Reduction of Galois representations and local constancy with respect to weight

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Crystalline representations

Notations

p : prime number, $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_p)$ with profinite topology.
$E/\mathbb{Q}_p$: finite extension of local fields.
$\mathcal{O}_E$: ring of integers in E.
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Colmez-Fontaine, 2000: Equivalence of categories

$$\boxed{\begin{array}{c}\text{Crystalline representations}\\ V \text{ of } G_p;\ \dim_E V = n\end{array}} \xrightarrow{\quad D_{cris}\quad} \boxed{\begin{array}{c}\text{Admissible filtered } \varphi\text{-modules}\\ D;\ \dim_E D = n\end{array}}$$

Crystalline representations of dimension 2 and their reduction

For $k \in \mathbb{Z}_{\geq 2}$ and $a \in \mathfrak{m}_E$, let $D_{k,a}$ be the irreducible filtered φ-module given by

$$D_{k,a} = Ee_1 \oplus Ee_2, \qquad \varphi e_1 = e_2,\ \varphi e_2 = -p^{k-1}e_1 + ae_2$$

$$\mathrm{Fil}^i D_{k,a} = \begin{cases} Ee_1 \oplus Ee_2, & i \leq 0 \\ Ee_1, & 1 \leq i \leq k-1 \\ 0, & k \leq i. \end{cases}$$

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With this notation, we are interested in the map

$$\boldsymbol{(k, a) \mapsto \bar{V}_{k,a}} \quad \in \mathrm{Rep}_{k_E}(G_p)$$

as $k \in \mathbb{Z}_{\geq 2}$ and $a \in \mathfrak{m}_{\bar{\mathbb{Q}}_p}$ vary.

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Recall $\mathbb{Z}_p^* = \langle \zeta_{p-1} \rangle \otimes \Gamma$, where $\Gamma = 1 + p\mathbb{Z}_p = \langle \gamma \rangle$. For any $\kappa \in \mathcal{W}$, the connected component of κ is determined by $\kappa(\zeta_{p-1})$ and inside the corresponding disc its coordinate is given by $\kappa(\gamma) - 1$.

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So the congruence class $k \mod (p-1)$ determines the connected component of $\mathcal{W}$ where a classical weight k lies. Thus two classical weights k_1 and k_2 are p-adically 'close' in $\mathcal{W}$ if

1. $k_1 \equiv k_2 \mod (p-1)$
2. $k_1 - k_2$ is divisible by high power of p.

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Let $v : \bar{\mathbb{Q}}_p^* \to \mathbb{Q}$ be the standard p-adic valuation normalised by $v(p) = 1$.

Thm A (local constancy with respect to a)

Let $k \geq 2$ be fixed. If $v(a - a') > 2v(a) + \lfloor (k-1)p/(p-1)^2 \rfloor$, then $\bar{V}_{k,a} \cong \bar{V}_{k,a'}$.

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Let $a \neq 0$ be fixed.

For all $k > \dfrac{3v(a)}{1 - \frac{p}{(p-1)^2}} + 1$, there exists $m = m(k,a) \in \mathbb{N}$ such that:

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- Berger gives no estimate for $m(k, a)$.
- It is not clear if the lower bound on k is necessary to ensure local constancy around a classical point $k \in \mathcal{W}$.

Main result: a simple version

Theorem(- , 2018)

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- Here S consists of 12 weights $k = p + 2, p + 4, 2p + 1, 2p + 2, 2p + 3, 2p + 5, 3p, 3p + 1, 3p + 2, 3p + 3, 3p + 4, 3p + 6$. However, the hypothesis $k \notin S$ is purely technical and thus the set S is not particularly significant.

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Further remarks

More notation: $G_{p^2} := \mathrm{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_{p^2})$ and $\omega_2 : G_{p^2} \to \bar{\mathbb{F}}_p^*$ is the fundamental character of level 2.

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For $\lambda \in \bar{\mathbb{F}}_p^*$, let μ_λ denote the unramified character of G_p sending the geometric Frobenius element to λ.

1. Recall that $a = 0$ was excluded in Berger's Theorem B quoted above. Analysing the known formula $\bar{V}_{k,0} \cong \mathrm{ind}_{G_{p^2}}^{G_p} \omega_2^{k-1} \otimes \mu_{\sqrt{-1}}$ closely, we note that local constancy fails at $a = 0$!

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   - $m(4, a) = \lceil v(a^2 - p^2) \rceil$ if $v = 1$ but $a \neq \pm p$.
   - $m(4, \pm p)$ does not exist, as $\bar{V}_{4,a}$ is irreducible but for large values of k p-adically close to 4, we observed that $\bar{V}_{k,\pm p}$ is reducible!

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We use the compatibility of the mod p and p-adic Local Langlands correspondence (LLC).

By-product of our proof

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- Let k be a classical weight satisfying the hypotheses in our main theorem. If the p-adic distance of k' from k in $\mathcal{W}$ is $\leq p^{-2v}$, then we manage to show
$$\bar{V}_{k',a} = \mathrm{ind}_{G_{p^2}}^{G_p}(\omega_2^{k-1}).$$

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For $V \in \mathrm{Rep}(\Gamma)$, the action of Γ can be inflated to K and further extended to KZ by making $p \in Z$ act trivially on the whole space. Then we consider the compact induction $\mathcal{I}(V) := \mathrm{ind}_{KZ}^{G} V \in \mathrm{Rep}(G)$.

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For all $r \geq 0$, let $V_r := \mathrm{Sym}^r \bar{\mathbb{F}}_p^2 \in \mathrm{Rep}_{\bar{\mathbb{F}}_p}(\Gamma)$ be modelled on the space of homogeneous polynomials $F(x, y)$ with $\deg(F) = r$. The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ acts on both variables $(x, y) \mapsto (ax + cy, bx + dy)$.

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- By the compatibility of p-adic and mod p LLC and its explicit nature in the crystalline case, determining $\bar{V}_{k',a}$ boils down to computing the mod p reduction of the following lattice in the automorphic side:

$$\Theta_{k',a} = \frac{\mathcal{I}(\mathrm{Sym}^{k'-2}\bar{\mathbb{Z}}_p^2)}{(T-a)\mathcal{I}(\mathrm{Sym}^{k'-2}\bar{\mathbb{Q}}_p^2) \cap \mathcal{I}(\mathrm{Sym}^{k'-2}\bar{\mathbb{Z}}_p^2)},$$

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- There is a natural surjection

$$P : \mathcal{I}(V_r) \twoheadrightarrow \bar{\Theta}_{k',a} = \Theta_{k',a} \otimes \bar{\mathbb{F}}_p,$$

  where $r = k' - 2$. To compute $\bar{\Theta}_{k',a}$, we study the map P closely and also use the fact that $\bar{\Theta}_{k',a}$ lies in the image of mod p LLC.

- For any $r \geq 0$, the Γ-representation V_r possesses a filtration defined using the Dickson polynomial $\theta = X^pY - XY^p$, each factor of the filtration $W_i = \frac{\theta^i V_{r-i(p+1)}}{\theta^{i+1} V_{r-(i+1)(p+1)}}$ is reducible of length 2:

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- Structure of W_c + mod p LLC + some standard analysis $\implies$ the shape of $\bar{\Theta}_{k',a}$.

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4. Remaining case: $v > 1$: similar results not available as both the weight and slope are unbounded. We expect $V_{k',a} \cong \mathrm{ind}_{G_{p^2}}^{G_p}(\omega_2^{p+2})$ if k' is close enough to $p + 3$. But how close?

Near $k = p + 3$: how close is close enough?

We prove:

1. If k' and $p+3$ lie in same component of $\mathcal{W}$ and further if $v(k' - (p+3)) \geq 2v$, then the map P induces a surjection

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So our answer: a distance of p^{-2v} in the weight space is close enough!

How do we get a map $\mathcal{I}(W_1) \twoheadrightarrow \bar{\Theta}_{k',a}$?

Answer: by eliminating possible non-zero contribution of
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- As $\langle \bar{F}_m \rangle_{KZ} = W_m$, the G-span of $[g, F_m]$ covers $\mathcal{I}(W_m)$. Hence all of $\mathcal{I}(W_m)$ contributes to $\ker P$ and not to $\bar{\Theta}_{k',a}$!

How does the functions look like?

The case $m = 0$: To eliminate W_0, which is a quotient of V_r of length 2, $r = k' - 2$, we use the following function in $\mathcal{I}(\mathrm{Sym}^r \bar{\mathbb{Q}}_p^2)$:

$$f = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sum_{\substack{p+1 \leq j < r \\ j \equiv 2 \mod (p-1)}} \frac{1}{pa} \binom{r}{j} x^{r-j} y^j \right]$$

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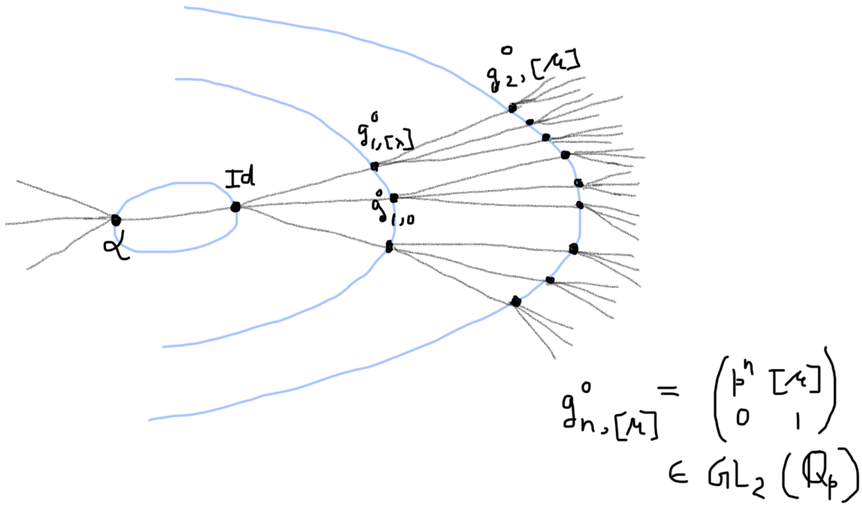
where $[\cdot]$ stands for the Teichmuller representative.

Tree of Gl(2)



Vertices of the tree are labelled as coset representatives of G/KZ. So
Elements in I(V) can be seen as V-valued functions on the tree with finite support. The Hecke operator T takes function on a vertex to one supported on its neighbour vertices, 1 backward and p-many forward neighbours.

We apply the formula for the Hecke operator

$$T([g,v]) = \sum_{\lambda\in\mathbb{F}_p} \left[g \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix},\ v\left(X, -[\lambda]X + pY\right)\right] + \left[g \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right),\ v(pX, Y)\right]$$

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Next step: For $m = 2, 3, \cdots$ etc., construct different functions in $\mathcal{I}(\mathrm{Sym}^r \bar{\mathbb{Q}}_p^2)$ to eliminate the contribution of $\mathcal{I}(W_m)$ in $\bar{\Theta}_{k',a}$ in a similar fashion as for $m = 0$.

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Consequence: the map $P : \mathcal{I}(V_r) \twoheadrightarrow \bar{\Theta}_{k',a}$ induces a surjection

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