# $p^r$-Selmer companion modular forms

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ICTS

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In "Selmer Companion Curve", Mazur-Rubin ask "the converse question": given a prime $p$ and a number field $K$, what information about $E/K$ can be obtained from the function $\mathrm{Hom}(G_K, \{\pm 1\}) \longrightarrow \mathbb{N} \cup \{0\}$ given by $\chi \longrightarrow \dim_{\mathbb{F}_p} S_p(E^\chi/K)$?

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## Definition (Mazur-Rubin)

$E_1, E_2$ are $n$-Selmer near-companions over $K$ if $\exists\ C = C(E_1, E_2, K)$ s.t. $\forall\ \chi$ of $G_K$, there is an abelian group $A_\chi$ and homomorphisms $S_n(E_1^\chi/K) \longrightarrow A_\chi$ and $S_n(E_2^\chi/K) \longrightarrow A_\chi$ with kernel and cokernel of order at most $C$.

## Conjecture (Mazur-Rubin)

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## Theorem (J., Majumdar, Shekhar)

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*Then for every quadratic character $\chi$ of $G_{\mathbb{Q}}$ and for every fixed $j$ with $0 \leq j \leq k-2$, we have an isomorphism of the $\pi^r$ - Bloch-Kato Selmer groups*

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

## Theorem (Mazur and Swinnerton-Dyer)

*E has good, ordinary reduction at p.* $\exists!\ g_E(T) \in \mathbb{Z}_p[[\Gamma]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *s. t.*

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IMC + (2) $\Rightarrow$ (1) for all possible quadratic $\chi$.

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For each $r$, $(f, f_r)$ satisfies all the conditions of our Theorem $\Rightarrow f$ and $f_r$ are $\pi$ Selmer companion for infinitely many $f_r$.

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$h \in \{f_1, f_2\}$. Let $\bar{\rho}_h : \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\frac{O_{K_{\mathfrak{p}}}}{\pi})$ be the residual representation of $\rho_h$.

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3. By Hida theory, there are infinitely many $f_r \in S_2(\Gamma_0(1246 \times 5^r), \psi_r)$ s.t. $(f, f_r)$ are $\pi$ Selmer companion.

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## References for computation:


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