PROBLEM SET 2

July 16, 2019

(1) Let $T : H_1 \to H_2$ be a closed, densely defined operator between two Hilbert spaces. Show that $T^* : H_2 \to H_1$ is also closed and densely defined. Further show that $(T^*)^* = T$.

(2) Prove the following improved version of Hormander's theorem in one variable:

Theorem 0.1 (Hörmander)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $\varphi \in C^2(\Omega)$ be a strictly subharmonic function, that is, φ is real-valued and $\Delta\varphi > 0$. If g is a function on Ω such that*

$$\int_\Omega \frac{|g|^2}{\Delta\varphi} e^{-\varphi}\, dV < \infty,$$

then there is a function u on Ω such that $\dfrac{\partial u}{\partial \overline{z}} = g$ and

$$\int_\Omega |u|^2 e^{-\varphi}\, dV \le \int_\Omega \frac{|g|^2}{\Delta\varphi} e^{-\varphi}\, dV. \tag{0.2}$$

(3) Definition:
  (a) A hypersurface (not necessarily smooth) is a subset $S \subset X$ such that for all $p \in X$, there exists a neighbourhood U of p and a function $f \in \mathcal{O}(U)$ such that
    (i) $S \cap U = \{f = 0\}$ and
    (ii) $df \not\equiv 0$ on any irreducible component of $S \cap U$.
  (b) A divisor on X is a formal sum $\sum_S a_S S$ where $a_S \in \mathbb{Z}$ and for all $x \in X$ and any neighbourhood U of x the set $\{S : S \cap U \neq 0\}$ is finite.

  Problems:
  (I) Show that for any divisor D there is a holomorphic line bundle L_D and a meromorphic section σ_D of L_D whose divisor is D
  (II) Let $L \to X$ be a holomorphic line bundle with meromorphic section σ, and let D be the divisor of σ. Show that $L \cong L_D$.

  (The divisor of a meromorphic function f is $\sum \mathrm{Ord}_V(f) \cdot V$ i.e., it is the hypersurface of zeros, counting multiplicity, minus the hypersurface of poles, counting multiplicity.)

(4) Let $\Omega \subset \mathbb{C}$ be a domain and let $\Gamma \subset \Omega$ be a locally finite subset. Let $f : \Gamma \to \mathbb{C}$ be any function. Show that there exists $F \in \mathcal{O}(\Omega)$ such that $F(\gamma) = f(\gamma)$ for all $\gamma \in \Gamma$.