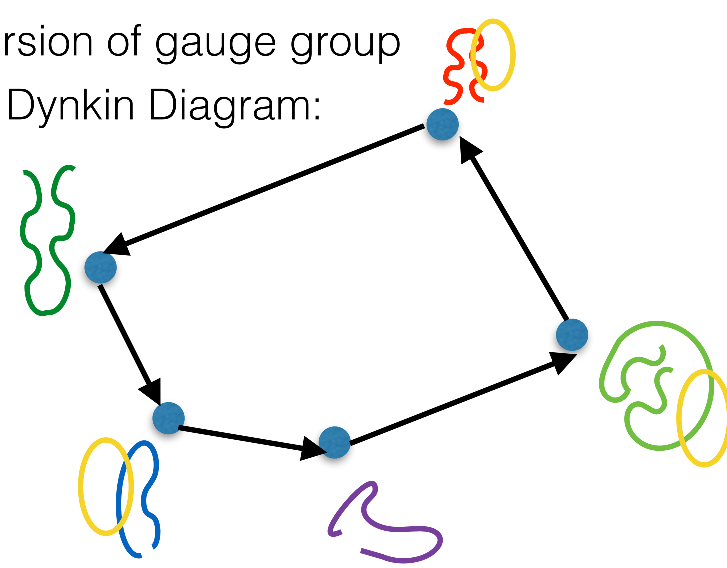
on the right: $L(p_\sigma +) = (B_{\sigma,RL} + \zeta B^\dagger_{\sigma,LR})(B_{\sigma,LR} - \zeta B^\dagger_{\sigma,RL}) - \left((\nu_1 + i\nu_2) + 2\zeta i\nu_3 - \zeta^2(\nu_1 - i\nu_2)\right)$

on the left: $L(p_\sigma -) = (B_{\sigma,LR} - \zeta B^\dagger_{\sigma,RL})(B_{\sigma,RL} + \zeta B^\dagger_{\sigma,LR}) - \left((\nu_1 + i\nu_2) + 2\zeta i\nu_3 - \zeta^2(\nu_1 - i\nu_2)\right)$



Reciprocal bow (cutting at λ-points instead) is of A_n type i.e. it is determined by the gauge group.

Affine version of gauge group $U^{aff}(n)$ Dynkin Diagram:

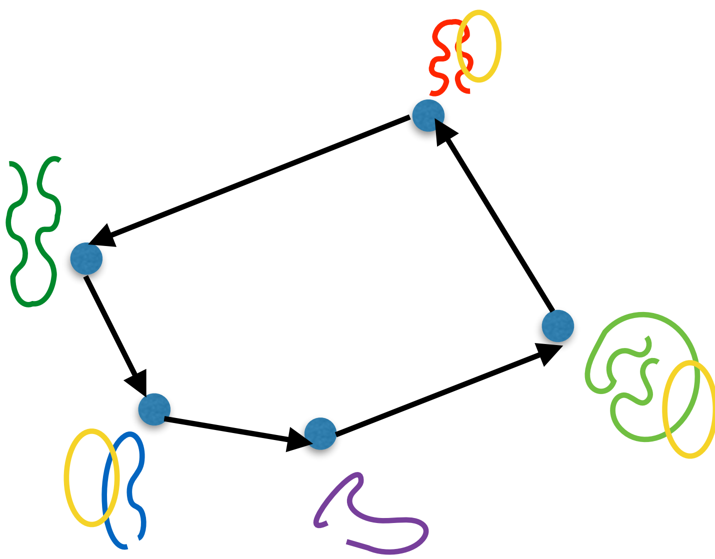


What is the significance of p-points?

Each p-point has an assigned moment map level: each determines a section of TP^1.

$$\eta = \left((\nu_1 + i\nu_2) + 2\zeta i\nu_3 - \zeta^2(\nu_1 - i\nu_2)\right)$$

Affine Dynkin diagram of the Instanton Gauge Group



- Each vertex of the affine Dynkin diagram carries a spectral curve.
- All of these curves are in $T\mathbb{P}^1$.
- p-points assign $\mathbb{P}^1$ curves (moment $\mathbb{P}^1{}_v$) to some vertices.
- Connected vertices => respective curve intersection divisor.
- A curve at a vertex intersection with the moment $\mathbb{P}^1$'s of that vertex.
- Ignore all other curve intersections.

Alternatively, one can view this as a single multi-component curve

Exact Metric via the Generalized Legendre Transform

w/ Roger Bielawski

In terms of finite HK quotient ingredients the symplectic structure on each interval is

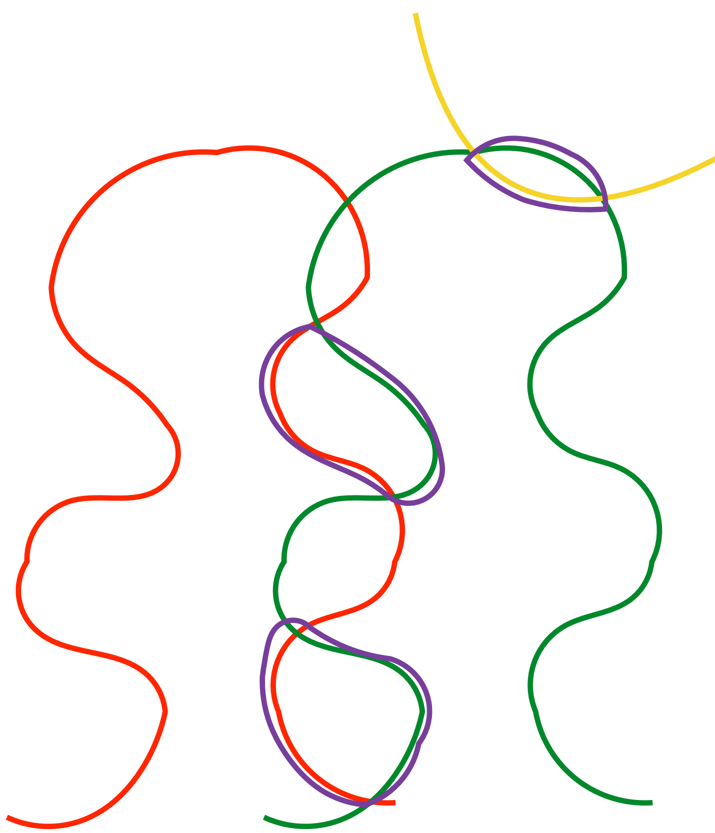
$$\omega = \mathrm{Tr}\left(H^{-1}dH \wedge dL + LH^{-1}dH \wedge H^{-1}dH\right)$$

Spectral curve on i[th] interval is

$$\eta_i^{r_i} + a_1^i(\zeta)\eta^{r_i-1} + \ldots + a_{r_i-1}^i(\zeta)\eta + a_{r_i-1}^i(\zeta) = 0$$

with polynomial coefficients

$$a_\alpha^i(\zeta) = z_i + v_i\zeta + w_{2,i}\zeta^2 + \ldots + w_{2r_i-2,i}\zeta^{2r_i-2} + (-1)^{r_i-1}\bar{v}_i\zeta^{2r_i-1} + (-1)^{r_i}\bar{z}_i\zeta^{2r_i}$$



Form Legendre potential, which is a function of coefficients of these polynomials

$$F = \sum_{i\in\text{Intervals}} l_i \frac{1}{2\pi i}\oint_0 \frac{\eta_i^2}{\zeta^3}d\zeta + \sum_{e\in\text{Edges}} \frac{1}{2\pi i}\oint_{\Gamma_e} (\eta_{h(e)} - \eta_{t(e)})\log(\eta_{h(e)} - \eta_{t(e)})\frac{d\zeta}{\zeta^2}$$

$$+ \sum_{\lambda\in\Lambda} \frac{1}{2\pi i}\oint_{\Gamma_e} (\eta_{i(\lambda)} - \nu_\lambda)\log(\eta_{i(\lambda)} - \nu_\lambda)\frac{d\zeta}{\zeta^2}$$

weights

more succinctly

$$F = -\frac{1}{2\pi i}\oint_0 \frac{\eta^2}{\zeta^3}d\zeta + \oint_C \frac{\eta}{\zeta^2}d\zeta$$

performing Legendre transform

$$u_i = \frac{\partial F}{\partial v_i}, \qquad \frac{\partial F}{\partial w_{\alpha,i}} = 0,$$

Constraints on the spectral curves.

gives the Kähler potential: $K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}) - uv - \bar{u}\bar{v}.$

using GLT of Hitchin, Karlhede, Lindstrom, Rocek '87