PROBLEM SET 1

July 15, 2019

(1) *Solving $\bar{\partial}$ in one variable for smooth data.* Let Ω be an open set in $\mathbb{C}$ and $g = g_1 d\bar{z} \in \Lambda^{0,1}(\Omega)$.

(a) Suppose $K \subset \Omega$ is a compact set, then there is an open set V with $K \subset V \subset\subset \Omega$. Let $\phi \in C_0^\infty(\Omega)$ be a function that is identically 1 on V. Extend the function ϕg_1 to $\mathbb{C}$ by zero. Let this extension be f. Then $f \in C_0^\infty(\mathbb{C})$. Let

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Show that $\frac{\partial u}{\partial \bar{z}}(z) = f(z)$ for all $z \in \mathbb{C}$ and therefore $\bar{\partial} u = g$ on V.

(b) Show that there exists a sequence of compact sets $\{K_n\}_{n=1}^\infty$ with the following properties:

- $K_n \subset \text{int}(K_{n+1})$.
- $\bigcup_{n=1}^\infty K_n = \Omega$
- Each bounded component of $\mathbb{C} \setminus K_n$ intersects some bounded component of $\mathbb{C} \setminus \Omega$.

(c) Now inductively construct a sequence $\{u_n\}_{n=1}^\infty$ of C^∞ functions such that $\bar{\partial} u = g$ on a neighbourhood of K_n and $|u_n(z) - u_{n-1}(z)| \leq 2^{-n}$ for all $z \in K_{n-1}$ as follows.

- Construct u_1 that solves $\bar{\partial} u_1 = g$ on a neighbourhood of K_1 as in part (a). Suppose that the functions $\{u_n\}_{n=1}^N$ have been constructed. Let h be a C^∞ function such that $\bar{\partial} h = g$ on a neighbourhood of K_{N+1}. Conclude that $h - u_N$ is holomorphic on K_N.
- By Runge's theorem, there is a rational function r with poles in $\mathbb{C} \setminus \Omega$ such that $|h(z) - u_N(z) - r(z)| \leq 2^{-N-1}$ for all $z \in K_{N+1}$. Let $u_{N+1} = h - r$.

(d) Show that the sequence $\{u_n\}$ converges uniformly on compact sets of Ω to a function u. Show that $u \in C^\infty(\Omega)$ and that $\bar{\partial} u = g$ on Ω.

(2) Show that if $g \in \mathcal{C}^\infty(\Omega)$ and $u \in \mathcal{C}^1(\Omega)$ is such that $\dfrac{\partial u}{\partial \bar{z}} = g$, then $u \in \mathcal{C}^\infty(\Omega)$.

(3) Let H be the Hilbert space $L^2(0,1)$ and define an unbounded operator T on H as follows. The domain of T consists of those $f \in L^2(0,1)$ such that the derivative $f' \in L^2(0,1)$ (where the derivative is in the sense of distributions.) For $f \in \text{Dom}(T)$ we set $Tf = f'$. Show that if $f \in \text{Dom}(T^*) \cap \mathcal{C}^1([0,1])$, then $f(0) = f(1) = 0$.

(4) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $L_1^2(\Omega)$ be the Hilbert space of 1-forms with coefficients in $L^2(\Omega)$. The inner product in $L_1^2(\Omega)$ is given by

$$(a, b) = \sum_{j=1}^n \int_\Omega a_j \bar{b}_j dV, \quad \text{for } a = \sum_{j=1}^n a_j dx_j, b = \sum_{j=1}^n b_j dx_j \in L_1^2(\Omega).$$

Let $D : L^2(\Omega) \dashrightarrow L^2_1(\Omega)$ be the linear operator with $\mathrm{Dom}(D) = C_0^\infty(\Omega)$ given by

$$Df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j, \quad \text{for all } f \in C_0^\infty(\Omega).$$

Find the formal adjoint D' of the operator D.

(5) Show that the operator J defined by

$$J\frac{\partial}{\partial x^i} := \frac{\partial}{\partial y^i} \quad \text{and} \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \qquad 1 \le i \le n$$

in any holomorphic coordinate system is independent of the holomorphic coordinate system and that it is intertwined with $\sqrt{-1}$ via the map $s^{1,0}$, i.e., $s^{1,0}J = \sqrt{-1}s^{1,0}$.

(6) Find all the holomorphic sections of the line bundles $T^{1,0}_{\mathbb{P}_1} \to \mathbb{P}_1$ and $(T^{1,0}_{\mathbb{P}_1})^* \to \mathbb{P}_1$.

(7) Show that the line bundle $K_{\mathbb{P}_n} \to \mathbb{P}_n$ is isomorphic to $\mathbb{U}^{\otimes(n+1)} \to \mathbb{P}_n$.