# On sign changes of the Fourier coefficients of modular forms over number fields

Dr. Narasimha Kumar

Department of Mathematics
Indian Institute of Technology
Hyderabad

September 18, 2019

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

# Outline

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

2. Fourier coefficients of Hilbert Modular Forms
   - Introduction
   - Simultaneous non-vanishing
   - Simultaneous sign-changes at prime powers

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

**Newforms of integral weight:**

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

**Newforms of integral weight:**

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms.

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

**Newforms of integral weight:**

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

## Newforms of integral weight:

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)
- The central values of the $L$-functions of newforms

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

## Newforms of integral weight:

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)
- The central values of the $L$-functions of newforms
  - (Luo-Ramakrishnan) twisted by certain Dirichlet characters or

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

## Newforms of integral weight:

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)
- The central values of the $L$-functions of newforms
  - (Luo-Ramakrishnan) twisted by certain Dirichlet characters or
  - (Kowalski, –) by a family of modular forms

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

## Newforms of integral weight:

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)
- The central values of the $L$-functions of newforms
  - (Luo-Ramakrishnan) twisted by certain Dirichlet characters or
  - (Kowalski, –) by a family of modular forms

  determine the newform uniquely.

# Introduction

Determination of modular forms is one of the fundamental and an active area of research in Number theory.

- vanishing or non-vanishing (eg., Sturm bound)
- mod $p$ - analogues (eg., Hecke bound)

**Newforms of integral weight:**

- The Hecke eigenvalues of $T_p (p \in \mathbb{P})$ acting on the space of newforms. (multiplicity one theorem)
- The central values of the $L$-functions of newforms
  - (Luo-Ramakrishnan) twisted by certain Dirichlet characters or
  - (Kowalski, –) by a family of modular forms

  determine the newform uniquely.

Similar results exists for various other class of modular forms.

# half-integral weight modular forms

A multiplicity one theorem in terms of the Hecke eigenvalues of $T_{p^2}$ is known[1]

[1] Manickam, Ramakrishnan, and Vasudevan. On the theory of newforms of half-integral weight. J. Number Theory 34 (1990), no. 2, 210–224.

# half-integral weight modular forms

A multiplicity one theorem in terms of the Hecke eigenvalues of $T_{p^2}$ is known[1]

### Theorem (Kohnen/MRV)

*For* $i = 1, 2$, *let* $g_i \in S^{\text{new}}_{k+\frac{1}{2}}(4N_i)$ *be newforms* ($N_i$*'s are (odd)* $\square$*-free).*

[1] Manickam, Ramakrishnan, and Vasudevan. On the theory of newforms of half-integral weight. J. Number Theory 34 (1990), no. 2, 210–224.

A multiplicity one theorem in terms of the Hecke eigenvalues of $T_{p^2}$ is known[1]

**Theorem (Kohnen/MRV)**

*For $i = 1, 2$, let $g_i \in S^{\mathrm{new}}_{k+\frac{1}{2}}(4N_i)$ be newforms ($N_i$'s are (odd) $\square$-free). If $g_1$ and $g_2$ have the same eigenvalues with respect to infinitely many $T_{p^2}$ for $(p, N_1N_2) = 1$.*

[1] Manickam, Ramakrishnan, and Vasudevan. On the theory of newforms of half-integral weight. J. Number Theory 34 (1990), no. 2, 210–224.

A multiplicity one theorem in terms of the Hecke eigenvalues of $T_{p^2}$ is known[1]

**Theorem (Kohnen/MRV)**

*For $i = 1, 2$, let $g_i \in S^{\text{new}}_{k+\frac{1}{2}}(4N_i)$ be newforms ($N_i$'s are (odd) $\square$-free). If $g_1$ and $g_2$ have the same eigenvalues with respect to infinitely many $T_{p^2}$ for $(p, N_1N_2) = 1$. Then $N_1 = N_2$ and $g_1$ is a constant multiple of $g_2$.*

[1] Manickam, Ramakrishnan, and Vasudevan. On the theory of newforms of half-integral weight. J. Number Theory 34 (1990), no. 2, 210–224.

A multiplicity one theorem in terms of the Hecke eigenvalues of $T_{p^2}$ is known[1]

**Theorem (Kohnen/MRV)**

*For $i = 1, 2$, let $g_i \in S^{\mathrm{new}}_{k+\frac{1}{2}}(4N_i)$ be newforms ($N_i$'s are (odd) $\square$-free). If $g_1$ and $g_2$ have the same eigenvalues with respect to infinitely many $T_{p^2}$ for $(p, N_1N_2) = 1$. Then $N_1 = N_2$ and $g_1$ is a constant multiple of $g_2$.*

In particular, the multiplicity one theorem holds on $S^{\mathrm{new}}_{k+\frac{1}{2}}(4N)$.

[1] Manickam, Ramakrishnan, and Vasudevan. On the theory of newforms of half-integral weight. J. Number Theory 34 (1990), no. 2, 210–224.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

## Theorem (Kowalski, Lau, Soundararajan, and Wu)

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free.*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density 0, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime $p$,*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime $p$, except those in a set $E$ of analytic density $\kappa$ with $\kappa \leq 1/32$,*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime $p$, except those in a set $E$ of analytic density $\kappa$ with $\kappa \leq 1/32$, it follows that $f_1 = f_2$.*

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime $p$, except those in a set $E$ of analytic density $\kappa$ with $\kappa \leq 1/32$, it follows that $f_1 = f_2$.*

For any sequence of signs $(\epsilon_p)_{p \in \mathbf{P}}$, there is at most one pair $(k, N)$ and one primitive form $f \in S_k(N)$ such that $\lambda_f(p)$ has sign $\epsilon_p$ for all $p$.

[1] Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (integral weight case):

For primitive forms $f$ of integral weight[1]:

**Theorem (Kowalski, Lau, Soundararajan, and Wu)**

*For $i = 1, 2$, let $f_i$ be primitive forms of weight $2k_i$ and level $N_i$, resp.,*

1. *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all $p$ except those in a set of analytic density $0$, then $f_1 = f_2$.*
2. *Assume that neither of $f_1$ and $f_2$ is of CM type, for instance assume that $N_1$ and $N_2$ are $\square$-free. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime $p$, except those in a set $E$ of analytic density $\kappa$ with $\kappa \leq 1/32$, it follows that $f_1 = f_2$.*

For any sequence of signs $(\epsilon_p)_{p\in\mathbf{P}}$, there is at most one pair $(k, N)$ and one primitive form $f \in S_k(N)$ such that $\lambda_f(p)$ has sign $\epsilon_p$ for all $p$. Matomäki proved a refined version of the above result.

[1]Kowalski, Lau, Soundararajan, and Wu. On modular signs. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 389–411.

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd)* $\square$*-free positive integer* $N$,

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd) $\square$-free positive integer $N$, the spaces $S^{\text{new}}_{k+\frac{1}{2}}(4N)$*

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd) $\square$-free positive integer $N$, the spaces $S^{\text{new}}_{k+\frac{1}{2}}(4N)$ and $S^{\text{new}}_{2k}(N)$*

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd) $\square$-free positive integer $N$, the spaces $S^{\text{new}}_{k+\frac{1}{2}}(4N)$ and $S^{\text{new}}_{2k}(N)$ are Hecke equivariantly isomorphic via suitable linear combinations of the Shimura liftings.*

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd) $\square$-free positive integer $N$, the spaces $S^{\text{new}}_{k+\frac{1}{2}}(4N)$ and $S^{\text{new}}_{2k}(N)$ are Hecke equivariantly isomorphic via suitable linear combinations of the Shimura liftings.*

- Hecke eigenvalues ✓

# By signs (half-integral weight case):

As an immediate consequence of the above results, one can deduce similar results for half-integral weight modular forms,

**Theorem (Kohnen)**

*For any (odd) $\square$-free positive integer $N$, the spaces $S_{k+\frac{1}{2}}^{\text{new}}(4N)$ and $S_{2k}^{\text{new}}(N)$ are Hecke equivariantly isomorphic via suitable linear combinations of the Shimura liftings.*

- Hecke eigenvalues ✓
- Fourier coefficients ?

# Outline

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

**Definition**

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$*

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

## Definition

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$ if*

- $f(\gamma z) = j(\gamma, z)^{2k+1} \chi(d) f(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *where* $j(\gamma, z) := (\frac{c}{d}) \epsilon_d^{-1} \sqrt{cz + d}$
- $f$ *is holomorphic at cusps of* $\Gamma_0(4N)$.

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

**Definition**

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* modular form *of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$ if*

- $f(\gamma z) = j(\gamma, z)^{2k+1} \chi(d) f(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *where* $j(\gamma, z) := (\frac{c}{d}) \epsilon_d^{-1} \sqrt{cz + d}$
- $f$ *is holomorphic at cusps of* $\Gamma_0(4N)$.

holomorphic at $\infty \implies f(z) = \sum_{n=0}^{\infty} a_f(n) q^n$, where $q = e^{2\pi i z}$.

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

**Definition**

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* modular form *of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$ if*

- $f(\gamma z) = j(\gamma, z)^{2k+1}\chi(d)f(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *where* $j(\gamma, z) := (\frac{c}{d})\epsilon_d^{-1}\sqrt{cz + d}$
- $f$ *is holomorphic at cusps of* $\Gamma_0(4N)$.

$$\text{holomorphic at } \infty \implies f(z) = \sum_{n=0}^{\infty} a_f(n)q^n, \text{ where } q = e^{2\pi i z}.$$

Let $N$ be an odd and $\square$-free integer and let $\chi$ be a Dirichlet character modulo $4N$.

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

**Definition**

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* modular form *of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$ if*

- $f(\gamma z) = j(\gamma, z)^{2k+1} \chi(d) f(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *where* $j(\gamma, z) := \left(\frac{c}{d}\right) \epsilon_d^{-1} \sqrt{cz + d}$
- $f$ *is holomorphic at cusps of* $\Gamma_0(4N)$.

$$\text{holomorphic at } \infty \implies f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \text{ where } q = e^{2\pi i z}.$$

Let $N$ be an odd and $\square$-free integer and let $\chi$ be a Dirichlet character modulo $4N$. For $k \in \mathbb{N}$,

$$S_{k+\frac{1}{2}}(4N, \chi) = \text{half} - \text{integral weight modular forms.}$$

# half-integral weight modular forms:

For any $k \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $\chi$ be Dirichlet character modulo $4N$.

**Definition**

*A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k + \frac{1}{2}$, level $4N$ and character $\chi$ if*

- $f(\gamma z) = j(\gamma, z)^{2k+1}\chi(d) f(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *where* $j(\gamma, z) := (\frac{c}{d})\epsilon_d^{-1}\sqrt{cz + d}$
- $f$ *is holomorphic at cusps of* $\Gamma_0(4N)$.

$$\text{holomorphic at } \infty \implies f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \text{ where } q = e^{2\pi i z}.$$

Let $N$ be an odd and $\square$-free integer and let $\chi$ be a Dirichlet character modulo $4N$. For $k \in \mathbb{N}$,

$$S_{k+\frac{1}{2}}(4N, \chi) = \text{half} - \text{integral weight modular forms}.$$

(For $k = 1$, orthogonal complement of the subspace spanned by single-variable unary theta functions)

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) \neq 0$.

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}^{\text{new}}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$.

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n) q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n) q^n \in S_{2k}^{\text{new}}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k+\frac{1}{2}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

## Relation between the Fourier coefficients of $f$ and $F_t$:

$$A_t(n) = \sum_{d|n,(d,2N)=1} \chi(d)d^{k-1}a\left(\frac{tn^2}{d^2}\right),$$

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n) q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n) q^n \in S_{2k}^{\text{new}}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

**Relation between the Fourier coefficients of $f$ and $F_t$:**

$$A_t(n) = \sum_{d \mid n, (d, 2N) = 1} \chi(d) d^{k-1} a\left(\frac{tn^2}{d^2}\right),$$

where $\chi$ is a quadratic character.

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k+\frac{1}{2}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

**Relation between the Fourier coefficients of $f$ and $F_t$:**

$$A_t(n) = \sum_{d|n,(d,2N)=1} \chi(d)d^{k-1}a\left(\frac{tn^2}{d^2}\right),$$

where $\chi$ is a quadratic character. For $p \in \mathbb{P}$ with $(p, 2N) = 1$,

$$a(tp^2) = A_t(p) - \chi(p)p^{k-1}.$$

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}^{\text{new}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}^{\text{new}}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

**Relation between the Fourier coefficients of $f$ and $F_t$:**

$$A_t(n) = \sum_{d|n,(d,2N)=1} \chi(d)d^{k-1}a\left(\frac{tn^2}{d^2}\right),$$

where $\chi$ is a quadratic character. For $p \in \mathbb{P}$ with $(p, 2N) = 1$,

$$a(tp^2) = A_t(p) - \chi(p)p^{k-1}.$$

Since $F_t$ is a primitive form with trivial nebentypus $\Longrightarrow$ the Fourier coefficients $a(tp^2) \in \mathbb{R}$,

# Shimura lift:

Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k+\frac{1}{2}}(4N, \chi_{\text{triv}})$ be a half-integral weight cuspidal eigenform with $a(t) = 1$.
Let $\text{Sh}_t(f) = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k}(2N, \chi_{\text{triv}})$ denote the Shimura lift corresponding to $f$ and $t$. We denote $F_t = \text{Sh}_t(f)$.

**Relation between the Fourier coefficients of $f$ and $F_t$:**

$$A_t(n) = \sum_{d|n,(d,2N)=1} \chi(d)d^{k-1}a\left(\frac{tn^2}{d^2}\right),$$

where $\chi$ is a quadratic character. For $p \in \mathbb{P}$ with $(p, 2N) = 1$,

$$a(tp^2) = A_t(p) - \chi(p)p^{k-1}.$$

Since $F_t$ is a primitive form with trivial nebentypus $\Longrightarrow$ the Fourier coefficients $a(tp^2) \in \mathbb{R}$, hence we can talk about signs and sign changes.

# Definitions:

## Sato-Tate measure:

# Definitions:

## Sato-Tate measure:

The Sato-Tate measure $\mu_{\mathrm{ST}}$ is the probability measure on $[-1, 1]$ given by $\frac{2}{\pi}\sqrt{1-t^2}dt$.

# Definitions:

## Sato-Tate measure:

The Sato-Tate measure $\mu_{\text{ST}}$ is the probability measure on $[-1, 1]$ given by $\frac{2}{\pi}\sqrt{1-t^2}dt$.

## Density:

# Definitions:

## Sato-Tate measure:

The Sato-Tate measure $\mu_{\mathrm{ST}}$ is the probability measure on $[-1, 1]$ given by $\frac{2}{\pi}\sqrt{1-t^2}dt$.

## Density:

Let $S$ be a subset of $\mathbb{P}$. The set $S$ has a natural density $d(S)$ (resp., analytic density $d_{\mathrm{an}}(S)$),

# Definitions:

## Sato-Tate measure:

The Sato-Tate measure $\mu_{\mathrm{ST}}$ is the probability measure on $[-1,1]$ given by $\frac{2}{\pi}\sqrt{1-t^2}dt$.

## Density:

Let $S$ be a subset of $\mathbb{P}$. The set $S$ has a natural density $d(S)$ (resp., analytic density $d_{\mathrm{an}}(S)$), if the limit

$$\lim_{x\to\infty} \frac{\#\{p \leq x : p \in S\}}{\pi(x)} \quad \left(\text{resp., } \lim_{s\to 1^+} \frac{\sum_{p\in S} \frac{1}{p^s}}{\log(\frac{1}{s-1})}\right)$$

exists

# Definitions:

## Sato-Tate measure:

The Sato-Tate measure $\mu_{\mathrm{ST}}$ is the probability measure on $[-1, 1]$ given by $\frac{2}{\pi}\sqrt{1-t^2}dt$.

## Density:

Let $S$ be a subset of $\mathbb{P}$. The set $S$ has a natural density $d(S)$ (resp., analytic density $d_{\mathrm{an}}(S)$), if the limit

$$\lim_{x\to\infty} \frac{\#\{p \leq x : p \in S\}}{\pi(x)} \quad \left(\text{resp., } \lim_{s\to 1^+} \frac{\sum_{p\in S} \frac{1}{p^s}}{\log(\frac{1}{s-1})}\right)$$

exists and is equal to $d(S)$ (resp., is equal to $d_{\mathrm{an}}(S)$), where $\pi(x) := \#\{p \leq x : p \in \mathbb{P}\}$.

# Matomäki's theorem (integral weight):

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp.,

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp.,

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki.

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222.

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

## Theorem (Matomäki)

*Suppose $F, G$ are primitive forms without CM.*

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

## Theorem (Matomäki)

*Suppose $F, G$ are primitive forms without CM. If $\lambda_F(p)$ and $\lambda_G(p)$ have same sign for every prime $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$*

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

**Theorem (Matomäki)**

*Suppose $F, G$ are primitive forms without CM. If $\lambda_F(p)$ and $\lambda_G(p)$ have same sign for every prime $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$ then $N_1 = N_2$, $k_1 = k_2$, and $F = G$.*

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

**Theorem (Matomäki)**

*Suppose $F, G$ are primitive forms without CM. If $\lambda_F(p)$ and $\lambda_G(p)$ have same sign for every prime $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$ then $N_1 = N_2$, $k_1 = k_2$, and $F = G$.*

We show that the Fourier coefficients on certain sub-families,

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

**Theorem (Matomäki)**

*Suppose $F, G$ are primitive forms without CM. If $\lambda_F(p)$ and $\lambda_G(p)$ have same sign for every prime $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$ then $N_1 = N_2$, $k_1 = k_2$, and $F = G$.*

We show that the Fourier coefficients on certain sub-families, which are accessible via the Shimura lift,

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Matomäki's theorem (integral weight):

Let $F, G$ be two primitive forms of (integral) weight $2k_1, 2k_2$, levels $N_1, N_2$ and with trivial nebentypus, resp., Let $\lambda_F(n), \lambda_G(n)$ denote the eigenvalue of $T_n (n \in \mathbb{N})$ for $F, G$, resp., Now, we state a Theorem of Matomäki[1].

**Theorem (Matomäki)**

*Suppose $F, G$ are primitive forms without CM. If $\lambda_F(p)$ and $\lambda_G(p)$ have same sign for every prime $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$ then $N_1 = N_2$, $k_1 = k_2$, and $F = G$.*

We show that the Fourier coefficients on certain sub-families, which are accessible via the Shimura lift, shall determine the half-integral weight modular form uniquely.

[1] On signs of Fourier coefficients of cusp forms. Mathematical Proceedings of the Cambridge Philosophical Society 152 (2012), no. 02, 207–222

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

2. Fourier coefficients of Hilbert Modular Forms
   - Introduction
   - Simultaneous non-vanishing
   - Simultaneous sign-changes at prime powers

# Hypothesis:

$$f = \sum_{n=1}^{\infty} a(n) q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}}) \underset{\text{Sh}_t(f)}{\rightsquigarrow} F_t = \sum_{n=1}^{\infty} A_t(n) q^n \in S^{\text{new}}_{2k_1}(2N_1)$$

# Hypothesis:

$$f=\sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1,\chi_{\text{triv}}) \underset{\text{Sh}_t(f)}{\rightsquigarrow} F_t=\sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k_1}(2N_1)$$

$$g=\sum_{n=1}^{\infty} b(n)q^n \in S^{\text{new}}_{k_2+\frac{1}{2}}(4N_2,\chi_{\text{triv}}) \underset{\text{Sh}_t(g)}{\rightsquigarrow} G_t=\sum_{n=1}^{\infty} B_t(n)q^n \in S^{\text{new}}_{2k_2}(2N_2),$$

# Hypothesis:

$$f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\mathrm{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\mathrm{triv}}) \underset{\mathrm{Sh}_t(f)}{\rightsquigarrow} F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\mathrm{new}}_{2k_1}(2N_1)$$

$$g = \sum_{n=1}^{\infty} b(n)q^n \in S^{\mathrm{new}}_{k_2+\frac{1}{2}}(4N_2, \chi_{\mathrm{triv}}) \underset{\mathrm{Sh}_t(g)}{\rightsquigarrow} G_t = \sum_{n=1}^{\infty} B_t(n)q^n \in S^{\mathrm{new}}_{2k_2}(2N_2),$$

where $f$ (resp., $g$) is a non-zero cuspidal eigenforms for operators $T_{p^2}$ for primes $p \nmid 2N_1$ (resp., $p \nmid 2N_2$), where $N_1, N_2$ are odd and $\square$-free integers.

# Hypothesis:

$$f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}}) \underset{\text{Sh}_t(f)}{\rightsquigarrow} F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k_1}(2N_1)$$

$$g = \sum_{n=1}^{\infty} b(n)q^n \in S^{\text{new}}_{k_2+\frac{1}{2}}(4N_2, \chi_{\text{triv}}) \underset{\text{Sh}_t(g)}{\rightsquigarrow} G_t = \sum_{n=1}^{\infty} B_t(n)q^n \in S^{\text{new}}_{2k_2}(2N_2),$$

where $f$ (resp., $g$) is a non-zero cuspidal eigenforms for operators $T_{p^2}$ for primes $p \nmid 2N_1$ (resp., $p \nmid 2N_2$), where $N_1, N_2$ are odd and $\square$-free integers.

Suppose there exists a $\square$-free integer $t \geq 1$ such that $a(t)b(t) \neq 0$.

# Hypothesis:

$$f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}}) \underset{\text{Sh}_t(f)}{\rightsquigarrow} F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k_1}(2N_1)$$

$$g = \sum_{n=1}^{\infty} b(n)q^n \in S^{\text{new}}_{k_2+\frac{1}{2}}(4N_2, \chi_{\text{triv}}) \underset{\text{Sh}_t(g)}{\rightsquigarrow} G_t = \sum_{n=1}^{\infty} B_t(n)q^n \in S^{\text{new}}_{2k_2}(2N_2),$$

where $f$ (resp., $g$) is a non-zero cuspidal eigenforms for operators $T_{p^2}$ for primes $p \nmid 2N_1$ (resp., $p \nmid 2N_2$), where $N_1, N_2$ are odd and $\square$-free integers.

Suppose there exists a $\square$-free integer $t \geq 1$ such that $a(t) = b(t) = 1$.

# Hypothesis:

$$f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}}) \underset{\text{Sh}_t(f)}{\rightsquigarrow} F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S^{\text{new}}_{2k_1}(2N_1)$$

$$g = \sum_{n=1}^{\infty} b(n)q^n \in S^{\text{new}}_{k_2+\frac{1}{2}}(4N_2, \chi_{\text{triv}}) \underset{\text{Sh}_t(g)}{\rightsquigarrow} G_t = \sum_{n=1}^{\infty} B_t(n)q^n \in S^{\text{new}}_{2k_2}(2N_2),$$

where $f$ (resp., $g$) is a non-zero cuspidal eigenforms for operators $T_{p^2}$ for primes $p \nmid 2N_1$ (resp., $p \nmid 2N_2$), where $N_1, N_2$ are odd and $\square$-free integers.
Suppose there exists a $\square$-free integer $t \geq 1$ such that $a(t) = b(t) = 1$.
Then, $F_t$ (resp., $G_t$) is a cuspidal eigenform for operators $T_p$ for primes $p \nmid 2N_1$ (resp., $p \nmid 2N_2$).

# Main result[1]

**Theorem (K.)**

[1]Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Main result[1]

## Theorem (K.)

*Let $f, g$ be two half-integral weight eigenforms.*

[1] Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Main result[1]

## Theorem (K.)

*Let $f, g$ be two half-integral weight eigenforms. If $a(tp^2)$ and $b(tp^2)$ have the same sign for every $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$,*

[1]Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Main result[1]

## Theorem (K.)

*Let $f, g$ be two half-integral weight eigenforms. If $a(tp^2)$ and $b(tp^2)$ have the same sign for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$, then $N_1 = N_2$, $k_1 = k_2$, and $f = g$, up to a positive scalar multiple.*

[1]Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Main result[1]

**Theorem (K.)**

*Let $f, g$ be two half-integral weight eigenforms. If $a(tp^2)$ and $b(tp^2)$ have the same sign for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$, then $N_1 = N_2$, $k_1 = k_2$, and $f = g$, up to a positive scalar multiple.*

For any prime $p$, let

$$C_{F_t}(p) := \frac{A_t(p)}{2p^{k_1 - \frac{1}{2}}}, D_{G_t}(p) := \frac{B_t(p)}{2p^{k_2 - \frac{1}{2}}}.$$

[1] Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Main result[1]

**Theorem (K.)**

*Let $f, g$ be two half-integral weight eigenforms. If $a(tp^2)$ and $b(tp^2)$ have the same sign for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$, then $N_1 = N_2$, $k_1 = k_2$, and $f = g$, up to a positive scalar multiple.*

For any prime $p$, let

$$C_{F_t}(p) := \frac{A_t(p)}{2p^{k_1 - \frac{1}{2}}}, D_{G_t}(p) := \frac{B_t(p)}{2p^{k_2 - \frac{1}{2}}}.$$

By the key relation, we have that

$$a(tp^2) < 0 \Longleftrightarrow C_{F_t}(p) < \frac{\chi_1(p)}{2\sqrt{p}}.$$

[1]Kumar, Narasimha. A variant of multiplicity one theorems for half-integral weight modular forms, Acta Arith. 190 (2019), no. 1, 7585.

# Key proposition

**Proposition (K.)**

*Let* $f = \sum_{n=1}^{\infty} a(n) q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}})$ *be as before.*

# Key proposition

## Proposition (K.)

*Let* $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k_1+\frac{1}{2}}^{\text{new}}(4N_1, \chi_{\text{triv}})$ *be as before.*

$$d\left(\left\{p \in \mathbb{P} | a(tp^2) C_{F_t}(p) < 0\right\}\right) = 0.$$

# Key proposition

## Proposition (K.)

*Let* $f = \sum_{n=1}^{\infty} a(n) q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}})$ *be as before.*

$$d\left(\{p \in \mathbb{P} | a(tp^2) C_{F_t}(p) < 0\}\right) = 0.$$

## Proof.

Observe that $F_t$ is without complex multiplication, since the level is $\square$-free.

# Key proposition

## Proposition (K.)

*Let* $f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}})$ *be as before.*

$$d\left(\{p \in \mathbb{P} | a(tp^2)C_{F_t}(p) < 0\}\right) = 0.$$

## Proof.

Observe that $F_t$ is without complex multiplication, since the level is $\square$-free. Analytic argument + Sato-Tate equi-distribution theorem for $F_t$.

# Key proposition

## Proposition (K.)

*Let* $f = \sum_{n=1}^{\infty} a(n) q^n \in S^{\text{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\text{triv}})$ *be as before.*

$$d\left(\{p \in \mathbb{P} | a(tp^2) C_{F_t}(p) < 0\}\right) = 0.$$

## Proof.

Observe that $F_t$ is without complex multiplication, since the level is □-free. Analytic argument + Sato-Tate equi-distribution theorem for $F_t$. □

By Proposition,

$$\text{sign}(a(tp^2)) = \text{sign}(C_{F_t}(p))$$

except on a set $E_f \subseteq \mathbb{P}$ with $d(E_f) = 0$.

# Key proposition

**Proposition (K.)**

*Let* $f = \sum_{n=1}^{\infty} a(n)q^n \in S^{\mathrm{new}}_{k_1+\frac{1}{2}}(4N_1, \chi_{\mathrm{triv}})$ *be as before.*

$$d\left(\{p \in \mathbb{P} | a(tp^2)C_{F_t}(p) < 0\}\right) = 0.$$

**Proof.**

Observe that $F_t$ is without complex multiplication, since the level is $\square$-free. Analytic argument + Sato-Tate equi-distribution theorem for $F_t$. □

By Proposition,

$$\mathrm{sign}(a(tp^2)) = \mathrm{sign}(C_{F_t}(p))$$

except on a set $E_f \subseteq \mathbb{P}$ with $d(E_f) = 0$. Similarly, for the eigenform $g$ and denote the set by $E_g$.

# Continued...

By Hypothesis, we have $\mathrm{sign}(a(tp^2)) = \mathrm{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$.

# Continued...

By Hypothesis, we have $\operatorname{sign}(a(tp^2)) = \operatorname{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$. Take $E = E_0 \cup E_f \cup E_g \subseteq \mathbb{P}$.

# Continued...

By Hypothesis, we have $\text{sign}(a(tp^2)) = \text{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$. Take $E = E_0 \cup E_f \cup E_g \subseteq \mathbb{P}$. This implies

$$\text{sign}(C_{F_t}(p)) = \text{sign}(D_{G_t}(p))$$

for every prime $p \notin E$ with $d_{\text{an}}(E) \leq 6/25$.

# Continued...

By Hypothesis, we have $\text{sign}(a(tp^2)) = \text{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\text{an}}(E_0) \leq 6/25$. Take $E = E_0 \cup E_f \cup E_g \subseteq \mathbb{P}$. This implies

$$\text{sign}(C_{F_t}(p)) = \text{sign}(D_{G_t}(p))$$

for every prime $p \notin E$ with $d_{\text{an}}(E) \leq 6/25$. By Matomäki's Theorem,

$$k_1 = k_2, N_1 = N_2, \& \ F_t = G_t.$$

# Continued...

By Hypothesis, we have $\operatorname{sign}(a(tp^2)) = \operatorname{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \leq 6/25$. Take $E = E_0 \cup E_f \cup E_g \subseteq \mathbb{P}$. This implies

$$\operatorname{sign}(C_{F_t}(p)) = \operatorname{sign}(D_{G_t}(p))$$

for every prime $p \notin E$ with $d_{\mathrm{an}}(E) \leq 6/25$. By Matomäki's Theorem,

$$k_1 = k_2, N_1 = N_2, \&\ F_t = G_t.$$

Since Shimura lift commutes with Hecke operators,

$$\text{for} \quad p \nmid 2N_1N_2, \quad T_pF_t = \lambda_p(f)F_t,$$

where $\lambda_p(f)$ is the $T_{p^2}$-eigenvalue of $f$.

# Continued...

By Hypothesis, we have $\mathrm{sign}(a(tp^2)) = \mathrm{sign}(b(tp^2))$ for every $p \notin E_0$ with $d_{\mathrm{an}}(E_0) \le 6/25$. Take $E = E_0 \cup E_f \cup E_g \subseteq \mathbb{P}$. This implies

$$\mathrm{sign}(C_{F_t}(p)) = \mathrm{sign}(D_{G_t}(p))$$

for every prime $p \notin E$ with $d_{\mathrm{an}}(E) \le 6/25$. By Matomäki's Theorem,

$$k_1 = k_2, N_1 = N_2, \,\&\, F_t = G_t.$$

Since Shimura lift commutes with Hecke operators,

$$\text{for} \quad p \nmid 2N_1N_2, \quad T_pF_t = \lambda_p(f)F_t,$$

where $\lambda_p(f)$ is the $T_{p^2}$-eigenvalue of $f$. By the multiplicity one theorem of MRV, we get that $f = g$ (up to a scalar).

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

2. Fourier coefficients of Hilbert Modular Forms
   - Introduction
   - Simultaneous non-vanishing
   - Simultaneous sign-changes at prime powers

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

2. Fourier coefficients of Hilbert Modular Forms
   - Introduction
   - Simultaneous non-vanishing
   - Simultaneous sign-changes at prime powers

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.
- $\psi$ be a Hecke character of $\mathbb{A}_F^{\times}$.

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.
- $\psi$ be a Hecke character of $\mathbb{A}_F^\times$.
- For an ideal $\mathfrak{b} \subseteq \mathcal{O}_F$, $M_k(\mathfrak{b}, \psi)$, $S_k(\mathfrak{b}, \psi)$

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.
- $\psi$ be a Hecke character of $\mathbb{A}_F^\times$.
- For an ideal $\mathfrak{b} \subseteq \mathcal{O}_F$, $M_k(\mathfrak{b}, \psi)$, $S_k(\mathfrak{b}, \psi)$
- For any $\mathbf{f} \in M_k(\mathfrak{b}, \psi)$,

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.
- $\psi$ be a Hecke character of $\mathbb{A}_F^\times$.
- For an ideal $\mathfrak{b} \subseteq \mathcal{O}_F$, $M_k(\mathfrak{b}, \psi)$, $S_k(\mathfrak{b}, \psi)$
- For any $\mathbf{f} \in M_k(\mathfrak{b}, \psi)$, $f_\nu(z) = \sum_{\xi \in F} a_{f_\nu}(\xi) \exp(2\pi i \xi z)$, where $\xi$ runs over all the totally positive elements in $t_\nu^{-1}\mathcal{O}_F$ and $\xi = 0$.

# Hilbert modular forms

- Let $F$ be a totally real number field with $\mathcal{O}_F$ as the ring of integers, $\mathbb{A}_F$ denote the adele ring of $F$, and $h$ is the narrow class number of $F$.
- $\psi$ be a Hecke character of $\mathbb{A}_F^\times$.
- For an ideal $\mathfrak{b} \subseteq \mathcal{O}_F$, $M_k(\mathfrak{b}, \psi)$, $S_k(\mathfrak{b}, \psi)$
- For any $\mathbf{f} \in M_k(\mathfrak{b}, \psi)$, $f_\nu(z) = \sum_{\xi \in F} a_{f_\nu}(\xi) \exp(2\pi i \xi z)$, where $\xi$ runs over all the totally positive elements in $t_\nu^{-1}\mathcal{O}_F$ and $\xi = 0$.
- We define the Fourier coefficients of $\mathbf{f}$ at ideal $\mathfrak{m}$ of $F$ as

$$C(\mathfrak{m}, \mathbf{f}) := \begin{cases} N(\mathfrak{m})^{\frac{k_0}{2}} a_\nu(\xi) \xi^{-(k+i\mu)/2} & \text{if} \quad \mathfrak{m} = \xi t_\nu^{-1}\mathcal{O}_F \subset \mathcal{O}_F, \xi \gg 0 \\ 0 & \text{if} \quad \mathfrak{m} \text{ is not integral} \end{cases}$$

where $k_0 = \max\{k_1, \ldots, k_n\}$.

# Primitive forms

- Let $\mathbf{f} \in S_k(\mathfrak{b}, \psi)^{\text{new}}$, then $\mathbf{f}$ is said to be a primitive form if $T_{\mathfrak{m}}\mathbf{f} = C(\mathfrak{m}, \mathbf{f})\mathbf{f}, \forall \mathfrak{m} \subseteq \mathcal{O}_F$ with $C(\mathcal{O}_F, \mathbf{f}) = 1$.

# Primitive forms

- Let $\mathbf{f} \in S_k(\mathfrak{b}, \psi)^{\text{new}}$, then $\mathbf{f}$ is said to be a primitive form if $T_{\mathfrak{m}}\mathbf{f} = C(\mathfrak{m}, \mathbf{f})\mathbf{f}, \forall \mathfrak{m} \subseteq \mathcal{O}_F$ with $C(\mathcal{O}_F, \mathbf{f}) = 1$.

## Notation

- $F$ to denote a totally real number field.

# Primitive forms

- Let $\mathbf{f} \in S_k(\mathfrak{b}, \psi)^{\text{new}}$, then $\mathbf{f}$ is said to be a primitive form if $T_{\mathfrak{m}}\mathbf{f} = C(\mathfrak{m}, \mathbf{f})\mathbf{f}, \forall \mathfrak{m} \subseteq \mathcal{O}_F$ with $C(\mathcal{O}_F, \mathbf{f}) = 1$.

## Notation

- $F$ to denote a totally real number field.
- $K$ denote a totally real number field of odd degree.

# Primitive forms

- Let $\mathbf{f} \in S_k(\mathfrak{b}, \psi)^{\text{new}}$, then $\mathbf{f}$ is said to be a primitive form if $T_{\mathfrak{m}}\mathbf{f} = C(\mathfrak{m}, \mathbf{f})\mathbf{f}, \forall \mathfrak{m} \subseteq \mathcal{O}_F$ with $C(\mathcal{O}_F, \mathbf{f}) = 1$.

## Notation

- $F$ to denote a totally real number field.
- $K$ denote a totally real number field of odd degree.
- $\mathbf{P}$ denote the set of all prime ideals of $\mathcal{O}_F$ (resp., $\mathcal{O}_K$).

# Primitive forms

- Let $\mathbf{f} \in S_k(\mathfrak{b}, \psi)^{\text{new}}$, then $\mathbf{f}$ is said to be a primitive form if $T_{\mathfrak{m}}\mathbf{f} = C(\mathfrak{m}, \mathbf{f})\mathbf{f}, \forall \mathfrak{m} \subseteq \mathcal{O}_F$ with $C(\mathcal{O}_F, \mathbf{f}) = 1$.

## Notation

- $F$ to denote a totally real number field.
- $K$ denote a totally real number field of odd degree.
- $\mathbf{P}$ denote the set of all prime ideals of $\mathcal{O}_F$ (resp., $\mathcal{O}_K$).
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_F$ (resp., $\mathcal{O}_K$) with odd inertia degree.

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$.

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$.

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

## Theorem (Barnet-Lamb, Gee, and Geraghty)

*Let* $\mathbf{f}$ *be a non-CM form as above.*

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

**Theorem (Barnet-Lamb, Gee, and Geraghty)**

*Let* $\mathbf{f}$ *be a non-CM form as above. Then* $\{\theta_{\mathfrak{p}}(\mathbf{f})\}_{\mathfrak{p}\in\mathbf{P},\mathfrak{p}\nmid\mathfrak{c}\mathfrak{D}_F}$ *is equi-distributed in* $[0, \pi]$

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

**Theorem (Barnet-Lamb, Gee, and Geraghty)**

*Let $\mathbf{f}$ be a non-CM form as above. Then $\{\theta_{\mathfrak{p}}(\mathbf{f})\}_{\mathfrak{p}\in\mathbf{P},\mathfrak{p}\nmid\mathfrak{c}\mathfrak{D}_F}$ is equi-distributed in $[0, \pi]$ with respect to $\mu_{\mathrm{ST}} = \frac{2}{\pi}\sin^2\theta d\theta$.*

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

## Theorem (Barnet-Lamb, Gee, and Geraghty)

*Let* $\mathbf{f}$ *be a non-CM form as above. Then* $\{\theta_{\mathfrak{p}}(\mathbf{f})\}_{\mathfrak{p}\in\mathbf{P}, \mathfrak{p}\nmid\mathfrak{c}\mathfrak{D}_F}$ *is equi-distributed in* $[0, \pi]$ *with respect to* $\mu_{\mathrm{ST}} = \frac{2}{\pi}\sin^2\theta d\theta$*. In other words, for any sub-interval* $I \subseteq [0, \pi]$*, we have*

$$\lim_{x\to\infty} \frac{\#\{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, N(\mathfrak{p}) \le x, \theta_{\mathfrak{p}}(\mathbf{f}) \in I\}}{\#\{\mathfrak{p} \in \mathbf{P} \mid N(\mathfrak{p}) \le x\}} = \mu_{\mathrm{ST}}(I) = \frac{2}{\pi}\int_I \sin^2\theta d\theta \tag{2.2}$$

# Sato-Tate equi-distribution Theorem

Let $\mathbf{f}$ be a primitive form over $F$ of level $\mathfrak{c}$, with trivial character and weight $2k$. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define $\beta(\mathfrak{a}, \mathbf{f}) := \frac{C(\mathfrak{a},\mathbf{f})}{N(\mathfrak{a})^{\frac{2k_0-1}{2}}}$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, we have

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\theta_{\mathfrak{p}}(\mathbf{f}), \tag{2.1}$$

for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.

**Theorem (Barnet-Lamb, Gee, and Geraghty)**

*Let $\mathbf{f}$ be a non-CM form as above. Then $\{\theta_{\mathfrak{p}}(\mathbf{f})\}_{\mathfrak{p}\in\mathbf{P},\mathfrak{p}\nmid\mathfrak{c}\mathfrak{D}_F}$ is equi-distributed in $[0, \pi]$ with respect to $\mu_{\mathrm{ST}} = \frac{2}{\pi}\sin^2\theta d\theta$. In other words, for any sub-interval $I \subseteq [0, \pi]$, we have*

$$\lim_{x\to\infty} \frac{\#\{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, N(\mathfrak{p}) \le x, \theta_{\mathfrak{p}}(\mathbf{f}) \in I\}}{\#\{\mathfrak{p} \in \mathbf{P} \mid N(\mathfrak{p}) \le x\}} = \mu_{\mathrm{ST}}(I) = \frac{2}{\pi}\int_I \sin^2\theta d\theta \tag{2.2}$$

*i.e., the natural density of $S = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, \theta_{\mathfrak{p}}(\mathbf{f}) \in I\}$ is $\mu_{\mathrm{ST}}(I)$.*

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp.,

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

## Conjecture

*Let $g_1, g_2$ be distinct non-CM primitive forms, as above, such that they are not twists of each other.*

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

## Conjecture

*Let $g_1, g_2$ be distinct non-CM primitive forms, as above, such that they are not twists of each other. For any two sub-intervals $I_1 \subseteq [-1, 1], I_2 \subseteq [-1, 1]$,*

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

## Conjecture

*Let $g_1, g_2$ be distinct non-CM primitive forms, as above, such that they are not twists of each other. For any two sub-intervals* $I_1 \subseteq [-1, 1], I_2 \subseteq [-1, 1]$,

$$d(S(I_1, I_2)) = \lim_{x \to \infty} \frac{\#S(I_1, I_2)(x)}{\pi(x)} = \mu_{\mathrm{ST}}(I_1)\mu_{\mathrm{ST}}(I_2),$$

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n)q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

## Conjecture

*Let $g_1, g_2$ be distinct non-CM primitive forms, as above, such that they are not twists of each other. For any two sub-intervals $I_1 \subseteq [-1, 1], I_2 \subseteq [-1, 1]$,*

$$d(S(I_1, I_2)) = \lim_{x \to \infty} \frac{\#S(I_1, I_2)(x)}{\pi(x)} = \mu_{\mathrm{ST}}(I_1)\mu_{\mathrm{ST}}(I_2),$$

*where* $S(I_1, I_2) = \{p \in \mathbb{P} : p \nmid N_1N_2, B_1(p) \in I_1, B_2(p) \in I_2\}$.

# Joint Sato-Tate conjecture (Classical Case)

For $i = 1, 2$, let $g_i = \sum_{n=1}^{\infty} b_i(n) q^n$ be primitive eigenforms of weight $2k_i$ and level $N_i$, resp., We let $B_i(p) := \frac{b_i(p)}{2p^{k_i - \frac{1}{2}}} \in [-1, 1]$.

**Conjecture**

*Let $g_1, g_2$ be distinct non-CM primitive forms, as above, such that they are not twists of each other. For any two sub-intervals $I_1 \subseteq [-1, 1], I_2 \subseteq [-1, 1]$,*

$$d(S(I_1, I_2)) = \lim_{x \to \infty} \frac{\# S(I_1, I_2)(x)}{\pi(x)} = \mu_{\mathrm{ST}}(I_1) \mu_{\mathrm{ST}}(I_2),$$

*where $S(I_1, I_2) = \{p \in \mathbb{P} : p \nmid N_1 N_2, B_1(p) \in I_1, B_2(p) \in I_2\}$.*

In other words, Fourier coefficients at primes are independently distributed with respect to the Sato-Tate distribution.

# Specific implications

- Take $I_1 = I_2 = (0, 1]$

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)
- For specific choice of intervals, we can also get information of

## Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)
- For specific choice of intervals, we can also get information of
  - Simultaneous non-vanishing at prime powers

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)
- For specific choice of intervals, we can also get information of
  - Simultaneous non-vanishing at prime powers
  - Simultaneous sign changes at prime powers

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)
- For specific choice of intervals, we can also get information of
  - Simultaneous non-vanishing at prime powers
  - Simultaneous sign changes at prime powers

  and their densities.

# Specific implications

- Take $I_1 = I_2 = (0, 1]$
  - Simultaneous non-vanishing at primes,
  - Simultaneous sign changes at primes

  and their densities.
- Take $I_1 = (0, 1]$ and $I_2 = [-1, 0)$ (Different Sign)
- For specific choice of intervals, we can also get information of
  - Simultaneous non-vanishing at prime powers
  - Simultaneous sign changes at prime powers

  and their densities.

What about similar formulation in the Hilbert modular case?

# Outline

1. Multiplicity one theorems
   - Literature
   - Half-integral weight modular forms
   - Results

2. Fourier coefficients of Hilbert Modular Forms
   - Introduction
   - Simultaneous non-vanishing
   - Simultaneous sign-changes at prime powers

**Theorem (Gun, Kumar, Paul)**

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n, g(\tau) = \sum_{n=1}^{\infty} a_g(n) q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weight* $k_1, k_2$*, respectively.*

# Modular forms over $\mathbb{Q}$

**Theorem (Gun, Kumar, Paul)**

*Let $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n, g(\tau) = \sum_{n=1}^{\infty} a_g(n)q^n$ be two distinct primitive forms of level $N_1, N_2$ and weight $k_1, k_2$, respectively. Then, for all $p \in \mathbb{P}$ with $(p, N_1N_2) = 1$,*

# Modular forms over $\mathbb{Q}$

**Theorem (Gun, Kumar, Paul)**

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n, g(\tau) = \sum_{n=1}^{\infty} a_g(n) q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weight* $k_1, k_2$*, respectively. Then, for all* $p \in \mathbb{P}$ *with* $(p, N_1 N_2) = 1$, $d(\{m \in \mathbb{N} \mid a_f(p^m) a_g(p^m) \neq 0\}) > 0$.

# Modular forms over $\mathbb{Q}$

**Theorem (Gun, Kumar, Paul)**

*Let $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n, g(\tau) = \sum_{n=1}^{\infty} a_g(n)q^n$ be two distinct primitive forms of level $N_1, N_2$ and weight $k_1, k_2$, respectively. Then, for all $p \in \mathbb{P}$ with $(p, N_1N_2) = 1$, $d(\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\}) > 0$.*

---

[a] Gun, Sanoli; Kumar, Balesh; Paul, Biplab. The First Simultaneous sign change and non-vanishing of Hecke Eigenvalues of newforms, J. Number Theory 200 (2019), 161–184.

# For Hilbert Modular forms

# For Hilbert Modular forms

**Theorem (Kaushik, -)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct* ***non-CM*** *primitive forms over* $F$ *with trivial character and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with weights* $2k_1, 2k_2$ *and Fourier coefficients* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$*, respectively.*

# For Hilbert Modular forms

**Theorem (Kaushik, -)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct* ***non-CM*** *primitive forms over* $F$ *with trivial character and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with weights* $2k_1, 2k_2$ *and Fourier coefficients* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$, *respectively. Then for any prime ideal* $\mathfrak{p}$ *such that* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$,

# For Hilbert Modular forms

## Theorem (Kaushik, -)

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct* ***non-CM*** *primitive forms over* $F$ *with trivial character and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with weights* $2k_1, 2k_2$ *and Fourier coefficients* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$*, respectively. Then for any prime ideal* $\mathfrak{p}$ *such that* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, $d(\{m \in \mathbb{N} \mid C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) > 0.$

[b] Kaushik, Surjeet; Kumar, Narasimha. Simultaneous behaviour of the Fourier coefficients of two Hilbert modular cusp forms, Archiv der Mathematik 112 (2019), no. 3, 241-248.

# For Hilbert Modular forms

**Theorem (Kaushik, -)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct* ***non-CM*** *primitive forms over* $F$ *with trivial character and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with weights* $2k_1, 2k_2$ *and Fourier coefficients* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$*, respectively. Then for any prime ideal* $\mathfrak{p}$ *such that* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, $d(\{m \in \mathbb{N} \mid C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) > 0$.

---

[b] Kaushik, Surjeet; Kumar, Narasimha. Simultaneous behaviour of the Fourier coefficients of two Hilbert modular cusp forms, Archiv der Mathematik 112 (2019), no. 3, 241-248.

Methodology of this proof is similar to the classical case,

# For Hilbert Modular forms

**Theorem (Kaushik, -)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct* ***non-CM*** *primitive forms over* $F$ *with trivial character and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with weights* $2k_1, 2k_2$ *and Fourier coefficients* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$*, respectively. Then for any prime ideal* $\mathfrak{p}$ *such that* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, $d(\{m \in \mathbb{N} \mid C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) > 0.$

[b] Kaushik, Surjeet; Kumar, Narasimha. Simultaneous behaviour of the Fourier coefficients of two Hilbert modular cusp forms, Archiv der Mathematik 112 (2019), no. 3, 241-248.

Methodology of this proof is similar to the classical case, but technically this is more difficult to prove.

# Recently...

## Theorem (Dalal, K.)

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively.*

# Recently...

## Theorem (Dalal, K.)

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*,*

# Recently...

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*, the set*

$$d(\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) = \frac{1}{2} \text{ or } 1.$$

# Recently...

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*, the set*

$$d(\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) = \frac{1}{2} \text{ or } 1.$$

- We showed that $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = 2\mathbb{N}$ or $\mathbb{N}$.

# Recently...

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*, the set*

$$d(\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) = \frac{1}{2} \text{ or } 1.$$

- We showed that $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = 2\mathbb{N}$ or $\mathbb{N}$.

**Corollary**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be as above. If* $\mathbf{f}, \mathbf{g}$ *are non-CM,*

# Recently...

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*, the set*

$$d(\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) = \frac{1}{2} \text{ or } 1.$$

- We showed that $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = 2\mathbb{N} \text{ or } \mathbb{N}$.

**Corollary**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be as above. If* $\mathbf{f}, \mathbf{g}$ *are non-CM, then there exists* $S \subseteq \mathbb{P}$ *with* $d(S) = 0$ *such that*

# Recently...

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two primitive forms over* $K$*, of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with characters* $\chi_1$ *and* $\chi_2$ *and weights* $2k, 2l$*, respectively. For all but finitely many prime ideals* $\mathfrak{p} \in \mathbb{P}$*, the set*

$$d(\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}) = \frac{1}{2} \text{ or } 1.$$

- We showed that $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = 2\mathbb{N}$ or $\mathbb{N}$.

**Corollary**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be as above. If* $\mathbf{f}, \mathbf{g}$ *are non-CM, then there exists* $S \subseteq \mathbb{P}$ *with* $d(S) = 0$ *such that*

$$\{m \in \mathbb{N} \mid C(\mathfrak{p}^m, \mathbf{f}) C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = \mathbb{N}$$

*for all prime ideals* $\mathfrak{p} \in \mathbb{P}$ *outside of* $S$*.*

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*.*

# Proof of the theorem

**Proposition**

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*,*

# Proof of the theorem

**Proposition**

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$, *with character* $\chi$ *and weight* $2k$. *Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$, *we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$.

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

Suppose $C(\mathfrak{p}, \mathbf{f}) = 0$ or $C(\mathfrak{p}, \mathbf{g}) = 0$

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}}M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

WLOG let $C(\mathfrak{p}, \mathbf{f}) = 0$

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$, *with character* $\chi$ *and weight* $2k$. *Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$, *we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$.

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

WLOG let $C(\mathfrak{p}, \mathbf{f}) = 0$

- $C(\mathfrak{p}^m, \mathbf{f}) = C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}^{m-1}, \mathbf{f}) - \chi_1(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}C(\mathfrak{p}^{m-2}, \mathbf{f})$,

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

WLOG let $C(\mathfrak{p}, \mathbf{f}) = 0$

- $C(\mathfrak{p}^m, \mathbf{f}) = C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}^{m-1}, \mathbf{f}) - \chi_1(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}C(\mathfrak{p}^{m-2}, \mathbf{f})$,
- $C(\mathfrak{p}^{2m+1}, \mathbf{f}) = 0$,

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

WLOG let $C(\mathfrak{p}, \mathbf{f}) = 0$

- $C(\mathfrak{p}^m, \mathbf{f}) = C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}^{m-1}, \mathbf{f}) - \chi_1(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}C(\mathfrak{p}^{m-2}, \mathbf{f})$,
- $C(\mathfrak{p}^{2m+1}, \mathbf{f}) = 0$, $C(\mathfrak{p}^{2m}, \mathbf{f}) \neq 0$

# Proof of the theorem

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

- If $K$ is Galois over $\mathbb{Q}$, then the above result holds $\forall \mathfrak{p} \in \mathbf{P}$.

## Proof

Suppose $C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}, \mathbf{g}) \neq 0$

- If $p \nmid M_{\mathbf{f}} M_{\mathbf{g}} \Rightarrow C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0, \forall m \in \mathbb{N}$.
- $A_{\mathfrak{p}} = \mathbb{N}$.

WLOG let $C(\mathfrak{p}, \mathbf{f}) = 0$

- $C(\mathfrak{p}^m, \mathbf{f}) = C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}^{m-1}, \mathbf{f}) - \chi_1(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}C(\mathfrak{p}^{m-2}, \mathbf{f})$,
- $C(\mathfrak{p}^{2m+1}, \mathbf{f}) = 0$, $C(\mathfrak{p}^{2m}, \mathbf{f}) \neq 0 \Rightarrow A_{\mathfrak{p}} = 2\mathbb{N}$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$*.*

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$*.*
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$, *with character* $\chi$ *and weight* $2k$. *Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$, *we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$.

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$.
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.
- $\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f})$ *and* $\alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$, *with character* $\chi$ *and weight* $2k$. *Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$, *we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$.

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$.
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.
- $\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f})$ *and* $\alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0$.
- $C(\mathfrak{p}^r, \mathbf{f}) = 0 \Leftrightarrow \left(\frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})}\right)^{r+1} = 1 \Rightarrow \alpha(\mathfrak{p}) = \zeta\beta(\mathfrak{p})$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$*.*
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.
- $\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f})$ *and* $\alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0$.
- $C(\mathfrak{p}^r, \mathbf{f}) = 0 \Leftrightarrow \left(\frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})}\right)^{r+1} = 1 \Rightarrow \alpha(\mathfrak{p}) = \zeta\beta(\mathfrak{p})$.
- $C(\mathfrak{p}, \mathbf{f}) = \pm\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})N(\mathfrak{p})^{(2k_0-1)/2} \in \mathbb{Q}(\mathbf{f})$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$*.*
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.
- $\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f})$ *and* $\alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0$.
- $C(\mathfrak{p}^r, \mathbf{f}) = 0 \Leftrightarrow \left(\frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})}\right)^{r+1} = 1 \Rightarrow \alpha(\mathfrak{p}) = \zeta\beta(\mathfrak{p})$.
- $C(\mathfrak{p}, \mathbf{f}) = \pm\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})N(\mathfrak{p})^{(2k_0-1)/2} \in \mathbb{Q}(\mathbf{f})$.
- $\mathbb{Q}(\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})p^{\frac{f(2k_0-1)}{2}}) \subseteq \mathbb{Q}(\mathbf{f})$.

# Proof of the proposition

## Proposition

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of level* $\mathfrak{c}$*, with character* $\chi$ *and weight* $2k$*. Then there exists an integer* $M_{\mathbf{f}} \geq 1$ *with* $N(\mathfrak{c}) \mid M_{\mathbf{f}}$ *such that for any prime* $p \nmid M_{\mathbf{f}}$ *and for any prime ideal* $\mathfrak{p} \in \mathbb{P}$ *over* $p$*, we have either* $C(\mathfrak{p}, \mathbf{f}) = 0$ *or* $C(\mathfrak{p}^r, \mathbf{f}) \neq 0$ *for all* $r \geq 1$*.*

## Sketch of the proof

- *Let* $\mathfrak{p} \in \mathbb{P}$ *s.t* $C(\mathfrak{p}, \mathbf{f}) \neq 0$ *but* $C(\mathfrak{p}^r, \mathbf{f}) = 0$ *for some* $r \geq 2$*.*
- $1 - C(\mathfrak{p}, \mathbf{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$.
- $\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f})$ *and* $\alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0$.
- $C(\mathfrak{p}^r, \mathbf{f}) = 0 \Leftrightarrow \left(\frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})}\right)^{r+1} = 1 \Rightarrow \alpha(\mathfrak{p}) = \zeta\beta(\mathfrak{p})$.
- $C(\mathfrak{p}, \mathbf{f}) = \pm\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})N(\mathfrak{p})^{(2k_0-1)/2} \in \mathbb{Q}(\mathbf{f})$.
- $\mathbb{Q}(\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})p^{\frac{f(2k_0-1)}{2}}) \subseteq \mathbb{Q}(\mathbf{f})$.
- $\mathbb{Q}(\sqrt{\zeta\chi(\mathfrak{p})}(1 + \zeta^{-1})\sqrt{p}) \subseteq \mathbb{Q}(\mathbf{f})$.

# Outline

# Question

| Sign changes | | |
|---|---|---|
| Fourier coefficients | $\mathfrak{p}$ fix, $r$ varying | $r$ fix, $\mathfrak{p}$ varying |
| $C(\mathfrak{p}^r, \mathbf{f})$ | | |
| $C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g})$ | | |

# Modular forms over $\mathbb{Q}$

**Theorem (Kohnen, Martin)**

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(N)$ *be a primitive form.*

# Modular forms over $\mathbb{Q}$

## Theorem (Kohnen, Martin)

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(N)$ *be a primitive form. There exists* $N = N(f) \in \mathbb{N}$ *such that for all primes* $p > N$,

**Theorem (Kohnen, Martin)**

*Let $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(N)$ be a primitive form. There exists $N = N(f) \in \mathbb{N}$ such that for all primes $p > N$, the sequence $\{a_f(p^r)\}_{r \in \mathbb{N}}$ has infinitely many sign changes.*

# Modular forms over $\mathbb{Q}$

**Theorem (Kohnen, Martin)**

*Let $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(N)$ be a primitive form. There exists $N = N(f) \in \mathbb{N}$ such that for all primes $p > N$, the sequence $\{a_f(p^r)\}_{r \in \mathbb{N}}$ has infinitely many sign changes.*

---

Kohnen, W; Martin, Y. Sign changes of Fourier coefficients of cusp forms supported on prime power indices. Int. J. Number Theory 10 (2014), no. 8, 1921–1927.

# Hilbert modular case

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of weight* $2k$*, level* $\mathfrak{c}$*, and with trivial character.*

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let $\mathbf{f}$ be a primitive form over $K$ of weight $2k$, level $\mathfrak{c}$, and with trivial character. Then, for all but finitely many $\mathfrak{p} \in \mathbb{P}$,*

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of weight* $2k$*, level* $\mathfrak{c}$*, and with trivial character. Then, for all but finitely many* $\mathfrak{p} \in \mathbb{P}$*, the Fourier coefficients* $\{C(\mathfrak{p}^r, \mathbf{f})\}_{r \in \mathbb{N}}$ *change signs infinitely often.*

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let $\mathbf{f}$ be a primitive form over $K$ of weight $2k$, level $\mathfrak{c}$, and with trivial character. Then, for all but finitely many $\mathfrak{p} \in \mathbb{P}$, the Fourier coefficients $\{C(\mathfrak{p}^r, \mathbf{f})\}_{r \in \mathbb{N}}$ change signs infinitely often.*

There are two ways to prove this Theorem.

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a primitive form over* $K$ *of weight* $2k$*, level* $\mathfrak{c}$*, and with trivial character. Then, for all but finitely many* $\mathfrak{p} \in \mathbb{P}$*, the Fourier coefficients* $\{C(\mathfrak{p}^r, \mathbf{f})\}_{r \in \mathbb{N}}$ *change signs infinitely often.*

There are two ways to prove this Theorem.

- Analytic argument.

# Hilbert modular case

Recall that

- $K$ denote a totally real number field of odd degree.
- $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_K$ with odd inertia degree.

**Theorem (Dalal, K.)**

*Let $\mathbf{f}$ be a primitive form over $K$ of weight $2k$, level $\mathfrak{c}$, and with trivial character. Then, for all but finitely many $\mathfrak{p} \in \mathbb{P}$, the Fourier coefficients $\{C(\mathfrak{p}^r, \mathbf{f})\}_{r\in\mathbb{N}}$ change signs infinitely often.*

There are two ways to prove this Theorem.

- Analytic argument.
- Signs of sin-function (Kohnen remarked this)

# Key Lemma

## Lemma

*Let* $\mathbf{f}$ *be a primitive form over* $F$ *of level* $\mathfrak{c}$, *with trivial character and weight* $2k$. *For any prime ideal* $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$, *let* $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$ *be defined as in* (2.1). *Then, for any* $r \geq 1$, *we have*

$$\beta(\mathfrak{p}^r, \mathbf{f}) = \begin{cases} (-1)^r(r+1) & \text{if } \theta_{\mathfrak{p}}(\mathbf{f}) = \pi, \\ r+1 & \text{if } \theta_{\mathfrak{p}}(\mathbf{f}) = 0, \\ \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin \theta_{\mathfrak{p}}(\mathbf{f})} & \text{if } 0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi. \end{cases} \tag{2.3}$$

- By induction

# Proof of the Theorem

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$,

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0, \pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p}, \mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.
- $\frac{C(\mathfrak{p}^r,\mathbf{f})}{N(\mathfrak{p}^r)^{\frac{2k_0-1}{2}}} = \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin\theta_{\mathfrak{p}}(\mathbf{f})}$ for $0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi$.

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.
- $\frac{C(\mathfrak{p}^r,\mathbf{f})}{N(\mathfrak{p}^r)^{\frac{2k_0-1}{2}}} = \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin\theta_{\mathfrak{p}}(\mathbf{f})}$ for $0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi$.
- $$C(\mathfrak{p}^r,\mathbf{f}) \gtrless 0 \iff \sin 2\pi(r+1)\frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi} \gtrless 0.$$

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.
- $\frac{C(\mathfrak{p}^r,\mathbf{f})}{N(\mathfrak{p}^r)^{\frac{2k_0-1}{2}}} = \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin\theta_{\mathfrak{p}}(\mathbf{f})}$ for $0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi$.
- $$C(\mathfrak{p}^r,\mathbf{f}) \gtrless 0 \iff \sin 2\pi(r+1)\frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi} \gtrless 0.$$
- Let $x = \frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi}$.

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.
- $\frac{C(\mathfrak{p}^r,\mathbf{f})}{N(\mathfrak{p}^r)^{\frac{2k_0-1}{2}}} = \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin\theta_{\mathfrak{p}}(\mathbf{f})}$ for $0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi$.
- $$C(\mathfrak{p}^r,\mathbf{f}) \gtrless 0 \iff \sin 2\pi(r+1)\frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi} \gtrless 0.$$
- Let $x = \frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi}$. For any $j \in \mathbb{N}$, the lengths of the intervals $(\frac{2j}{2x}, \frac{(2j+1)}{2x})$ and $(\frac{(2j-1)}{2x}, \frac{2j}{2x})$ are bigger than 1, as $\frac{1}{2x} > 1$.

# Proof of the Theorem

- $\frac{C(\mathfrak{p},\mathbf{f})}{N(\mathfrak{p})^{\frac{2k_0-1}{2}}} = 2\cos\theta_{\mathfrak{p}}(\mathbf{f})$, for some $\theta_{\mathfrak{p}}(\mathbf{f}) \in [0,\pi]$.
- If $\theta_{\mathfrak{p}}(\mathbf{f}) = 0$ or $\pi$, then $C(\mathfrak{p},\mathbf{f}) = \pm 2N(\mathfrak{p})^{\frac{2k_0-1}{2}} \in \mathbb{Q}(\mathbf{f})$, can happen only for finitely many $\mathfrak{p} \in \mathbb{P}$.
- $\frac{C(\mathfrak{p}^r,\mathbf{f})}{N(\mathfrak{p}^r)^{\frac{2k_0-1}{2}}} = \frac{\sin((r+1)\theta_{\mathfrak{p}}(\mathbf{f}))}{\sin\theta_{\mathfrak{p}}(\mathbf{f})}$ for $0 < \theta_{\mathfrak{p}}(\mathbf{f}) < \pi$.
- $$C(\mathfrak{p}^r,\mathbf{f}) \gtrless 0 \iff \sin 2\pi(r+1)\frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi} \gtrless 0.$$
- Let $x = \frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi}$. For any $j \in \mathbb{N}$, the lengths of the intervals $(\frac{2j}{2x}, \frac{(2j+1)}{2x})$ and $(\frac{(2j-1)}{2x}, \frac{2j}{2x})$ are bigger than 1, as $\frac{1}{2x} > 1$.
- There exists $n_j, m_j \in \mathbb{Z}$ such that $n_j + 1 \in (\frac{2j}{2x}, \frac{2j+1}{2x})$ and $m_j + 1 \in (\frac{2j-1}{2x}, \frac{2j}{2x})$. Therefore, we have $\sin((n_j+1)\theta_{\mathfrak{p}}(\mathbf{f})) > 0$ and $\sin((m_j+1)\theta_{\mathfrak{p}}(\mathbf{f})) < 0$.

# So far...

| Sign changes | | |
|---|---|---|
| Fourier coefficients | $\mathfrak{p}$ fix, $r$ varying | $r$ fix, $\mathfrak{p}$ varying |
| $C(\mathfrak{p}^r, \mathbf{f})$ | $\checkmark(K, \mathbb{P})$ | |
| $C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g})$ | | |

# Hilbert modular case

## Theorem (Dalal, K.)

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$*, with trivial character and weight* $2k$*. For any* $r \geq 1$*, we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

# Hilbert modular case

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$*, with trivial character and weight* $2k$*. For any* $r \geq 1$*, we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$*, then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.

# Hilbert modular case

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$*, with trivial character and weight* $2k$*. For any* $r \geq 1$*, we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$*, then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$*, then*

# Hilbert modular case

## Theorem (Dalal, K.)

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$*, with trivial character and weight* $2k$*. For any* $r \geq 1$*, we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$*, then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$*, then*
   $d(\mathbf{P}(r)_{>0}) = \frac{r+2}{2(r+1)} - \frac{1}{2\pi} \tan\left(\frac{\pi}{r+1}\right)$, and

# Hilbert modular case

## Theorem (Dalal, K.)

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$*, with trivial character and weight* $2k$*. For any* $r \geq 1$*, we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$*, then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$*, then*
   $d(\mathbf{P}(r)_{>0}) = \frac{r+2}{2(r+1)} - \frac{1}{2\pi} \tan\left(\frac{\pi}{r+1}\right)$, and
   $d(\mathbf{P}(r)_{<0}) = \frac{r}{2(r+1)} + \frac{1}{2\pi} \tan\left(\frac{\pi}{r+1}\right)$.

# Hilbert modular case

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$, *with trivial character and weight* $2k$. *For any* $r \geq 1$, *we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$, *then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$, *then*
   $d(\mathbf{P}(r)_{>0}) = \frac{r+2}{2(r+1)} - \frac{1}{2\pi}\tan\left(\frac{\pi}{r+1}\right)$, and
   $d(\mathbf{P}(r)_{<0}) = \frac{r}{2(r+1)} + \frac{1}{2\pi}\tan\left(\frac{\pi}{r+1}\right)$.

*In particular, for any* $r \in \mathbb{N}$, *the sequence* $\{C(\mathfrak{p}^r, \mathbf{f})\}_{\mathfrak{p} \in \mathbf{P}}$ *changes sign infinitely often.*

# Hilbert modular case

## Theorem (Dalal, K.)

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$, *with trivial character and weight* $2k$. *For any* $r \geq 1$, *we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$, *then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$, *then*
   $d(\mathbf{P}(r)_{>0}) = \frac{r+2}{2(r+1)} - \frac{1}{2\pi}\tan\left(\frac{\pi}{r+1}\right)$, and
   $d(\mathbf{P}(r)_{<0}) = \frac{r}{2(r+1)} + \frac{1}{2\pi}\tan\left(\frac{\pi}{r+1}\right)$.

*In particular, for any* $r \in \mathbb{N}$, *the sequence* $\{C(\mathfrak{p}^r, \mathbf{f})\}_{\mathfrak{p} \in \mathbf{P}}$ *changes sign infinitely often.*

- Sato-Tate equi-distribution theorem for $\mathbf{f}$

# Hilbert modular case

## Theorem (Dalal, K.)

*Let* $\mathbf{f}$ *be a non-CM primitive form over* $F$ *of level* $\mathfrak{c}$, *with trivial character and weight* $2k$. *For any* $r \geq 1$, *we define*
$\mathbf{P}(r)_{\gtrless 0} = \{\mathfrak{p} \in \mathbf{P} \mid \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, C(\mathfrak{p}^r, \mathbf{f}) \gtrless 0\}$.

1. *If* $r \equiv 1 \pmod 2$, *then* $d(\mathbf{P}(r)_{>0}) = d(\mathbf{P}(r)_{<0}) = \frac{1}{2}$.
2. *If* $r \equiv 0 \pmod 2$, *then*
$d(\mathbf{P}(r)_{>0}) = \frac{r+2}{2(r+1)} - \frac{1}{2\pi} \tan\left(\frac{\pi}{r+1}\right)$, and
$d(\mathbf{P}(r)_{<0}) = \frac{r}{2(r+1)} + \frac{1}{2\pi} \tan\left(\frac{\pi}{r+1}\right)$.

*In particular, for any* $r \in \mathbb{N}$, *the sequence* $\{C(\mathfrak{p}^r, \mathbf{f})\}_{\mathfrak{p} \in \mathbf{P}}$ *changes sign infinitely often.*

- Sato-Tate equi-distribution theorem for $\mathbf{f}$
- A similar result exists for modular forms over $\mathbb{Q}$ due to Meher, Shankhadhar and Viswanadham.

# So far...

| Sign changes | | |
|---|---|---|
| Fourier coefficients | $\mathfrak{p}$ fix, $r$ varying | $r$ fix, $\mathfrak{p}$ varying |
| $C(\mathfrak{p}^r, \mathbf{f})$ | ✓ $(K, \mathbb{P})$ | ✓ $(F, \mathbf{P})$ |
| $C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g})$ | | |

# Classical case

**Theorem (Amri)**

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n$ *and* $g = \sum_{n=1}^{\infty} a_g(n) q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weights* $k_1, k_2$ *respectively.*

# Classical case

## Theorem (Amri)

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g = \sum_{n=1}^{\infty} a_g(n)q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weights* $k_1, k_2$ *respectively. For any prime* $p \nmid N_1N_2$*, we define* $\theta_p(f), \theta_p(g) \in [0, \pi]$ *by* $\frac{a_p(f)}{p^{\frac{k_1-1}{2}}} = 2\cos\theta_p(f)$ *(resp.* $\theta_p(g)$ *).*

# Classical case

## Theorem (Amri)

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g = \sum_{n=1}^{\infty} a_g(n)q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weights* $k_1, k_2$ *respectively. For any prime* $p \nmid N_1N_2$, *we define* $\theta_p(f), \theta_p(g) \in [0, \pi]$ *by* $\frac{a_p(f)}{p^{\frac{k_1-1}{2}}} = 2\cos\theta_p(f)$ *(resp.* $\theta_p(g)$ *). Then* *for all but finitely many primes* $p$

# Classical case

## Theorem (Amri)

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g = \sum_{n=1}^{\infty} a_g(n)q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weights* $k_1, k_2$ *respectively. For any prime* $p \nmid N_1N_2$*, we define* $\theta_p(f), \theta_p(g) \in [0, \pi]$ *by* $\frac{a_p(f)}{p^{\frac{k_1-1}{2}}} = 2\cos\theta_p(f)$ *(resp.* $\theta_p(g)$ *). Then for all but finitely many primes* $p$ *the linear independence of* $1, \frac{\theta_p(f)}{2\pi}, \frac{\theta_p(g)}{2\pi}$ *over* $\mathbb{Q}$

# Classical case

## Theorem (Amri)

*Let* $f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g = \sum_{n=1}^{\infty} a_g(n)q^n$ *be two distinct primitive forms of level* $N_1, N_2$ *and weights* $k_1, k_2$ *respectively. For any prime* $p \nmid N_1N_2$*, we define* $\theta_p(f), \theta_p(g) \in [0, \pi]$ *by* $\frac{a_p(f)}{p^{\frac{k_1-1}{2}}} = 2\cos\theta_p(f)$ *(resp.* $\theta_p(g)$ *). Then for all but finitely many primes* $p$ *the linear independence of* $1, \frac{\theta_p(f)}{2\pi}, \frac{\theta_p(g)}{2\pi}$ *over* $\mathbb{Q}$ *implies*

$$\lim_{x\to\infty} \frac{\#\{r \leq x : a_f(p^r)a_g(p^r) \gtrless 0\}}{x} = \frac{1}{2}.$$

---

Amri, MA. Simultaneous sign change and equidistribution of signs of Fourier coefficients of two cusp forms. Arch. Math. (Basel) 111 (2018), no. 3, 257–266

# Hilbert modular case

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct non-CM primitive forms over* $F$ *of levels* $\mathfrak{c}_1, \mathfrak{c}_2$*, with trivial characters, and weights* $2k, 2l$*, respectively.*

# Hilbert modular case

**Theorem (Dalal, K.)**

*Let* $\mathbf{f}, \mathbf{g}$ *be two distinct non-CM primitive forms over* $F$ *of levels* $\mathfrak{c}_1, \mathfrak{c}_2$, *with trivial characters, and weights* $2k, 2l$, *respectively. Then, for a natural density* $1$ *set of primes* $\mathfrak{p} \in \mathbf{P}$ , *the linear independence of* $1, \frac{\theta_{\mathfrak{p}}(\mathbf{f})}{2\pi}, \frac{\theta_{\mathfrak{p}}(\mathbf{g})}{2\pi}$ *over* $\mathbb{Q}$ *implies*

$$\lim_{x \to \infty} \frac{\#\{r \leq x : C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g}) \gtrless 0\}}{x} = \frac{1}{2}.$$

# So far...

| Sign changes | | |
|---|---|---|
| Fourier coefficients | $\mathfrak{p}$ fix, $r$ varying | $r$ fix, $\mathfrak{p}$ varying |
| $C(\mathfrak{p}^r, \mathbf{f})$ | $\checkmark (K, \mathbb{P})$ | $\checkmark (F, \mathbf{P})$ |
| $C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g})$ | $\checkmark (F, \mathbf{P})$ | |

# Classical case

**Theorem (Amri,2018)**

*Let* $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n$ *and* $g(z) = \sum_{n=1}^{\infty} a_g(n) q^n$ *be distinct non-CM primitive forms of weight* $2k_1, 2k_2$ *and level* $N_1, N_2$*, respectively, such that they are not twist of each other.*

# Classical case

**Theorem (Amri,2018)**

*Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ and $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n$ be distinct non-CM primitive forms of weight $2k_1, 2k_2$ and level $N_1, N_2$, respectively, such that they are not twist of each other. If joint Sato-Tate conjecture holds, then for any $r \equiv 1 \pmod 2$,*

$$d\{p \in \mathbb{P} | a_f(p^r)a_g(p^r) \gtreqless 0\} = \frac{1}{2}.$$

# Classical case

## Theorem (Amri,2018)

*Let* $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n$ *be distinct non-CM primitive forms of weight* $2k_1, 2k_2$ *and level* $N_1, N_2$*, respectively, such that they are not twist of each other. If joint Sato-Tate conjecture holds, then for any* $r \equiv 1 \pmod 2$,

$$d\{p \in \mathbb{P} | a_f(p^r)a_g(p^r) \gtrless 0\} = \frac{1}{2}.$$

## Remark

Let $f, g$ be same as in the above theorem.

# Classical case

## Theorem (Amri,2018)

*Let* $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n$ *be distinct non-CM primitive forms of weight* $2k_1, 2k_2$ *and level* $N_1, N_2$*, respectively, such that they are not twist of each other. If joint Sato-Tate conjecture holds, then for any* $r \equiv 1 \pmod 2$,

$$d\{p \in \mathbb{P} | a_f(p^r)a_g(p^r) \gtrless 0\} = \frac{1}{2}.$$

## Remark

Let $f, g$ be same as in the above theorem.Then for any $r \equiv 0 \pmod 2$,

# Classical case

## Theorem (Amri,2018)

*Let* $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ *and* $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n$ *be distinct non-CM primitive forms of weight* $2k_1, 2k_2$ *and level* $N_1, N_2$*, respectively, such that they are not twist of each other. If joint Sato-Tate conjecture holds, then for any* $r \equiv 1 \pmod 2$,

$$d\{p \in \mathbb{P} | a_f(p^r)a_g(p^r) \gtrless 0\} = \frac{1}{2}.$$

## Remark

Let $f, g$ be same as in the above theorem.Then for any $r \equiv 0 \pmod 2$,

- $d\{p \in \mathbb{P} | a_f(p^r)a_g(p^r) \gtrless 0\} = \frac{1}{2} + \frac{1}{2(r+1)^2} - \frac{2}{r+1}c \pm 2c^2$ where $c = \frac{1}{2\pi}\tan(\frac{\pi}{r+1})$.

# Final Picture

| Sign changes | | |
|---|---|---|
| Fourier coefficients | $\mathfrak{p}$ fix, $r$ varying | $r$ fix, $\mathfrak{p}$ varying |
| $C(\mathfrak{p}^r, \mathbf{f})$ | $\checkmark(K, \mathbb{P})$ | $\checkmark(F, \mathbf{P})$ |
| $C(\mathfrak{p}^r, \mathbf{f})C(\mathfrak{p}^r, \mathbf{g})$ | $\checkmark(F, \mathbf{P})$ | Yes.. with Joint Sato-Tate |

# at Integral ideals

**Theorem (Dalal,K.)**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$*, trivial character and different weights* $2k$, $2l$*, respectively.*

**Theorem (Dalal,K.)**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$*, trivial character and different weights* $2k$*,* $2l$*, respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$*, we assume that* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *are real numbers.*

**Theorem (Dalal,K.)**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$*, trivial character and different weights* $2k$*,* $2l$*, respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$*, we assume that* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *are real numbers. Suppose that for every ideal* $\mathfrak{n} \subseteq \mathcal{O}_F$*, there exists an ideal* $\mathfrak{r} \subseteq \mathcal{O}_F$ *such that* $(\mathfrak{n}, \mathfrak{r}) = 1$ *such that* $C(\mathfrak{r}, \mathbf{f})C(\mathfrak{r}, \mathbf{g}) \neq 0$*.*

**Theorem (Dalal,K.)**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$*, trivial character and different weights* $2k$*,* $2l$*, respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$*, we assume that* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *are real numbers. Suppose that for every ideal* $\mathfrak{n} \subseteq \mathcal{O}_F$*, there exists an ideal* $\mathfrak{r} \subseteq \mathcal{O}_F$ *such that* $(\mathfrak{n}, \mathfrak{r}) = 1$ *such that* $C(\mathfrak{r}, \mathbf{f})C(\mathfrak{r}, \mathbf{g}) \neq 0$*. Then there exist infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) > 0$

**Theorem (Dalal,K.)**

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$, *trivial character and different weights* $2k$, $2l$, *respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$, *we assume that* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *are real numbers. Suppose that for every ideal* $\mathfrak{n} \subseteq \mathcal{O}_F$, *there exists an ideal* $\mathfrak{r} \subseteq \mathcal{O}_F$ *such that* $(\mathfrak{n}, \mathfrak{r}) = 1$ *such that* $C(\mathfrak{r}, \mathbf{f})C(\mathfrak{r}, \mathbf{g}) \neq 0$. *Then there exist infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) > 0$ *and infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) < 0$.

## Theorem (Dalal,K.)

*Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$*, trivial character and different weights* $2k$*,* $2l$*, respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$*, we assume that* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *are real numbers. Suppose that for every ideal* $\mathfrak{n} \subseteq \mathcal{O}_F$*, there exists an ideal* $\mathfrak{r} \subseteq \mathcal{O}_F$ *such that* $(\mathfrak{n}, \mathfrak{r}) = 1$ *such that* $C(\mathfrak{r}, \mathbf{f})C(\mathfrak{r}, \mathbf{g}) \neq 0$*. Then there exist infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) > 0$ *and infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) < 0$*.*

## Remark

The condition of simultaneous non-vanishing of Fourier coefficients is required only to ensure that the certain $L$-function is non-zero.


THANK YOU
FOR YOUR LISTENING