The Geometric Langlands conjecture and non-abelian Hodge theory

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- Hitchin's system, and
- non abelian Hodge theory.

Joint work with Tony Pantev, and partially with Carlos Simpson.

The abelian case

Motivation: The Langlands program is the *non-abelian* extension of class field theory (CFT). The abelian case is well understood. Its geometric version, or geometric CFT, is essentially the theory of a curve C and its Jacobian $J = J(C)$. This abelian case of the Geometric Langlands Conjecture (GLC) amounts to the well known result that any rank 1 local system (or: line bundle with flat connection) on the curve C extends uniquely to J, and this extension is natural with respect to the Abel-Jacobi map.

The abelian case

Topological proof: A rank 1 local system is a representation of π_1. For the Jacobian, $\pi_1(J) = H_1(C, \mathbf{Z}) = \pi_1(C)/([\ ,\ ])$. A representation of $\pi_1(C)$ into the abelian group $\mathbf{C}^*$ factors uniquely through $\pi_1(J)$.

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Algebraic proof: For every $d > 0$, the rank 1 local system L on C induces L^d on the Cartesian product C^d and $Sym^d(L)$ on the symmetric product $Sym^d(C)$. By RR, for $d >> 0$, the Abel-Jacobi map $Sym^d(C) \to J$ is a bundle, with simply connected fiber $\mathbf{P}^{d-g}$. So $Sym^d(L)$ descends to a rank 1 local system on J.

The abelian case

Naturality: Replacing J by $Pic := Pic(C)$, geometric CFT can be reformulated in terms of the Abel-Jacobi map

$$AJ : Pic \times C \to Pic$$

$$(L, x) \to L(x) := L \otimes \mathcal{O}(x)$$

It says: for every rank 1 local system $\mathbb{L}$ on C there is a unique rank 1 local system $c_{\mathbb{L}}$ on Pic such that:

$$AJ^*(c_{\mathbb{L}}) = c_{\mathbb{L}} \otimes \mathbb{L}.$$

The general case

GLC is the attempt to extend these classical results from C^* to all complex reductive groups G. This goes as follows.

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Correspondences

A correspondence between varieties A, B is a subscheme $H \subset A \times B$. It induces a transform from objects on A to objects on B, by pull-push. (May have to specify a kernel.) Each Hecke correspondence $H^{\mu} \subset \mathbf{Bun} \times \mathbf{Bun} \times C$ induces a transform from D-modules on $\mathbf{Bun}$ to D-modules on $\mathbf{Bun} \times C$.

Geometric Langlands

The Geometric Langlands Conjecture says that an irreducible G-local system on C determines a perverse sheaf on **Bun** which is a simultaneous eigensheaf for the action of the Hecke operators - this turns out to be the right generalization of naturality with respect to the Abel-Jacobi map. (A perverse sheaf is, roughly, a local system on a Zariski open subset of **Bun**, extended in a natural way across the complement.) Fancier versions of the conjecture recast this as an equivalence of derived categories: of D-modules on **Bun** vs. coherent sheaves on the moduli space **Loc** of local systems. There are many related conjectures and extensions, notably to punctured curves via parabolic bundles and local systems. Some of these make an appearance below.

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A fancier version of the Geometric Langlands conjecture predicts the existence of a canonical equivalence of categories

$$\boxed{\mathfrak{c} : D_{\mathrm{coh}}(\mathbf{Loc}, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}({}^L\mathbf{Bun}, \mathcal{D}),} \tag{GLC}$$

which is uniquely characterized by the property that $\mathfrak{c}$ sends the structure sheaves of points $\mathbb{V}$ in **Loc** to Hecke eigen $\mathcal{D}$-modules $\mathfrak{c}(\mathcal{O}_{\mathbb{V}})$ (corresponding to the above $c_{\mathbb{V}}$) on ${}^L\mathbf{Bun}$:

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Here μ is an appropriate character, and ${}^LH^{\mu}$ is the Hecke correspondence bounded by μ.

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Great progress has been made towards understanding these conjectures, through works of Drinfeld, Laumon, Beilinson, Lafforgue, Frenkel, Gaitsgory, Vilonen, Heinloth, ... Some versions are known for $GL(n)$. The conjecture is unknown for other groups. There are some more recent results of Heinloth in the parabolic case. Even for $GL(n)$, the proof is somewhat indirect: no explicit construction of non-abelian Hecke eigensheaves is known. Here we use non-abelian Hodge theory to reduce the construction of Hecke eigensheaves to solution of explicit differential equations

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HMS interpretation: the desired Hecke eigensheaf on **Bun** can be obtained by applying family Floer homology to Hitchin's system $\Longrightarrow$ c_L should be the relative Floer homology between two Lagrangians in $\mathbf{Higgs}_{C,{}^LG}$, one fixed, the other moving over **Bun**:

- View L as a point of $\mathbf{Higgs}_{C,G}$. Its Fourier-Mukai dual is a Lagrangian (with line bundle) in $\mathbf{Higgs}_{C,{}^LG}$.
- A general cotangent fiber of $\mathbf{Higgs}_{C,{}^LG}$.

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The main result of [DP1] is formulated as a duality of the Hitchin system: There is a canonical isomorphism between the bases $B, {}^LB$ of the Hitchin system for the group G and its Langlands dual ${}^LG$, taking the discriminant in one to the discriminant in the other. Away from the discriminants, the corresponding fibers are abelian varieties, and we exhibit a canonical duality between them.

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Abelianized Hecke eigensheaves: π_* of degree-0 line bundles on Hitchin fibers $h^{-1}(b)$. We are thinking of **Bun** as the base space and of $h^{-1}(b)$ as its spectral cover.

Abelianization via Hitchin's system

It is very tempting to try to understand the relationship of this abelianized result to the full GLC. The view of the GLC pursued in [BeDr] is that it is a quantum theory. The emphasis in [BeDr] is therefore on quantizing Hitchin's system, which leads to the investigation of *opers*. One possibility, discussed in [DP1] and [Ar], is to view the full GLC as a quantum statement whose classical limit is the result in [DP1]. The idea then would be to try to prove GLC by deforming both sides of the result of [DP1] to higher and higher orders. Arinkin has carried out some deep work in this direction. But there is another path.

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This leads us to non-abelian Hodge theory.

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Non Abelian Hodge theory (NAHT), as developed by Hitchin, Donaldson, Corlette, Simpson, Saito, Sabbah, Mochizuki, and others, establishes under appropriate assumptions the equivalence of local systems and Higgs bundles. A richer object (harmonic bundle or twistor structure) is introduced, which specializes to both local systems and Higgs bundles. This is closely related to Deligne's notion of a λ-connection: at $\lambda = 1$ we have ordinary connections (or local systems), while at $\lambda = 0$ we have Higgs bundles. Depending on the exact context, these specialization maps are shown to be diffeomorphisms or categorical equivalences.

NAHT

The projective (or compact Kahler) case and the one dimensional open case were settled by Simpson twenty years ago - but the open case in higher dimension had to await the recent breakthroughs by Saito, Sabbah, Mochizuki, Jost-Yang-Zuo, Biquard, etc. This higher dimensional theory produces an equivalence of parabolic local systems and parabolic Higgs bundles. This is quite analogous to what is obtained in the compact case, except that the objects involved are required to satisfy three key conditions discovered by Mochizuki. Below we review these exciting developments, and outline our proposal for using NAHT to construct the automorphic sheaves required by the GLC.

NAHT, GLC, and QFT

This approach is purely mathematical of course, but it is parallel to physical ideas that have emerged from the collaborations of Witten with Kapustin, Gukov and Frenkel [KW, GW, W3, FW], where the GLC was placed firmly in the context of quantum field theory.

NAHT and GLC

Completion of these ideas depends on verification that Mochizuki's conditions are satisfied in situations arising from GLC. This requires a detailed analysis of instability loci in moduli spaces.

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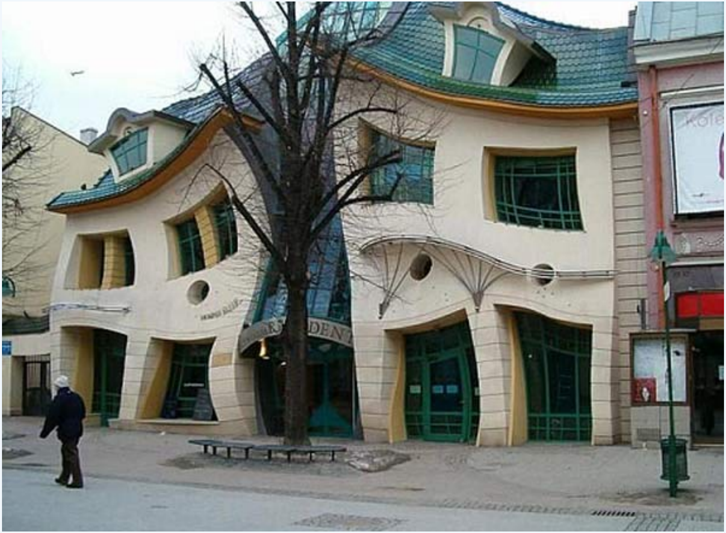
A bundle V is *wobbly* if it is stable but not very stable.

A bundle V is *shaky* if it is stable but there is a stable Higgs bundle (V', ϕ) over it with V' unstable. (More details below.)

Wobbly



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We will describe some known results about these loci for rank 2 bundles. These lead in some cases to an explicit construction (modulo solving the differential equations inherent in the non-abelian Hodge theory) of the Hecke eigensheaf demanded by the GLC.

NAHT: the theorems

Originally Corlette and Simpson proved the non-abelian Hodge theorem for projective manifolds:

Theorem [Corlette, Simpson] *Let* $(X, \mathcal{O}_X(1))$ *be a smooth complex projective variety. Then there is a natural equivalence of dg* $\otimes$*-categories:*

$$\mathbf{nah}_X : \begin{pmatrix} \text{finite rank } \mathbb{C}\text{-} \\ \text{local systems} \\ \text{on } X \end{pmatrix} \longrightarrow \begin{pmatrix} \text{finite rank } \mathcal{O}_X(1)\text{-semistable} \\ \text{Higgs bundles on } X \text{ with} \\ ch_1 = 0 \text{ and } ch_2 = 0 \end{pmatrix}$$

NAHT: the theorems

Mochizuki proved a version of the non-abelian Hodge correspondence which allows for singularities of the objects involved:

Theorem [Mochizuki] *Let $(X, \mathcal{O}_X(1))$ be a smooth complex projective variety and let $D \subset X$ be an effective divisor. Suppose that we have a closed subvariety $Z \subset X$ of codimension ≥ 3, such that $X - Z$ is smooth and $D - Z$ is a normal crossing divisor. Then there is a canonical equivalence of dg $\otimes$-categories:*

$$\mathbf{nah}_{X,D} : \left(\begin{array}{l} \textit{finite rank tame} \\ \textit{parabolic } \mathbb{C}\textit{-local} \\ \textit{systems on } (X, D) \end{array} \right) \longrightarrow \left(\begin{array}{l} \textit{finite rank locally abelian} \\ \textit{tame parabolic Higgs} \\ \textit{bundles on } (X, D) \\ \textit{which are } \mathcal{O}_X(1)\textit{-} \\ \textit{semistable and satisfy} \\ \textit{parch}_1 = 0 \textit{ and parch}_2 = 0 \end{array} \right)$$

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(2) a local condition: tameness (the Higgs field is allowed to have at most logarithmic poles along D) and compatibility of filtrations (the parabolic structure is locally isomorphic to a direct sum of rank one objects); and
(3) a global condition: vanishing of parabolic Chern classes.

NAHT: the theorems

A feature of the non-abelian Hodge correspondence that is specific to the open case is captured in another result of Mochizuki:

Theorem [Mochizuki's Extension Theorem] *Let U be a quasi-projective variety and suppose U has two compactifications*

$$\phi : U \to X, \ \psi : U \to Y$$

where:

- *X, Y are projective and irreducible;*
- *X is smooth and $X - U$ is a normal crossing divisor away from codimension 3;*

Then the restriction from X to U followed by the middle perversity extension from U to Y gives an equivalence of abelian categories:

$$\phi_{*!} \circ \psi^* : \begin{pmatrix} \textit{irreducible} & \textit{tame} \\ \textit{parabolic} & \mathbb{C}\textit{-local} \\ \textit{systems on } (X, D) & \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{simple } \mathcal{D}\textit{-modules on } Y \\ \textit{which are smooth on } U \end{pmatrix}$$

The plan

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As we saw, non-abelian Hodge theory provides a natural approach to constructing the geometric Langlands correspondence $\mathfrak{c}$. The big hope is that the known eigensheaf of the abelianized Heckes, which is a Higgs-type object $(\mathcal{E}, \varphi)$, extends by non abelian Hodge theory to a twistor eigensheaf on ${}^L\mathbf{Bun}$. The original Higgs sheaf appears at $z = 0$, while at the opposite end $z = 1$ we can expect to find precisely the Hecke eigensheaf postulated by the GLC.

The plan

The situation is essentially non-compact: In the moduli space ${}^L\mathbf{Bun}^s$ of stable bundles there is a locus $\mathbf{S}$ of shaky bundles along which our Higgs field φ blows up. This can be traced back, essentially, to the difference between the notions of stability for bundles and Higgs bundles.

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The plan

So the heart of the matter amounts to verification of the Mochizuki conditions: we need to find where the Higgs field blows up, resolve this locus to obtain a normal crossing divisor, lift the objects to this resolution, and verify that the parabolic chern classes of these lifts vanish upstairs. This would provide the crucial third step in the following six step recipe for producing the candidate automorphic sheaf:

G-local system (V, ∇) on C

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$\Downarrow$ (1)

G-Higgs bundle (E, θ) on C

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This is the Corlette-Simpson non-abelian Hodge correspondence $(E, \theta) = \mathbf{nah}_C(V, \nabla)$ on the smooth compact curve C.

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⇓ (1)

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⇓ (2)

$_{\mathrm{ab}}{}^{L}$**Hecke**-eigensheaf on ^{L}**Higgs**

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G-Higgs bundle (E, θ) on C

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${}_{\mathrm{ab}}{}^{L}\mathbf{Hecke}$-eigensheaf on ${}^{L}\mathbf{Higgs}$

The functor (2) sends $(E, \theta) \in \mathbf{Higgs}$ to $\mathbf{FM}(\mathcal{O}_{(E,\theta)})$ where $\mathbf{FM}$ is a Fourier-Mukai transform for coherent sheaves on $T^{\vee}\mathbf{Bun} = \mathbf{Higgs}$. In fact $\mathbf{FM}$ is the integral transform with kernel the Poincare sheaf constructed (away from the discriminant) in [DP1]. This sheaf is supported on the fiber product of the two Hitchin fibrations $h : \mathbf{Higgs}_0 \to B$ and ${}^{L}h : {}^{L}\mathbf{Higgs} \to B$.

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This part is known only in special cases, see below.

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$\Downarrow$ (4)

parabolic local system on ${}^{L}\mathbf{Bun}^{s}$ satisfying MC (1)-(3)

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parabolic local system on ${}^{L}\mathbf{Bun}^{s}$ satisfying MC (1)-(3)

The functor (4) is the parabolic non-abelian Hodge correspondence $\mathbf{nah}_{{}^{L}\mathbf{Bun}^{ss},\mathbf{S}}$ of Mochizuki. Here ${}^{L}\mathbf{Bun}^{ss}$ denotes the (rigidified) stack of semistable bundles. Note that here we are applying the first Mochizuki theorem not to a projective variety but to a smooth proper Deligne-Mumford stack with a projective moduli space. In fact Mochizuki's proof works in this generality with no modifications.

G-local system (V, ∇) on C

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G-Higgs bundle (E, θ) on C

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ordinary local system on Zariski open in ${}^{L}\mathbf{Bun}$

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Step 5 is just restriction.

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parabolic Higgs sheaf on ${}^{L}\mathbf{Bun}^{s}$ satisfying MC (1)-(3)

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parabolic local system on $^{L}\mathbf{Bun}^{s}$ satisfying MC (1)-(3)

$\Downarrow$ (5)

ordinary local system on Zariski open in $^{L}\mathbf{Bun}$

$\Downarrow$ (6)

$\mathcal{D}$-module on $^{L}\mathbf{Bun}$

The plan

Note that all of the other steps in this process are essentially already in place. The functor (1) is given by the Corlette-Simpson non-abelian Hodge correspondence $(E, \theta) = \mathbf{nah}_C(V, \nabla)$ on the smooth compact curve C.

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The functors (5) and (6) are the pullback and middle extension functors applied to the two compactifications ${}^L\mathbf{Bun}^{ss} \supset {}^L\mathbf{Bun}^{s} \subset {}^L\mathbf{Bun}$. In order to conclude that the composition $(6) \circ (5)$ is an equivalence we need a strengthening of Mochizuki's extension theorem which would allow for Y to be an Artin stack which is only locally of finite type.

Three papers

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- in [DP3] in a concrete example, namely P^1 with 5 parabolic points, to exhibit explicit solutions of GLC.
- in [DPS2] for C a curve of genus 2, with no parabolic points.

THANK YOU FOR YOUR ATTENTION!!!