PROBLEM SET 4

July 18, 2019

(1) Let X be a compact complex manifold with a Kähler metric g, $E \to X$ a holomorphic vector bundle, and h Hermitian metric on E. Assume $\Theta_g(h) \geq cI$ for some $c > 0$.
   (a) Show that for any $\phi \in L^2_{n,q}(g,h)$, there exists $v \in \Gamma(X, \wedge_X^{n,q} \otimes E)$ such that $\overline{\partial} v$, $\overline{\partial}^* v \in L^2_{n,q}(g,h)$ and $\Box v = \phi$ in the weak sense, i.e., for all $\eta \in \Gamma(X, \wedge_X^{n,q} \otimes E)$ smooth,
$$(\overline{\partial} v, \overline{\partial} \eta) + (\overline{\partial}^* v, \overline{\partial}^* \eta) = (\phi, \eta).$$
   (b) Show that if $\overline{\partial}\phi = 0$ then $\overline{\partial}^* \overline{\partial} v = 0$ in the weak sense, $(\overline{\partial} v, \overline{\partial} \eta) = 0$ for smooth η.
   (c) From (b), prove Hörmander for compact complex manifolds.

(2) Show that if X is a compact complex manifold and $L \to X$ is a holomorphic line bundle admitting a metric of positive curvature then
   (a) X is Kähler, and
   (b) Any $\Box$-harmonic form is identically equal to 0.

(3) Explani, using the complex geometry we have developed, the close relationship between dual connnections and formal adjoints.

(4) Let X be a compact Kähler manifold, $E \to X$ a holomorphic vector bundle with Hermitian metric h. Assume $\Theta_g(h) \geq cI$. Let $\phi \in \Gamma(X, \wedge_X^{n,q} \otimes E)$ be a smooth form satisfying $\overline{\partial}\phi = 0$. Consider
$$\mathcal{A}_\phi := \{\phi + \overline{\partial}\eta : \eta \in \Gamma(X, \wedge_X^{n,q} \otimes E)\}.$$
Find the element of $\mathcal{A}_\phi$ whose L^2-norm is minimal.

(5) Let X be a compact complex manifold and $L \to X$ a holomorphic line bundle with a Hermitian metric h whose curvature is positive. Let $x_1, \ldots, x_n \in X$ be distinct points and $v_i \in L_{x_i}$, $1 \leq i \leq n$ be such that $h(v_i, v_i) = 1$. Show that there exists m sufficiently large and a holomorphic section s of $L^{\otimes m} \to X$ such that $s(x_i) = v_i$, $1 \leq i \leq n$.