PROBLEM SET 9

July 25, 2019

(1) Show that $\xi \in \Gamma_{\mathcal{O}}(\Omega, \mathcal{H}(\phi)^*)$ if and only if for all $f \in \mathcal{H}(\phi)_0 = \mathcal{H}(\phi)_t$ the function $\Omega \ni t \mapsto \langle \xi_t, f \rangle$ is holomorphic.

(2) Show that if $\{q_j\} \subset \mathcal{H}(\phi)_t$ is an orthonormal basis then $\overline{K}^x_t(y)(= K_t(x,y)) = \sum_{j=1}^n g_j(x) \otimes \overline{g_j(y)}$.

(3) Show that

$$K_t(x,x)e^{-\phi_t(x)} = \sup\{|f(x)|^2 e^{-\phi_t(x)} : f \in \mathcal{H}(\phi)_t \text{ and } \|f\|_t = 1\}$$

(4) Let $E \to \Omega$ be a holomorphic vector bundle with Hermitian metric h, and let $\Theta(h)$ be a the curvature of its Chern connection. Show that the following are equivalent:

(a) for all $t \in \Omega$, $x \in T_{\Omega,t}$ and each $v \in E_t$,

$$h(\Theta(h)_{x,x}(v), v) \geq 0.$$

(b) For each $\sigma \in \Gamma_{\mathcal{O}}(\Omega, E^*)$ the function $\log h^*(\sigma, \sigma)$ is plurisubharmonic on Ω.

(5) Let $\phi_t^{\langle N \rangle}(z) = |t|^2|z|^2 + (N+n)\log(1+|z|^2)$.

(a) Show that

$$\mathcal{H}(\phi_t^{\langle N \rangle}) = \{f \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |f|^2 e^{-\phi_t^{\langle N \rangle}} dV < +\infty\}$$

is finite dimensional if and only if $t = 0$.

(b) What is $\mathcal{H}(\phi_0^{\langle N \rangle})$?