PROBLEM SET 7

July 23, 2019

(1) Show that if $X = \mathbb{P}_1$, $L = H \to \mathbb{P}_1$ is the hyperplane bundle, $S = \{[P_1 = [1,0], P_2 = [0,1], P_3 = [1,1]\} \subset \mathbb{P}_1$, $\lambda_i \in \mathbb{H}_{P_i}$ defined by $\lambda_1(t,0) = t$, $\lambda_2(0,t) = 0$, $\lambda_3(t,t) = -t$, then there does not exist $F \in \Gamma_{\mathcal{O}}(\mathbb{P}_1, \mathbb{H})$ such that $F(p_i) = \lambda_i$.

(2) Suppose $R_S : L^2(X, e^{-\psi} dV_g) \cap \mathcal{O}(X) \to L^2(S, e^{-\psi} dA_g) \cap \mathcal{O}(S)$ is surjective. Show that there exists $C > 0$ such that for all $f \in L^2(S, e^{-\psi} dA_g) \cap \mathcal{O}(S)$ there exists $F \in L^2(X, e^{-\psi} dV_g) \cap \mathcal{O}(X)$ such that $F|_S = f$ and

$$\int_X |F|^2 e^{-\psi} dV_g \leq C \int_S |f|^2 e^{-\psi} dA_g.$$

(3) Let X be a strongly pseudoconvex manifold. Show that for all $x \in X$, $c \in C$, and $\alpha \in T^{1,0*}_{X,x}$, there exists $f \in \mathcal{O}(X)$ such that $f(x) = c$ and $df(x) = \alpha$.

(4) (a) Show that on a projective manifold X, every holomorphic line bundle has a meromorphic section which is not identically zero.
(b) Show that every holomorphic line bundle on X is the line bundle of a divisor.
(c) Show that (a) is false if we replace "meromorphic" by "holomrophic".

(5) Let X be a compact Riemann surface.
(a) Let $L \to X$ be a holomorphic line bundle. Show that for any metric h for L, $\int_X \Theta(h)$ is independent of h.
(b) Show that for any metric h on L_p,

$$\int_X \frac{\sqrt{-1}}{2\pi} \Theta(h) = 1.$$

(c) Show that for any holomophic line bundle $L \to X$ and any metric h for $L \to X$,

$$\frac{\sqrt{-1}}{2\pi} \int_X \Theta(h) \in \mathbb{Z}.$$