Rigid Analytic Vectors in Locally Analytic Representations: Exactness and Applications

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Layout of the Talk

1. Introduction: Context of the Problem
2. Main Theorem
3. Applications And Future Directions

Context of The Problem

- Schneider-Stuhler in 1997 functorially associated a sheaf on the Bruhat-Tits building of a connected p-adic reductive group G to a smooth representation* of G . The first step of the construction was exactness of the functor $V \rightsquigarrow V^H$ for a uniform pro-p subgroup H.

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- Schneider-Teitelbaum (early 2000's) introduced the theory of *locally analytic representations* which has since become important in p-adic local Langlands.

Question

Can we mimic Schneider-Stuhler's construction for locally analytic representations?

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- V is a locally analytic **admissible** representation of H
- $V_{\mathbb{H}_n^\circ - an} := \{v \in V \big| f_v : h \to h.v \in \mathcal{O}(\mathbb{H}_n^\circ, V)\}$

Main Theorem

- The main result for this talk is,

Theorem

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- We actually concentrate on proving $(V)'_b \rightsquigarrow (V_{\mathbb{H}_n^\circ-\mathrm{an}})'_b$ is exact, because it lets us algebraize the problem via distribution algebras and coamdissible modules over them.

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3. An isomorphism of topological A-modules $A \cong \varprojlim_n A_n$, where each of the maps $A \to A_n$ has dense image. The right hand side is given a projective limit topology from the transition maps induced from part (2).

Examples

- The distribution algebra $D^{\mathrm{la}}(\mathbb{Z}_p, E) := C^{\mathrm{la}}(\mathbb{Z}_p, E)'_b = \varprojlim_n A_n$, has a WFS structure with

$$A_n := (C^{\mathrm{la}}(\mathbb{Z}_p, E)_{p^n\mathfrak{m}_{\mathbb{C}_p} - an})'_b = \Big(\bigoplus_{a \in \mathbb{Z}/p^{n+1}\mathbb{Z}} \mathcal{O}(a + p^n \mathfrak{m}_{\mathbb{C}_p}) \Big)'_b .$$

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- There exists $0 < r_n < 1$'s.t the A_n's can be described as,

$$A_n := \mathcal{O}_{r_n}(X)^{\dagger} := \left\{ \sum_{n=0}^{\infty} a_n T^n \;\middle|\; \lim_{n \to \infty} a_n R^n = 0 \text{ for some } R > r_n \right\}$$

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Theorem (Emerton)

The distribution algebra $D^{\mathrm{la}}(H, E) := (C^{\mathrm{la}}(H, E))'_b$ is a weak Fréchet-Stein algebra with a weak Fréchet-Stein structure

$$D^{\mathrm{la}}(H, E) \xrightarrow{\simeq} \varprojlim_n D(\mathbb{H}_n^{\circ}, H) := \varprojlim_n (C^{\mathrm{la}}(H, E)_{\mathbb{H}_n^{\circ}-an})'_b$$

Coadmissible module

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2. An isomorphism of topological A-modules $M \cong \varprojlim_n M_n$ with $A_n \hat{\otimes}_{A_{n+1}} M_{n+1} \cong M_n$, for each $n \geq 1$.

The Idea of Proof

- By definition, a locally analytic representation V of H is admissible if $M := V_b'$ is a coadmissible $D^{\mathrm{la}}(H, E)$ module.

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- The coadmissible structure is given by $M = \varprojlim_n M_n$ with $M_n := (V_{\mathbb{H}_n^\circ - an})_b'$
- Emerton shows, $M_n \cong A_n \widehat{\otimes}_A M$,

  with $A_n := D(\mathbb{H}_n^\circ, H)$,

  $M := V_b', A = D^{\mathrm{la}}(H, E)$.

The Idea of Proof Cont.

- A crucial step for the proof of main theorem is

$$\boxed{(V_{\mathbb{H}_n^\circ - an})_b^{'} \cong D^{\mathrm{la}}(\mathbb{H}_n^\circ, H) \otimes_{D^{\mathrm{la}}(H)} V_b^{'}}$$

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- A crucial step for the proof of main theorem is
$$\boxed{(V_{\mathbb{H}_n^\circ - an})'_b \cong D^{\mathrm{la}}(\mathbb{H}_n^\circ, H) \otimes_{D^{\mathrm{la}}(H)} V'_b}$$
- This result is an extension of a similar result of S-T, we do it for Weak Fréchet-Stein algebras which are also equipped with a Fréchet-Stein structure.

The Idea of Proof Cont.

- Another crucial observation is that the natural map

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  is flat.
- This flatness is result of a tour-de-force in commutative algebra and relies deeply on the interaction between the WFS and FS structure of the distribution algebra. And it is due to Emerton.

Future Directions of Research

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- By the work of Remy-Thuillier-Werner there is associated to every point x of the Bruhat-Tits building $BT(G)$ of G a rigid-analytic affinoid group $\mathbb{G}_x$. This gives rise to a sheaf $U \mapsto \mathbb{G}(U) := \bigcup_{x \in U} \mathbb{G}_x$ of rigid analytic groups on the Bruhat-Tits building $BT(G)$.

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- Given such a representation V we consider for any open subset $U \subset BT(G)$ the subspace $V_{\mathbb{G}(U)-\mathrm{an}} \subset V$ of rigid analytic vectors for U, and its continuous dual $\mathcal{M}_V(U)$. Then

$$U \to \mathcal{M}_V(U)$$

is a sheaf on $BT(G)$.

Question

Does the Schneider-Stuhler procedure, when applied to the sheaf $\mathcal{M}_V$ lead to a canonical resolution of $M = V'$ in the category of $D(G)$-modules.

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Thank You