A Quasi-Topological Gauged Sigma Model, the Geometric Langlands Program, and Knots

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Scope of Presentation

- Introduction
- Motivation
- Background Requirements
- Summary of Main Results
- Detailed Explanation of Main Results
- Conclusion

Introduction

- Based on paper **arXiv:1111.0691**

- Built upon insights from author's work in

"Two-Dimensional Twisted Sigma Models And The Theory of Chiral Differential Operators", **arXiv:hep-th/0604179**,

"Chiral Equivariant Cohomology and the Half-Twisted Gauged Sigma Model", **arXiv:hep-th/0612164**,

"Chiral Algebra of (0,2) Sigma Models: Beyond Perturbation Theory - II", (with J. Yagi) **arXiv:0805.1410**,

"Supersymmetric Surface Operators, 4-Manifold Theory, and Invariants in Various Dimensions", **arXiv:1006.3313**.

Motivation

- To furnish a physical interpretation of the recently formulated mathematical theory of "Twisted Chiral Differential Operators".

- To furnish an alternative, non gauge-theoretic, physical interpretation of the geometric Langlands correspondence, so as to be able to

(i) make contact with the original mathematical formulation of the correspondence by Beilinson-Drinfeld which utilizes algebraic CFT,

Motivation

(ii) gain, via the physics, mathematical insights into the correspondence at genus 1 and 0 which were not addressed in the gauge-theoretic approach by Kapustin-Witten,

(iii) furnish physical proofs of mathematical conjectures which relate geometric Langlands to knots

(iv) Furnish physical proofs of mathematical conjectures relating quantum groups to the correspondence.

Background Required (Physics)

- 2d supersymmetry and SCFT
- Twisted sigma models
- Anomalies
- Current algebra
- Worldsheet Instantons
- Supersymmetric quantum mechanics
- WZW models/Chern-Simons Theory

Background Required (Maths)

- **Differential Geometry**
  - Riemannian Geometry
  - de Rham/Dolbeault Cohomology
- **Algebraic geometry**
  - Sheaves and Sheaf cohomology
  - Intersection theory
- **Group representation theory**
  -Group extensions and group cohomology
  -Representation theory
- **Algebraic Topology**
  -Elliptic Genera
- **Knots, Knot Homology and Symplectic Geometry**
  - Jones Polynomial
  - Khovanov Homology
  - Lagrangian intersection Floer Homology

Summary of Main Results

- **We first study the physical features of a non-dynamically gauged quasi-topological (0,2) sigma model with target a G-manifold in perturbation theory**

  - Quasi-topological model.

  - Invariant under Weyl scalings of worldsheet.

  -Conformal anomaly unless target space is Ricci-flat.

  -Infinite tower of excited states/operator observables that have holomorphic weights.

Summary of Main Results

-Elliptic genus of local operators is therefore nontrivial in general.

-Operator observables span a holomorphic chiral algebra.

-Ground operators span a topological chiral ring.

-Operator observables can be described as classes in a certain Cech-cohomology group.

- For Cartan gauge group, and target space a flag manifold of G, model physically manifests the mathematical theory of "Twisted Chiral Differential Operators.

Summary of Main Results

- Taking the infinite target volume limit, one can see that the model possesses a generalized T-duality.

- From this T-duality, we have an isomorphism of W-algebras, and going back to the finite volume limit, we have only a Feigin-Frenkel isomorphism of W-algebras, i.e., affine G-algebra at critical level parameterized by ^LG-opers on worldsheet.

- A physical interpretation of the geometric Langlands correspondence for G, where complex curve is the worldsheet.

- Correlation functions of certain purely bosonic operators are Hecke eigensheaves.

- Hecke operators are certain nonlocal operators. We insert them into the correlation functions to effect a Hecke modification.

Summary of Main Results

- **We then study the nonperturbative effects of worldsheet twisted instantons in the sigma model**

  - For worldsheet of genus 0 with less than 3 punctures, instanton effects trivialize the chiral algebra completely.

  - Consistent with the fact that there are no Hecke eigensheaves at genus 0 with less than 3 punctures.

  - Via supersymmetric gauged quantum mechanics, chiral algebra observables correspond to harmonic spinors on the loop space of the flag manifold of G.

  - Can therefore connect the condition for the existence of Hecke eigensheaves to the Hohn-Stolz conjecture in algebraic topology that says that there are no harmonic spinors on the flag manifold of G.

Summary of Main Results

- **Finally, we study the infinite volume limit of the sigma model with target the flag manifold of simply-laced G.**

  - 2D Lagrangian is the same as that for a WZW model for G.

  - Via the WZW-Chern-Simons connection, can express knot invariants, in particular the Jones polynomial, in terms of the correlation functions of the chiral algebra observables. In turn, we can express the Khovanov homology in terms of quantum "ramified" D-modules and therefore, Lagrangian intersection homology. As such, we have a physical proof of a conjecture by Seidel-Smith.

  - Via the isomorphism of W-algebras in the infinite volume limit, we have a physical interpretation of the "ramified" geometric Langlands for G - which relates quantum "ramified" D-modules associated with G and ^LG - as a generalized T-duality.

Summary of Main Results

- As such, via the simultaneous interpretation of the correlation functions of the chiral algebra observables as quantum “ramified” D-modules and generators of knot invariants, we have a physical interpretation of a conjecture by Gaitsgory - which relates “ramified” D-modules to quantum groups associated with G and ^LG, respectively – as a generalized T-duality.

The Lagrangian of the quasi-topological gauged (0,2) model

$$S_{gauged} = \int_{\Sigma} |d^2 z| \; g_{i\bar{j}} (D_{\bar{z}}\phi^i D_z \phi^{\bar{j}} + \psi^i_{\bar{z}} \widehat{D}_z \psi^{\bar{j}}),$$

$$D_{\bar{z}}\phi^i = \partial_{\bar{z}}\phi^i - A^a_{\bar{z}} V^i_a \qquad D_z\phi^{\bar{i}} = \partial_z \phi^{\bar{i}} + A^a_z V^{\bar{i}}_a \qquad \widehat{D}_z \psi^{\bar{i}} = D_z \psi^{\bar{i}} + A^a_z \nabla_{\bar{k}} V^{\bar{i}}_a \psi^{\bar{k}}$$

$$\nabla_{\bar{k}} V^{\bar{i}}_a = \partial_{\bar{k}} V^{\bar{i}}_a + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} V^{\bar{j}}_a. \qquad \frac{\partial V^i_a}{\partial \phi^{\bar{j}}} = \frac{\partial V^{\bar{i}}_a}{\partial \phi^j} = 0. \qquad [V_a, V_b] = f_{ab}{}^c V_c,$$

$$\nabla_i V_{ja} + \nabla_j V_{ia} = 0, \quad \nabla_i V_{\bar{j}a} + \nabla_{\bar{j}} V_{ia} = 0,$$

$$\mathcal{L}_a(\partial V_b) - \mathcal{L}_b(\partial V_a) = [\partial V_a, \partial V_b] - f_{ab}{}^c \partial V_c \quad \text{and} \quad \mathcal{L}_a(\bar{\partial} V_b) - \mathcal{L}_b(\bar{\partial} V_a) = [\bar{\partial} V_a, \bar{\partial} V_b] - f_{ab}{}^c \bar{\partial} V_c,$$

The Lagrangian of the quasi-topological gauged (0,2) model

$$S_{gauged} = \int_{\Sigma} |d^2 z| \; g_{i\bar{j}} (D_{\bar{z}} \phi^i D_z \phi^{\bar{j}} + \psi^i_{\bar{z}} \widehat{D}_z \psi^{\bar{j}}),$$

$$\delta_\epsilon A^a_z = \partial_z \epsilon^a - f_{bc}{}^a \epsilon^b A^c_z \quad \text{and} \quad \delta_\epsilon A^a_{\bar{z}} = \partial_{\bar{z}} \epsilon^a + f_{bc}{}^a \epsilon^b A^c_{\bar{z}},$$

$$\delta_\epsilon \phi^i = \epsilon^a V^i_a \quad \text{and} \quad \delta_\epsilon \phi^{\bar{i}} = -\epsilon^a V^{\bar{i}}_a.$$

$$\delta_\epsilon \psi^i_{\bar{z}} = \epsilon^a \partial_k V^i_a \psi^k_{\bar{z}} \quad \text{and} \quad \delta_\epsilon \psi^{\bar{i}} = -\epsilon^a \partial_{\bar{k}} V^{\bar{i}}_a \psi^{\bar{k}}.$$

The Lagrangian of the quasi-topological gauged (0,2) model

$$S_{gauged} = \int_\Sigma |d^2z| \; g_{i\bar{j}}(D_{\bar{z}}\phi^i D_z\phi^{\bar{j}} + \psi^i_{\bar{z}}\widehat{D}_z\psi^{\bar{j}}),$$

$$S_{gauged} = \int_\Sigma |d^2z|\{Q, \mathscr{V}\}, \qquad \mathscr{V} = -g_{i\bar{j}}\psi^i_{\bar{z}}D_z\phi^{\bar{j}},$$

$$\begin{aligned} \delta\phi^i &= 0, & \delta\phi^{\bar{i}} &= \psi^{\bar{i}}, \\ \delta\psi^{\bar{i}} &= 0, & \delta\psi^i_{\bar{z}} &= -D_{\bar{z}}\phi^i, \\ \delta A^a_z &= 0, & \delta A^a_{\bar{z}} &= 0. \end{aligned}$$

Physical Features of the quasi-topological gauged (0,2) model

$$T_{z\bar{z}} = 0.$$

-Invariant under Weyl scalings of worldsheet

$$T_{zz} = g_{i\bar{j}}\partial_z\phi^i D_z\phi^{\bar{j}}, \qquad T_{\bar{z}\bar{z}} = \{Q, -g_{i\bar{j}}\psi^i_{\bar{z}}\partial_{\bar{z}}\phi^{\bar{j}}\},$$

-A quasi-topological sigma model

$$[Q, T_{zz}] = g_{i\bar{j}}\partial_z\phi^i \widehat{D}_z\psi^{\bar{j}} = 0 \ \ \text{(on-shell)}.$$

-At the classical level, field transformations generated by holomorphic stress tensor map elements in the Q-cohomology to other elements in the Q-cohomology.

Physical Features of the quasi-topological gauged (0,2) model

$$\bar{L}_0 = \{Q, V_0\} \text{ for some } V_0. \qquad \{Q, \mathcal{O}] = 0.$$

$$[\bar{L}_0, \mathcal{O}] = \{Q, \{V_0, \mathcal{O}]]. \qquad [\bar{L}_0, \mathcal{O}] = m\mathcal{O}.$$

-Antiholomorphic conformal weights are zero in the Q-cohomology.

-Infinite tower of local operators with positive holomorphic conformal weights in the Q-cohomology.

Physical Features of the quasi-topological gauged (0,2) model

$$\bar{L}_{-1} = \{Q, V_{-1}\} \text{ for some } V_{-1}. \qquad \{Q, \mathcal{O}] = 0,$$

$$\partial_{\bar{z}}\mathcal{O} = \{Q, \{V_{-1}, \mathcal{O}]]$$

- Local operators in the Q-cohomology vary holomorphically in z.

- Since a class which vanishes classically in Q-cohomology continues to do so in the quantum theory, the last two statements about the conformal weights and holomorphicity of the local operators continue to hold in the quantum theory.

Physical Features of the quasi-topological gauged (0,2) model

One-loop correction:

$$[Q, T_{zz}] = \partial_z (R_{i\bar{j}} D_z \phi^i \psi^{\bar{j}}).$$

- If X is not Ricci-flat, field transformations generated by holomorphic stress tensor NO LONGER map elements in the Q-cohomology to other elements in the Q-cohomology.

Physical Features of the quasi-topological gauged (0,2) model

Nevertheless,

$$[Q, L_{-1}] = 0,$$

- Operators remain in Q-cohomology after global translations on worldsheet.

$$[S, Q] = 0, \qquad S = L_0 - \bar{L}_0. \qquad \bar{L}_0 = 0. \qquad [Q, L_0] = 0$$

-Operators remain in the Q-cohomology after dilatations of the worldsheet coordinates.

Physical Features of the quasi-topological gauged (0,2) model

Moreover,

$$\bar{L}_{-1} = \oint d\bar{z}\, T_{\bar{z}\bar{z}}, \text{ where } T_{\bar{z}\bar{z}} = \{\overline{Q}_+, \dots\},$$

$$\partial_{\bar{z}} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\dots\mathcal{O}_s(z_s)\rangle$$

$$\oint d\bar{z}\,\langle\{Q,\dots\}\;\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\dots\mathcal{O}_s(z_s)\rangle = \oint d\bar{z}\,\langle\{Q,\cdots\prod_i \mathcal{O}_i(z_i)]\rangle = 0,$$

-Correlation functions of local operators in the Q-cohomology vary holomorphically in z. Holomorphic chiral algebra:

$$\mathcal{O}(z)\widetilde{\mathcal{O}}(z') \sim \sum_k f_k(z-z')\mathcal{O}_k(z'),$$

Physical Features of the quasi-topological gauged (0,2) model

$$T_{z\bar{z}} = \{\overline{Q}_+, G_{z\bar{z}}\} \text{ for some } G_{z\bar{z}}$$

Correlation functions of local operators invariant under Weyl scalings of worldsheet.

Sheaf of Perturbative Observables

- A general local operator looks like

$$\mathcal{F}(\phi^i, \partial_z\phi^i, \partial_z^2\phi^i, \ldots; \phi^{\bar{i}}, \partial_z\phi^{\bar{i}}, \partial_z^2\phi^{\bar{i}}, \ldots; A_z^a, \partial_z A_z^a, \partial_z^2 A_z^a \ldots; \psi^{\bar{i}}),$$

- Examples : q_R = k operators can be interpreted as (0,k)-forms on X valued in certain sum of vector bundles

$$\mathcal{F}(\phi^i, \phi^{\bar{i}}; \psi^{\bar{j}}) \;=\; f_{\bar{j}_1,\ldots,\bar{j}_k}(\phi^i, \phi^{\bar{i}})\psi^{\bar{j}_i} \ldots \psi^{\bar{j}_k}$$

$$\mathcal{F}(\phi^l, \phi^{\bar{l}}; \partial_z\phi^i; A_z^a; \psi^{\bar{j}}) \;=\; f_{i,\bar{j}_1,\ldots,\bar{j}_k}(\phi^l, \phi^{\bar{l}})D_z\phi^i\psi^{\bar{j}_1} \ldots \psi^{\bar{j}_k}$$

$$\mathcal{F}(\phi^l, \phi^{\bar{l}}; \partial_z\phi^{\bar{s}}; A_z^a; \psi^{\bar{j}}) \;=\; f^i{}_{\bar{j}_1,\ldots,\bar{j}_k}(\phi^l, \phi^{\bar{l}})g_{i\bar{s}}D_z\phi^{\bar{s}}\psi^{\bar{j}_i} \ldots \psi^{\bar{j}_k}$$

Sheaf of Perturbative Observables

- Generally, the local operator is not invariant under gauge transformations.

- Nevertheless, the transformed operator remains Q-closed, and its U(1) R-charge is the same as before.

- Can interpret a gauge transformation as a change in the basis of the space of operators. (Will be able to see this in the supersymmetric gauged quantum mechanics picture too)

Topological Chiral Ring

$$\mathcal{F}_a(z)\mathcal{F}_b(z') = \sum_{q_c=q_a+q_b} \frac{C_{abc}\ \mathcal{F}_c(z')}{(z-z')^{h_a+h_b-h_c}},$$

$$\widetilde{\mathcal{F}}_a(z)\widetilde{\mathcal{F}}_b(z') = \sum_{q_c=q_a+q_b} \frac{C_{abc}\ \mathcal{F}_c(z')}{(z-z')^{-h_c}}.$$

$$\widetilde{\mathcal{F}}_a(z)\widetilde{\mathcal{F}}_b(z') = \sum_{q_c=q_a+q_b} C_{abc}\ \widetilde{\mathcal{F}}_c(z,z').$$

$$\widetilde{\mathcal{F}}_a\widetilde{\mathcal{F}}_b = \sum_{q_c=q_a+q_b} C_{abc}\ \widetilde{\mathcal{F}}_c.$$

Topological Chiral Ring

- Operators in the ring are generally not gauge-invariant.

- However, grading by conformal dimension and U(1) R-charge is preserved. Thus, ring structure is well-defined.

- But for chiral ring to be unambiguously-defined, must consider only gauge-invariant local operators.

Sheaf of Chiral Algebras

- Can express local operators as elements of the Cech-cohomology group generated by the sheaf of chiral algebra $\widehat{\mathcal{A}}$:

$$\bigoplus_{q_R} H^{q_R}_{\text{Cech}}(X, \widehat{\mathcal{A}})$$

- Approach is to treat Q as the co-boundary operator in Cech-cohomology.

- Important points to note during formulation

$$H^{0,k}_{\bar{\partial}}(U, \widehat{F}) \text{ on } U. \qquad H^{k}_{\text{Cech}}(U, \widehat{F}) = 0 \text{ for } k > 0.$$

Via Cech-Dolbeault isomorphism $H^{0,k}_{\bar{\partial}}(U, \widehat{F}) = 0 \text{ for } k > 0.$

Sheaf of Chiral Algebras

- Since quantum perturbative corrections can only annihilate and not create cohomology classes, the last point means that local operators with positive q_R must vanish in Q-cohomology over U.

$$\mathcal{F}_1 = [Q, \mathcal{C}_e] \quad \mathcal{F}_1 = [Q, \mathcal{C}_e] = [Q, \mathcal{C}_f] \text{ over the intersection } U_e \cap U_f$$

$$[Q, \mathcal{C}_e - \mathcal{C}_f] = 0. \quad \mathcal{C}_{ef} = \mathcal{C}_e - \mathcal{C}_f \quad [Q, \mathcal{C}_{ef}] = 0 \quad \mathcal{C}_{ef} = -\mathcal{C}_{fe}, \quad \mathcal{C}_{ef} + \mathcal{C}_{fg} + \mathcal{C}_{ge} = 0.$$

$$[Q, \mathcal{K}_e] = [Q, \mathcal{K}_f] = 0, \quad \mathcal{C}_{ef} \sim \mathcal{C}'_{ef} = \mathcal{C}_{ef} + \mathcal{K}_e - \mathcal{K}_f. \quad H^1_{\check{\mathrm{C}}\mathrm{ech}}(X, \widehat{\mathcal{A}}).$$

- Can run derivation backwards using partition of unity subordinates approach that is standard in algebraic geometry.

Relation to Perturbed $\beta\ \gamma$ System

- We next study the model with abelian subgroup T of G on a local patch U of the target space X

$$I = \frac{1}{2\pi}\int_\Sigma |d^2 z| \sum_{i,\bar{j},a,b} \delta_{i\bar{j}} \left(\partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i + \psi^i_{\bar{z}} \partial_z \psi^{\bar{j}} - A^a_{\bar{z}} V^i_a A^b_z V^{\bar{j}}_b \right).$$

Let

$$\beta_i = \delta_{i\bar{j}} \partial_z \phi^{\bar{j}} \text{ and } \gamma^i = \phi^i$$

then the fields obey the standard OPE

$$\beta_i(z)\gamma^j(z') = -\frac{\delta_i^j}{z - z'} + \text{regular}.$$

Relation to the Perturbed $\beta\ \gamma$ Systems

Since on a local patch, we have fermion-free operators

$$\widehat{\mathcal{F}}(\phi^i, \partial_z \phi^i, \ldots; \phi^{\bar{i}}, \partial_z \phi^{\bar{i}}, \ldots; A^a_z, \partial_z A^a_z, \ldots)$$

the model which describes them is a free, perturbed βγ system:

$$I_{\beta\gamma} = \frac{1}{2\pi} \int |d^2 z| \sum_{i,a,b} \left(\beta_i \partial_{\bar{z}} \gamma^i - A^a_{\bar{z}} V^i_a A^b_z V_{ib}\right).$$

so we have

$$\widehat{\mathcal{F}}(\gamma, \partial_z \gamma, \partial^2_z \gamma, \ldots, \beta, \partial_z \beta, \partial^2_z \beta, \ldots A^a_z, \partial_z A^a_z, \partial^2_z A^a_z, \ldots).$$

"Gluing" the Perturbed $\beta\ \gamma$ System

- By "gluing" on overlaps the open patches in X using the symmetries of the perturbed βγ system, we will arrive at the (0,2) model over all of X.

- To "glue" geometrically over X, use conformal currents of the perturbed βγ system:

$$K_V = \oint J_V dz. \qquad J_V = -V^i \beta_i.$$

$$J_V(z)\gamma^k(z') \sim \frac{V^k(z')}{z-z'}. \qquad \delta\gamma^k = i\epsilon V^k$$

$$J_V(z)\beta_k(z') \sim -\frac{\partial_k V^i \beta_i(z')}{z-z'}. \qquad \delta_\epsilon \beta_k = -i\epsilon \partial_k V^i \beta_i$$

"Gluing" the Perturbed $\beta\ \gamma$ System

- To 'glue' the theory algebraically, use the conformal currents of the perturbed βγ system constructed using the moduli:

$$B = \sum_i B_i(\gamma) d\gamma^i \quad J_C = B_i D_z \gamma^i; \quad K_C = \oint J_C dz. \quad J_C = J_F + J_E$$

$$J_E(z)\beta_k(z') \sim \frac{\partial_k B_i \partial_z \gamma^i(z')}{z - z'} - \frac{\beta_k(z')}{(z-z')^2}. \qquad \delta_\epsilon \beta_k = -i\epsilon V_a^i \partial_k B_i A_z^a$$

$$J_F(z)\beta_k(z') \sim -\frac{V_a^i \partial_k B_i A_z^a(z')}{z - z'}. \qquad \delta_\epsilon \beta_k = i\epsilon \partial_k B_i \partial_z \gamma^i$$

holomorphic $(2,0)$-form $C = \partial B$,

- C must be a local holomorphic section of the sheaf $\Omega^{2,cl}$

The Automorphism Relations of the Sheaf of Chiral Algebras

$$\tilde{\gamma}^i = g^i(\gamma),$$
$$\tilde{A}^a_z = A^a_z,$$
$$\tilde{\beta}_i = D_i{}^k \beta_k + \partial_z \gamma^j E_{ij} - F_{i,a} A^a_z,$$

$$[E]_{ij} = \partial_i B_j$$

a $\mathfrak{t}$-valued holomorphic one-form $F_a = F_{l,a}\, d\gamma^l = (\partial_l f_a) d\gamma^l$ (where $f_a = V^k_a B_k$) on X.

The One-Loop Beta Function in Terms of Holomorphic Data

- Consider the case of X = CP1, which is non Ricci-flat.

$$\mathcal{T}(z) = - :\beta_i \partial_z \gamma^i:(z). \qquad \widetilde{\mathcal{T}}(z) = - :\tilde{\beta}_i \partial_z \tilde{\gamma}^i:(z),$$

Expected to be element of $H^0(X, \widehat{\mathcal{A}})$ since

$$T_{zz} = g_{i\bar{j}} \partial_z \phi^i D_z \phi^{\bar{j}},$$

But

$$\widetilde{\mathcal{T}}(z) - \mathcal{T}(z) = \partial_z \left(\frac{\partial_z \gamma}{\gamma} - \frac{V_a A^a_z}{\gamma} \right)(z).$$

which is a Cech description of

$$[Q, T_{zz}] = \partial_z \left(R_{i\bar{j}} \partial_z \phi^i \psi^{\bar{j}} - R_{i\bar{j}} V^i_a A^a_z \psi^{\bar{j}} \right).$$

Relation to the theory of "Twisted Chiral Differential Operators"

- Let gauge group be a Cartan subgroup of G, and X a flag manifold of G. Then automorphism relations become

$$g_{ef}^{tw}|_{\mathcal{O}_{U_e \cap U_f}} = g_{ef}|_{\mathcal{O}_{U_e \cap U_f}},$$
$$g_{ef}^{tw}(\lambda_a^*) = \lambda_a^*,$$
$$g_{ef}^{tw}(\xi) = g_{ef}(\xi) - \textstyle\sum_a (\iota_\xi \lambda_a^{(1)}(U_e \cap U_f))\lambda_a^*.$$

$A_z^a = \lambda_a^*.$ a holomorphic vector field $\xi = \xi^k \partial / \partial \gamma^k$ on X.

Relation to the theory of "Twisted Chiral Differential Operators"

- The perturbed βγ system on a local patch in X is now a section of the sheaf of TCDO's defined by Arakawa et al.

- The symmetries of the perturbed βγ CFT that we have used to 'glue' the local descriptions together are therefore the vertex algebra automorphisms used to 'glue' the sheaves of TCDO's.

- Standard Cech-cohomological analysis of the sheaves of chiral algebras of shows that obstruction to `gluing' vanishes if

$$\frac{1}{2}p_1^T(X) = 0.$$

Sheaf of "Twisted Chiral Differential Operators" on flag manifold of G

$$I = \frac{1}{2\pi} \int |d^2 z| \sum_{i=1}^{|\Delta_+|} \left(\beta_i \partial_{\bar{z}} \gamma^i - \sum_{a,b=1}^{l} A_z^a V_{ia} A_{\bar{z}}^b V_b^i \right), \qquad \beta_i(z)\gamma^j(z') \sim -\frac{\delta_i^j}{z - z'}.$$

$$\tilde{I} = \frac{1}{2\pi} \int |d^2 z| \sum_{i=1}^{|\Delta_+|} \left(\tilde{\beta}_i \partial_{\bar{z}} \tilde{\gamma}^i - \sum_{a,b=1}^{l} \tilde{A}_z^a V_{ia} \tilde{A}_{\bar{z}}^b V_b^i \right), \qquad \tilde{\beta}_i(z)\tilde{\gamma}^j(z') \sim -\frac{\delta_i^j}{z - z'}.$$

$$H^0(X, \widehat{\Omega}_{X;1}^{ch,tw})$$

$$J_{e_{\alpha_i}} = \beta_{\alpha_i} + \sum_{j=1}^{|\Delta_+|} : P_j^i(\gamma)\beta_j :, \quad i = 1, 2, \ldots, l;$$

$$J_{h_a} = -\sum_{j=1}^{|\Delta_+|} D_j : \gamma^j \beta_j : + A_z^a, \quad a = 1, 2, \ldots, l;$$

$$J_{f_{\alpha_i}} = \sum_{j=1}^{|\Delta_+|} : Q_j^i(\gamma)\beta_j : + C_i \partial_z \gamma^{\alpha_i} + A_z^a \gamma^{\alpha_i}, \quad a = i = 1, 2, \ldots, l.$$

Sheaf of "Twisted Chiral Differential Operators" on flag manifold of G

$$S^{(s_i)}(z) =: \tilde{d}^{\zeta_1\zeta_2\zeta_3\ldots\zeta_{s_i}}(k)(J_{\zeta_1}J_{\zeta_2}\ldots J_{\zeta_{s_i}})(z) : .$$

$$S^{(s_i)}(z) = (k+h^\vee)T^{(s_i)}(z),$$

But we have an affine G algebra at critical level

$$k = -h^\vee$$

So $S^{(si)}(z)$ does not exist as a quantum operator, i.e.,
It is purely classical, and ought to be generated only by the non-dynamical gauge field A.

The point is that the $S^{(si)}(z)$'s generate a constant multiple of the transformations associated with the stress tensor and its higher spin Segal-Sugawara analogs. It being zero quantum mechanically means that it is not in the Q-cohomology, as we saw earlier in the X= CP1 case.

Sheaf of "Twisted Chiral Differential Operators" on flag manifold of G

For example when G = SL(2), we have

$$S(z) = \frac{1}{4}A_z^2(z) - \frac{1}{2}\partial_z A_z(z).$$

For example when G = SL(3), we have

$$S(z) = \frac{1}{2}\left[(d_{ab}A_z^a A_z^b)(z) - \sum_{i=1}^{3}(V_a^i \partial_z A_z^a)(z)\right],$$

$$S^{(3)}(z) = \sqrt{-\frac{6}{15}}\sum_{i<j<k}\left((\epsilon_i \cdot A_z)(\epsilon_j \cdot A_z)(\epsilon_k \cdot A_z)\right)(z) - \sum_{i<j}(i-1)\partial_z\left((\epsilon_i \cdot A_z)(\epsilon_j \cdot A_z)\right)(z)$$
$$-\sum_{i<j}(j-i-1)\partial_z\left((\epsilon_i \cdot A_z)(\epsilon_j \cdot \partial_z A_z)\right)(z) + \frac{1}{2}\sum_i (i-1)(i-2)(\epsilon_i \cdot \partial_z^2 A_z)(z)$$
$$-\frac{1}{4}\partial_z(A_z \cdot A_z)(z) - \frac{1}{2}\sum_i (i-1)\epsilon_i \cdot A_z(z), \qquad (5.41)$$

Sheaf of "Twisted Chiral Differential Operators" on flag manifold of G

Since they are classical and do not have nontrivial OPE's with the current fields and themselves, their Laurent modes would obey

$$[S_n^{(s_i)}, J_m^{\alpha}] = [S_n^{(s_i)}, S_m^{(s_i)}] = 0.$$

This means that the $S^{(si)}(z)$'s can be identified as the center

$$\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}_{\mathbb{C}}))$$

of the chiral algebra generated by $\{J_{G_{\mathbb{C}}}\}$ and its z-derivatives.

T-Duality of Infinite volume limit sigma model and isomorphism of W-algebras

By taking the infinite volume limit of the sigma model, one can show that it enjoys a generalized T-duality which will allow us to derive the following identity:

$$\mathcal{W}_k(\mathfrak{g}_\mathbb{C}) \cong \mathcal{W}_{{}^L k}({}^L\mathfrak{g}_\mathbb{C}) \qquad \text{where} \qquad ({}^L k + {}^L h^\vee)^{-1} = r^\vee (k + h^\vee).$$

By going back to the finite volume limit, the symmetries of the sigma model greatly reduce, and we only have

$$\mathcal{W}_{-h^\vee}(\mathfrak{g}_\mathbb{C}) \cong \mathcal{W}_\infty({}^L\mathfrak{g}_\mathbb{C}).$$

which means that

$$\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}_\mathbb{C})) \cong \mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}_\mathbb{C}}(D).$$

Affine G-algebras at critical level parameterized by ^LG-Opers

Recall that we had

$$J_{e_{\alpha_i}} = \beta_{\alpha_i} + \sum_{j=1}^{|\Delta_+|} : P_j^i(\gamma)\beta_j :, \quad i = 1, 2, \ldots, l;$$

$$J_{h_a} = -\sum_{j=1}^{|\Delta_+|} D_j : \gamma^j \beta_j : + A_z^a, \quad a = 1, 2, \ldots, l;$$

$$J_{f_{\alpha_i}} = \sum_{j=1}^{|\Delta_+|} : Q_j^i(\gamma)\beta_j : + C_i \partial_z \gamma^{\alpha_i} + A_z^a \gamma^{\alpha_i}, \quad a = i = 1, 2, \ldots, l.$$

So thru the A-dependence and $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}_{\mathbb{C}})) \cong \mathrm{Fun}\,\mathrm{Op}_{{}^L\mathfrak{g}_{\mathbb{C}}}(D)$.

we have an affine G-algebra at critical level parameterized by opers associated with the Langlands dual of G.

Holomorphic G-bundles on Σ

We have a current algebra on Σ

$$J_a(z)J_b(w) \sim -\frac{h^\vee d_{ab}}{(z-w)^2} + \sum_c f_{ab}{}^c \frac{J_c(w)}{(z-w)},$$

whose zero modes obey the Lie algebra of G

$$[J_a^0(w), J_b^0(w)] = \sum_c f_{ab}{}^c J_c^0(w).$$

which we can then exponentiate to define a principal G-bundle over Σ. This bundle would be holomorphic because the J generators are holomorphic in w.

A Sheaf of correlation functions over Bun(G)

Consider the bosonic operators Φ in the chiral algebra that are also primary field operators of highest weight zero

$$J_a(z)\Phi^0(z') \sim -\frac{(t_a^0)\ \Phi^0(z')}{z-z'},$$

Under a infinitesimal motion in Bun(G), an n-point correlation function varies as

$$\delta_\nu \langle \Phi^0(z_1)\ldots\Phi^0(z_n)\rangle = -\sum_{k=1}^{n} \frac{1}{2\pi i}\oint_{\mathcal{C}_k} \frac{dz}{z-z_k}\,\eta^a(z)\,t_a^0\,\langle \Phi^0(z_1)\ldots\Phi^0(z_n)\rangle$$

$$= -\sum_{k=1}^{n} \eta^a(z_k)\,t_a^0\,\langle \Phi^0(z_1)\ldots\Phi^0(z_n)\rangle.$$

A Sheaf of correlation functions over Bun(G)

Since

$$\Psi_n = \langle \Phi^0(z_1) \dots \Phi^0(z_n) \rangle$$

changes as

$$\Psi_n \to \Psi_n', \text{ where } \Psi_n' = \alpha \Psi_n \text{ for some constant } \alpha.$$

can define a sheaf of correlation functions over Bun(G).

D-module on Bun(G)

Consider the case of G = SL(2). We have

$$J_a(z)\mathbb{V}_\lambda(x,z') \sim \frac{D_a \mathbb{V}_\lambda(x,z')}{z-z'},$$

where for λ = 0, like in our case, we have

$$D_+ = -x^2\partial_x,$$

$$D_0 = -x\partial_x,$$

$$D_- = -\partial_x.$$

D-module on Bun(G)

What this means is that the n-point correlation function is also a function of x's, i.e.

$$\Psi_n(x_1|z_1, \ldots, x_n|z_n)$$

and since it is a sheaf over Bun(G), one can naturally interpret the x's as holomorphic coordinates on Bun(G).

In turn, this means that S(z), which is second order in the J's, will act as a second-order differential operator on Ψ in Bun(G).

D-module on Bun(G)

For the case of general G, we can thus write

$$\mathcal{D}^{s_i} \cdot \Psi_n(x_1^1|z_1, \ldots, x_n^{|\Delta_+|}|z_n) = \Omega_\Sigma^{s_i} \, \Psi_n(x_1^1|z_1, \ldots, x_n^{|\Delta_+|}|z_n),$$

where the $D^{(si)}$'s are s_i-th order holomorphic differential operator on Bun(G) associated with the $S^{(si)}(z)$'s, while the Ω's are degree s_i differentials on Σ.

Note that the $D^{(si)}$'s act only to multiply by a c-number because the $S^{(si)}(z)$'s are purely classical fields, as explained earlier.

Clearly, Ψ is a D-module on Bun(G).

A geometric Langlands correspondence for G

1. The $S^{(s_i)}(z)$'s have nonsingular OPE's with the J's. So by the formula

$$\delta_\nu S^{(s_i)}(x) = \oint_{\mathcal{C}} \eta^a(t) \{ J_a(t) \cdot S^{(s_i)}(x) \} \, dt,$$

we find that the $S^{(s_i)}$'s and therefore the $D^{(s_i)}$'s are constant and thus globally well-defined on Bun(G)

2. Since the $S^{(s_i)}$'s are effectively c-numbers, they and therefore the $D^{(s_i)}$'s, commute with one another.

A geometric Langlands correspondence for G

3. Because dim $D^{(si)}$'s = dim $S^{(si)}$'s = dim Bun(G), we have a holonomic differential equation and thus Ψ.

4. Recall that we had the isomorphism

$$\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}_{\mathbb{C}})) \cong \mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}_{\mathbb{C}}}(D).$$

which means that we have the isomorphism

$$\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}_{\mathbb{C}}}(X) \xrightarrow{\sim} \mathcal{D}.$$

A geometric Langlands correspondence for G

5. Recall that we have a family of J currents of G parametrized by ^LG opers or bundles on Σ. In turn, since the realization of Ψ depends on the J's, for every ^LG-bundle, we have a Ψ.

Altogether from 1-5, we have a **GEOMETRIC LANGLANDS CORRESPONDENCE FOR G!**

The infinite volume limit of the flag manifold model and CS theory

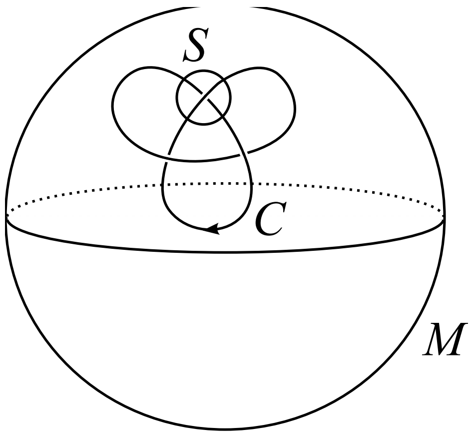
Let G be simply-laced. Then, in the infinite volume limit of the sigma model, its action will be

$$I_{\infty,\text{eff}} = \frac{1}{\pi}\int_{\Sigma} |d^2 z|\,\sqrt{g}\,e^{-2\sigma(z,\bar{z})}\left[\sum_{i=1}^{|\Delta_+|}\{\beta_i\partial_{\bar{z}}\gamma^i + \partial_{\bar{z}}(V^i\cdot Y)\partial_z(V_i\cdot Y)\} - i\frac{\mathcal{R}_{\bar{z}z}}{\sqrt{\widehat{k}+h^\vee}}(\rho\cdot Y)\right]. \quad (9.4)$$

This is a WZW model for (compact) G at level $\widehat{k}$

So, via the seminal work of Witten, correlation functions of local operators of the sigma model carry information about knots in the CS theory of the 3-space split along Σ.

The infinite volume limit of the flag manifold model and CS theory



The holomorphic conformal blocks

$$\mathscr{C} = \langle \Phi(z_1) \ldots \Phi(z_n) \rangle_{\widehat{\Sigma}} \,,$$

of local primary operators which obey

$$J_a(z)\Phi_r^\lambda(z') \sim -\sum_s \frac{(t_a^\lambda)_{rs}\ \Phi_s^\lambda(z')}{z-z'},$$

can be associated with knot invariants.

The relation to knot invariants

When there are no knots:

$$\mathscr{C} = \mathscr{C}_{\text{empty}} = \left\langle \Phi^0(z_1) \dots \Phi^0(z_n) \right\rangle_{\widehat{\Sigma}} = \langle 1 \rangle_{\widehat{\Sigma}} \,.$$

When there are knots in a representation λ

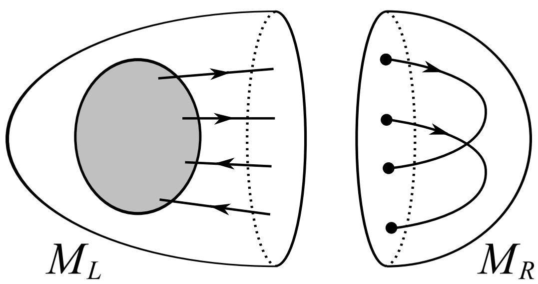
$$\mathscr{C}_{\text{knots}} = \mathscr{C}^{\lambda}_{\text{knots}} = \left\langle \Phi^{\lambda}_{s_1}(z_1)\, \Phi^{\lambda}_{s_2}(z_2)\, \Phi^{\bar{\lambda}}_{s_3}(z_3)\, \Phi^{\bar{\lambda}}_{s_4}(z_4) \right\rangle_{\widehat{\Sigma}} ,$$

The monodromy around each puncture is

$$g_{\lambda_i} = \exp\left(-\frac{2\pi i \lambda_i^*}{\widehat{k}} \right) , \qquad g_{\lambda_1} \dots g_{\lambda_p} = 1;$$

and one can interpret C_{knots} as quantum "ramified" D -modules, as they are sections of $H^0(\mathcal{M}_{G;z_1,\dots,z_p}, \mathscr{L}^{\widehat{k}})$

The relation to knot invariants



$$Z_M^\lambda(C) = \langle \chi | \psi \rangle = (\chi, \psi).$$

$$\chi = \sum_{s=1}^{\dim \mathcal{H}} a_\chi^s \mathscr{D}_s \qquad \text{and} \qquad \psi = \sum_{s=1}^{\dim \mathcal{H}} a_\psi^s \mathscr{D}_s,$$

$\mathscr{D}_s \in \mathcal{D}^c_{mod}(\mathcal{M}_{G;z_1,\ldots,z_4})$ spans an orthogonal basis in $\mathcal{H}$.

$$Z_M^\lambda(C) = \sum_{s=1}^{\dim \mathcal{H}} a_\chi^s a_\psi^s.$$

Jones Polynomial and Khovanov Homology

For G = SU(2), and M = S^3, Z_M is the Jones Polynomial. This can be related to Khovanov homology, so we can write

$$\sum_{a,b} (-1)^a q^b \dim \mathcal{K}^{a,b}(C) = \sum_{c \in \mathscr{C}} a^c_\chi a^c_\psi, \qquad q = \exp\left(\frac{2\pi i}{\widehat{k} + h^\vee}\right).$$

Thus, the (weighted) count of the Khovanov homology of the knot C, would be given by the (weighted) count of the number of components of the quantum “ramified” D-modules χ and ψ that coincide.

Lagrangian intersection Floer homology and a conjecture by Seidel-Smith

Now notice that we can also write

$$\sum_{n=1}^{\dim\mathcal{H}}\sum_{m=1}^{\dim\mathcal{H}}\delta^n_m\langle\chi|\mathscr{D}_m\rangle\langle\mathscr{D}_n|\psi\rangle=(\chi,\psi). \tag{9.19}$$

In terms of the vectors $\varphi_n, \varphi_m \in \mathcal{H}$, where $\varphi_n = \sum_{n=1}^{\dim\mathcal{H}} \mathscr{D}_n$ and $\varphi_m = \sum_{m=1}^{\dim\mathcal{H}} \mathscr{D}_m$, this is

$$\delta^n_m\,(\chi,\varphi_m)(\varphi_n,\psi)=(\chi,\psi). \tag{9.20}$$

Compare this with the relation between Lagrangian intersection Floer homology groups

$$HF^*_{\text{symp}}(L_0,L_1)\otimes HF^*_{\text{symp}}(L_1,L_2)\to HF^*_{\text{symp}}(L_0,L_2), \tag{9.21}$$

Generalization of Seidel-Smith to arbitrary links

$$Z_{\mathbf{S}^3}^{\lambda_1 \dots \lambda_{n-1}}(L) = \sum_j {S_0}^j \sum_{s_j=1}^{\dim \mathcal{H}_j} (\mathscr{D}_j^{s_j}, \phi_{\widehat{B}}(\mathscr{D}_j^{s_j})), \tag{9.30}$$

where the $\mathscr{D}_j^{s_j}$'s are "quantum" ramified $\mathcal{D}$-modules $\mathcal{D}^c_{mod}(\mathcal{M}_{G;z_1,\dots,z_{n-1},z_n})$ that span an orthogonal basis in $\mathcal{H}_j$; (,) is the usual natural pairing in $\mathcal{H}_j$; and $\phi_{\widehat{B}}$ is the operator $\widehat{B}$ – with eigenvector $\mathscr{D}_j^{s_j}$ – representing an autoequivalence of $\mathcal{C}$. Thus, the link invariant $Z_{\mathbf{S}^3}^{\lambda_1 \dots \lambda_{n-1}}(L)$ just counts (with appropriate weights) the number of linearly-independent "quantum" ramified $\mathcal{D}$-modules $\mathscr{D}_j^{s_j}$.

Specialize to G = SU(2), we get connection to Khovanov Homology

$$\sum_{a,b} (-1)^a q^b \dim \mathcal{K}^{a,b}(L) = \sum_j {S_0}^j \sum_{s_j=1}^{\dim \mathcal{H}_j} (\mathscr{D}_j^{s_j}, \phi_{\widehat{B}}(\mathscr{D}_j^{s_j})),$$

${S_0}^j$ are components of the modular transformation matrix acting in the Hilbert space of quantum "ramified" D-modules.

Conclusion

- Nice interplay between physics and mathematics.
- The physical interpretation provides a novel way of looking at the mathematical results and proving the conjectures, and connects seemingly unrelated areas of mathematics via the physics.
- Four-dimensional gauge theory not the only physical manifestation of the Geometric Langlands correspondence.

THANK YOU FOR YOUR ATTENTION!