PROBLEM SET 8

July 24, 2019

(1) Let $\Omega \subset\subset \mathbb{C}^n$ be bounded pseudoconvex, $l : \mathbb{C}^n \to \mathbb{C}$ linear such that $\ker l \cap \Omega \neq \emptyset$. Let

$$Z = \{l = 0\} \subset \Omega, \quad \phi \in \mathrm{PSH}\,(\Omega).$$

Show that for all $\in \mathcal{O}(Z)$ such that $\int_Z |f|^2 e^{-\phi} dA < \infty$, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_Z = f$ and

$$\int_\Omega |F|^2 e^{-\phi}\, dV \leq C \int_Z |f|^2 e^{-\phi}\, dA$$

where C depends only on $\sup_\Omega |l|$.

(2) Let $X = \mathbb{C}$, $Z = \Gamma$ a closed discrete subset of $\mathbb{C}$, $T \in \mathcal{O}(\mathbb{C})$ such that $T(z) = 0$ if and only if $z \in \Gamma$ and $dT(\gamma) \neq 0$, for all $\gamma \in \Gamma$. Let $\phi = |z|^2$ and

$$\lambda(z) = \lambda_r(z) := \frac{1}{\pi r^2} \int_{D_r(z)} \log |T|^2\, dA, g = |dz|^2$$

(a) Show that λ_r is subharmonic.
(b) Show that $|T|^2 e^{-\lambda r} \leq 1$ on $\mathbb{C}$.
(c) Compute the curvature of $e^{-\lambda_r}$. (Metric for trivial bundle)
(d) What does the condition

$$\partial\overline{\partial}\phi + \mathrm{Ricci}\,(g) \geq (1+\delta)\partial\overline{\partial}$$

mean?
(e) Show that if

$$\sup_{\mathbb{C}} \frac{\#\Gamma \cap D_r(z)}{r^2} < 1$$

then for all $f : \Gamma \to \mathbb{C}$ such that

$$\sum_{\gamma\in\Gamma} \frac{|f(\gamma)|^2 e^{-|\gamma|^2}}{|dT(\gamma)|^2 e^{-\lambda_r(\gamma)}} < +\infty,$$

there exists $F \in \mathcal{O}(\mathbb{C})$ such that

$$\int_{\mathbb{C}} |F|^2 e^{-|z|^2} dA < +\infty,$$

and $F|_\Gamma = f$.
(f) Show that the functions $|dT|^2 e^{-\lambda_r}$ and $\Delta\lambda_r$ depend only on Γ, and not on T. In fact, show that $|dT(0)|^2 e^{-\lambda_r(\gamma)}$ depends only on $\Gamma \cap D_r(\gamma)$.
(g) In view of (f), for all $\gamma_0 \in \Gamma$ one can use

$$T(z) = \prod_{\gamma\in D_r(z)} (z-\gamma).$$

Show that if Γ is uniformly seperated, i.e.,

$$\delta_\Gamma := \inf\{|\gamma - \mu| : \gamma, \mu \in \Gamma, \gamma \neq \mu\} > 0,$$

then $|dT|^2 e^{-\lambda_r} : \Gamma \to \mathbb{R}_+$ is bounded below by a positive constant (depending on γ).