PROBLEM SET 3

July 17, 2019

(1) (a) Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain, annd ρ, r be two defining functions for Ω. Show that there is a smooth positive function h near $b\Omega$ such that $\rho = hr$.
(b) Show that if the Levi form is positive definite with respect to ρ then its is positive definite with respect to r also.

(2) Show that the unit ball $\{|z_1|^2 + \cdots + |z_n|^2 < 1\}$ in $\mathbb{C}^n$ is strongly pseudoconvex by computing its Levi-form.

(3) Let $\Omega \subset \mathbb{C}^n$ be a domain. Show that Ω is pseudoconvex if and only if for all $p \in \mathbb{C}^n$, there exists a neighbourhood U of p such that $U \cap \Omega$ is pseudoconvex.

(4) Show that $H_p(\partial\Omega) = T_{\partial\Omega,p} \cap J(T_{\partial\Omega,p})$

(5) Let X be a complex manifold, $L \to X$ holomorphic line bundle, with metric h. The dual bundle $L^* \to X$ has dual metric h^*.

$$D(h^*) = \{\alpha \in L^* : h^*(\alpha, \alpha) < 1\} \subset L^*.$$

Show the $D(h^*)$ is pseudoconvex if and only if $\Theta(h)$ is positive $(1,1)$-form.

(6) The line bundle $\mathbb{U} \to \mathbb{P}_n$ has the following metric : If $v \in \mathbb{U}_l = l$ then $h(v, v) = |v|^2$, where $\| \cdot \|^2$ is the Euclidean square-norm.
(a) Show that a neighbourhood of the boundary of $D(h^*)$ in $\mathbb{U}$ is biholomorphic to a neighbourhood of the unit sphere in $\mathbb{C}^{n+1}$.
(b) What is the curvature of the dual metric h for $\mathbb{H} \to \mathbb{P}_n$?