p-adic modular forms and Completed Cohomology

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Topics

1. Recollections
2. Completed Cohomology : p-adic automorphic forms
3. Galois representations

Completed Cohomology

Definition

- Recall the definition of completed cohomology -

$$\tilde{H}^i(K^p, \mathbb{Z}/p^N\mathbb{Z}) := \varinjlim_{K_p} H^i(Y(K_f^p K_p), \mathbb{Z}/p^N\mathbb{Z})$$

$$\tilde{H}^i(K^p) := \varprojlim_{N} \tilde{H}^i(K^p, \mathbb{Z}/p^N\mathbb{Z})$$

- $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ is a fixed tame level.

Completed Cohomology
(Compactly supported)

- We need the version for compactly supported cohomology, which is defined in the same way -

$$\tilde{H}^i_c(K^p) := \varprojlim_N \varinjlim_{K_p} H^i_c(K^p, \mathbb{Z}/p^N\mathbb{Z})$$

- Today we will try and understand the theorem whose rough assertion is -

Theorem

All Hecke eigenvalues appearing in $\tilde{H}^i_c(K^p)$ can be p-adically interpolated by Hecke eigenvalues coming from classical cusp forms.

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Galois representations

- The group in question is a group giving rise to a Shimura variety of Hodge type. We will assume it to be Sp_{2g} for ease of exposition.
- Combining this theorem with existence of Galois representations in the case of Symplectic (or Unitary) Shimura varieties, we can see that there are associated Galois representations to eigensystems appearing in $\tilde{H}^i_c(K^p)$ in this case.
- Our interest in these groups also spans from the fact that the group $Res_{\mathcal{O}_F/\mathbb{Z}}GL_n$ can be realized as a Levi subgroup of these groups.

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- It is a standard fact that this Levi subgroup contributes to the boundary cohomology.
- Looking at this boundary contribution, we get a desired Galois representation associated to the group $Res_{\mathcal{O}_F/\mathbb{Z}}GL_n$, except that this representation would be $2n+1$ or $2n$ dimensional.
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Roadmap
Outline of proof

- Make the statement of the theorem precise.
- The proof has the following key ingredients -
  (1) A comparison isomorphism from p-adic Hodge theory of rigid analytic varieties for cohomology with torsion coefficients.
- The result is roughly that one can compute the compactly supported cohomology groups as the étale cohomology groups of the sheaf of cusp forms of infinite level.

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- (2) Existence of a perfectoid Shimura varieties as constructed in previous talks.
- Using a version of almost purity, this implies the étale cohomology can be computed by a Cech complex of an affinoid cover of (the minimal compactification of) the Shimura variety.
- This complex whose terms are 'cusp forms of infinite level on affinoid subsets' should be approximated by cusp forms of finite level *defined over all of the Shimura variety*, without disturbing the Hecke eigenvalues.

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- Classically, one has the situation where these forms are defined over the ordinary locus and multiplying by an appropriate power of the Hasse invariant to get rid of poles.
- The crucial property of Hasse invariant here is that it commutes with all Hecke operators away from p, so as to not mess up the eigensystem.
- The third key ingeredient is (3) the Hodge-Tate period map which in particular yields analogues of the Hasse invariant on arbitrary subsets of the Shimura variety.

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Perfectoid Shimura varieties

Notation

- Assume we have a Shimura datum (G, D) with $G = Sp_{2g}$ and D is the Siegel upper half space, which is the symmetric space associated to G.
- We denote by Y_K the locally symmetric space

$$Y_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

and by Y_K^* its compactification. We will assume that the tame level K^p is contained in the level-N subgroup for some $N \geq 3$ prime to p.
- Under a fixed isomorphism $\bar{Q}_p \cong \mathbb{C}$, we can base change to a fixed algebraic closure C of $\bar{Q}_p$. Let $\mathcal{Y}_K^*$ be the associated adic space to the compactification over C.

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$$\mathcal{Y}^*_{K^p} \sim \varprojlim_{K_p} \mathcal{Y}^*_{K^p K_p}.$$

- Moreover, there is a $G(\mathbb{Q}_p)$-equivariant map

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p-adic automorphic forms
First and second Key ingredients

- Let $\mathcal{I}_{\mathcal{Y}^*_{K^p}}$ denote the ideal sheaf of the boundary.

Theorem

There is a natural isomorphism of almost-$\mathcal{O}_C$-modules

$$\tilde{H}^i_c(\mathbb{Z}/p^m\mathbb{Z}) \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}^a_c/p^m \cong H^i(\mathcal{Y}^*_{K^p}, \mathcal{I}^{+a}_{\mathcal{Y}^*_{K^p}}/p^m).$$

- This isomorphism is compatible with natural inclusions and trace maps as the tame level varies.
- The right hand side is essentially (almost) the cohomology of sheaf of p-adic cusp forms modulo p^m at infinite level. It is computed on the topological space of $\mathcal{Y}^*_{K^p}$.

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- Let $j_K : \mathcal{Y}^*_K \setminus Z_K \hookrightarrow \mathcal{Y}^*_K$ be the open embedding of the complement of the boundary Z_K. By various comparison theorems at finite level, we have -

$$H^i_c(Y_{K^pK_p}, \mathbb{Z}/p^m\mathbb{Z}) \cong H^i_{et}(\mathcal{Y}^*_{K^pK_p}, j_{K!}\mathbb{Z}/p^m\mathbb{Z})$$

- Using Scholze's comparison theorem for rigid analytic varieties, and passing to an inverse limit, we get -

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- Then the proof follows after one more application of Scholze's comparison to pass from étale cohomology to the analytic topology after some homological algebra +
- Strong Zariski-closed property of the boundary, which is part of the existence of the perfectoid Shimura variety.

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Completed cohomology
Vanishing results

- As a corollary, one can derive the following result.

Corollary

If d is the dimension of Y_K over $\mathbb{C}$, the groups $\tilde{H}^i_c(K^p, \mathbb{Z}/p^m\mathbb{Z})$ (and thus $\tilde{H}^i_c(K^p, \mathbb{Z}_p)$) vanish for $i > d$ for all m.

- One can also prove the cohomological vanishing for $\tilde{H}^i$ using this and Poincaré duality.

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p-adic automorphic forms

Hecke algebras and eigensystems

- Recall that we have defined the Hecke algebra as the algebra of double coset operators -

$$\mathbb{T} := \mathbb{T}_{K^p} = \mathbb{Z}_p[G(K^p)\backslash G(\mathbb{A}_f^p)/G(K^p)]$$

which also has an avatar as an algebra of compactly supported, bi-$G(K^p)$-invariant functions on $G(\mathbb{A}_f^p)$.

- There is a line bundle ω_K on Y_K given by the dual of the determinant of the Lie algebra of the universal abelian variety on Y_K.
- $\mathbb{T}$ acts on the spaces of sections $H^0(Y_{K^pK_p}, \omega_{K^pK_p}^{\otimes k})$ naturally.

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p-adic automorphic forms
Key result

- The key result roughly says that all Hecke eigenvalues appearing in $\tilde{H}^i_c(K^p, \mathbb{Z}_p)$ come via p-adic interpolation from $H^0(\mathcal{Y}_{K^pK_p}, \omega^{\otimes k}_{K^pK_p} \otimes \mathcal{I})$.

Theorem

Fix some integer $r \geq 1$. Let $\mathbb{T} = \mathbb{T}_{cl}$ denote $\mathbb{T}$ with the weakest topology for which all the maps

$$\mathbb{T} \to End_C(H^0(Y_{K^pK_p}, \omega^{\otimes rk}_{K^pK_p} \otimes I)$$

are continuous for varying $k \geq 1$ and K_p. Then the map

$$\mathbb{T} \to End_{\mathbb{Z}/p^m\mathbb{Z}}(\tilde{H}^i_c(K^p, \mathbb{Z}/p^m\mathbb{Z}))$$

is continuous.

p-adic automorphic forms
Key result

- The action of $\mathbb{T}$ on $H^0(\mathcal{Y}_{K^pK_p}, \omega_{K^pK_p}^{\otimes k} \otimes \mathcal{I})$ is given by trace maps.
- The proof begins with the previous theorem -

$$\tilde{H}_c^i(\mathbb{Z}/p^m\mathbb{Z}) \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_c^a/p^m \cong H^i(\mathcal{Y}_{K^p}^*, \mathcal{I}_{\mathcal{Y}_{K^p}^*}^{+a}/p^m).$$

- The assertion can be reduced using this to the claim that

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Third key ingredient

- At this point, we make use of the Hodge-Tate period map.
- One obtains an affinoid cover $\mathcal{V}_J$ of the perfectoid Shimura variety via pullback from $\mathcal{FL}$ for $J \subset \{1, 2, \cdots, \binom{2g}{g}\}$ nonempty, and reduces via Cech complex computations to show that it suffices to prove that

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Third key ingredient : finishing the proof

- At this point, we again use the Hodge-Tate period map to construct the analogues of the Hasse invariant in the classical case.
- We are again led via a series of computations and reductions (involving a 'strange' formal model of the finite level Shimura variety) to showing that

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is continuous, which holds by assumption.

Recollections

Some notation

- Note that even though we chose $G = Sp_{2g}$ for ease of exposition, we can have (G, D) of Hodge type for the key result.
- In this section, we restrict to the case F totally real, $G = Res_{F/\mathbb{Q}} Sp_{2n}$. In the case F is CM, one could deal with a similar restriction of a unitary group. In all cases, (G, D) is of Hodge type. Also, G admits $Res_{F/\mathbb{Q}} GL_n$ as a maximal Levi.
- Also, G is a twisted endoscopic group of $H := Res_{F/\mathbb{Q}} H_0$, where $H_0 = GL_h/F$ with $h = 2n + 1$. (Similar properties exist for the CM case with $h = 2n$.)

Recollections

Endoscopic Transfer and Galois representations

- We know that all Hecke eigenvalues in completed cohomology interpolate from classical automorphic forms. So we would like to obtain our Galois representations by interpolation from the associated automorphic forms as well.
- Before we can do this, we mention that endoscopic transfer results are required in order to get to GL_n from G. These are available by work of Arthur in the case we want. (Mok in the unitary case.)
- To reduce notation and save time, we directly combine these results with existence of Galois representations and state the following theorem.

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Recollections

Galois representations

Theorem

Let Π be a cuspidal automorphic representation of $Res_{F/\mathbb{Q}}GL_n$ such that Π is self-dual and $\Pi|.|^{k/2}$ is regular L-algebraic for some integer k. Then there exists a continuous semisimple representation

$$\sigma_\Pi : Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_p)$$

such that $\sigma_\Pi^\vee \cong \sigma_\pi \chi_p^k$, where χ_p is the p-adic cyclotomic character, such that the Langlands parameters at all unramified places match.

- We next record the Satake isomorphism in order to characterize representations by their Hecke polynomials.

Hecke algebra
Notations

- Let S be the finite set of finite places of F containing all ramified places of F and all places above p.
- Let $\mathbb{T} = \mathbb{T}^S$ be the abstract Hecke algebra

$$\mathbb{T} = \bigotimes_{\nu \notin S} \mathbb{T}_\nu$$

where

$$\mathbb{T}_\nu := \mathbb{Z}_p G_0(K_\nu \backslash G_0(F_\nu)/K_\nu)$$

is the Hecke algebra as before. (K_ν is the level as always.)

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Hecke algebra
Satake isomorphism

- The Satake isomorphism in this case is -

Theorem

Fix $\nu \notin S$. Let q_ν be the cardinality of the residue field. Then the Satake transform gives a canonical isomorphism

$$\mathbb{T}_\nu[q_\nu^{1/2}] \cong \mathbb{Z}_p[q_\nu^{1/2}][X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n}.$$

The unramified endoscopic transfer from $G_0(F_\nu)$ to $GL_{2n+1}(F_\nu)$ is given by

$$\mathbb{Z}_p[q_\nu^{1/2}][Y_1^{\pm 1}, \cdots, Y_{2n+1}^{\pm 1}]^{S_{2n+1}} \to \mathbb{Z}_p[q_\nu^{1/2}][X_i^{\pm 1}]_{i=1,2,\ldots,n}^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n}$$

sending the set $\{Y_1, \ldots, Y_{2n+1}\}$ to $\{X_1^{\pm 1}, \ldots, X_n^{\pm 1}, 1\}$.

Hecke algebra

Satake isomorphism

- (Continued) Let $T_i \in \mathbb{Z}_p[q_\nu^{1/2}][Y_1^{\pm 1}, \cdots, Y_{2n+1}^{\pm 1}]^{S_{2n+1}}$ be the i-th symmetric polynomial in the variables Y_j's for $i = 1, 2, \ldots, 2n+1$, and let $T_{i,\nu}$ be its image in $\mathbb{T}_\nu[q_\nu^{1/2}]$. Then $T_{i,\nu} \in \mathbb{T}_\nu$.
- Fix $k > n$. Let $\mathbb{T}_{K,k}$ be the image of $\mathbb{T}$ in

$$End_C(H^0(Y_K^*, \omega_K^{\otimes k} \otimes \mathcal{I}) \otimes_{\mathbb{C}} C).$$

Then, for any $x \in (\text{Spec } \mathbb{T}_{K,k})(\bar{\mathbb{Q}}_p)$, we have the following corollary from the previous discussion.

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Then, for any $x \in (\text{Spec } \mathbb{T}_{K,k})(\bar{\mathbb{Q}}_p)$, we have the following corollary from the previous discussion.

Hecke algebra

Galois representation associated to eigensystems

Corollary

There exists a continuous semisimple representation

$$\sigma_x : G_{F,S} \to GL_h(\bar{\mathbb{Q}}_p)$$

such that σ_x is self-dual, and such that for any $\nu \notin S$, the 'number theorist's characteristic polynomial' of the geometric Frobenius $Frob_\nu$ is given by

$$det(1 - X.Frob_\nu|_{\sigma_x}) = 1 - T_{1,\nu}(x)X + T_{2,\nu}(x)X^2 - \ldots - T_{h,\nu}(x)X^h.$$

$(h = 2n + 1)$.

- This can be stated in terms of Chenevier's determinants.

Hecke algebra
Determinants

- We have the following definition -

Definition

Let A be a topological ring and G be a topological group. A d-dimensional determinant is an A-polynomial law $D : A[G] \to A$ which is multiplicative and homogenous of degree d. For any $g \in G$, we call $D(1 - Xg) \in A[X]$ the characteristic polynomial of g.

- An A-polynomial law between two A-modules M and N is a natural transformation $M \otimes B \to N \otimes B$ on the category of B-modules for A-algebras B.
- Multiplicative means that it commutes with multiplication of elements, and homogenous of degree d means $D(bx) = b^d D(x)$.

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Hecke algebra

Existence of determinants

- If $\rho : G \to GL_d(A)$ is a continuous representation, $D = \det \circ \rho$ gives an example of a determinant in the above sense.
- Then we can record the following corollary -

Corollary

Let $\mathbb{T} = \mathbb{T}_{cl}$ be the Hecke algebra from before. Then for any continuous quotient $\mathbb{T} \to A$ with A discrete, there exists a unique continuous h-dimensional determinant D of $G_{F,S}$ with values in A such that

$$D(1 - X.Frob_\nu) = 1 - T_{1,\nu}X + T_{2,\nu}X^2 - \ldots - T_{h,\nu}X^h$$

for all finite places $\nu \notin S$.

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Galois representations
At the boundary

- Recall that our group G contains $M = Res_{F/\mathbb{Q}} GL_n$ as a Levi. This implies that the cohomology of locally symmetric spaces associated to M contributes to the cohomology of Y_K.
- We directly write the result of computations at the boundary below. These computations in particular use the previous corollary and the key automorphic result we saw in Section 2.
- Fix n. Let F, h, S be as before, with S being invariant under complex conjugation. Define $d = [F : \mathbb{Q}]((n^2 + n)/2)$. Let $K \subset GL_n(\mathbb{A}_{F,f})$ be 'sufficiently small' with K_S torsion-free and $K^S = \prod_{\nu \notin S} GL_n(\mathcal{O}_\nu)$.

Galois representations
At the boundary

- Let

$$X_K = GL_n(F)\backslash[(GL_n(F\otimes\mathbb{R})/\mathbb{R}_{\geq 0}K_\infty)\times GL_n(\mathbb{A}_{F,f})/K]$$

denote the locally symmetric space associated with GL_n/F.

- Let $\mathbb{T}_{F,S}(K,i,m) := im(\mathbb{T}_{F,S}\to End_{\mathbb{Z}/p^m\mathbb{Z}}H^i_!(X_K,\mathbb{Z}/p^m\mathbb{Z}))$ be the Hecke algebra.

Corollary

Then there is an ideal $I\subset\mathbb{T}_{F,S}(K,i,m)$ with $I^{2d+1}=0$ such that there is a continuous h-dimensional determinant $\tilde{D}$ of $G_{F,S}$ with values in $\mathbb{T}_{F,S}(K,i,m)/I$, with a suitable matching of characteristic polynomials.

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Galois representations

Going from h to n

- As mentioned before, the theory of endoscopic transfer only gives us access to a $(2n+1)$-dimensional representation as seen before.
- The choice of S is made with a corresponding choice of characteristic polynomials (which we did not write explicitly in the previous slide) so as to be able to isolate the 'correct' n-dimensional part from it.
- We can prove the following desired theorem.

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Galois representations

Main Theorems

- Let $P_\nu(X) = 1 - T_{1,\nu}X + T_{2,\nu}X^2 - \ldots + (-1)^n T_{n,\nu}X^n$.

Theorem

There is an ideal $J \subset \mathbb{T}_{F,S}(K,i,m)$ with $J^{4(d+1)} = 0$ such that there is a continuous n-dimensional determinant D of $G_{F,S}$ with values in $\mathbb{T}_{F,S}(K,i,m)/J$ such that

$$D(1 - X.Frob_\nu) = P_\nu(X)$$

for all $\nu \notin S$. (The equality is modulo the ideal J.)

Galois representations
Main Theorems

- The proof of this theorem follows the idea of using the existence of $\tilde{D}$ (from the previous corollary) for many cyclotomic twists to carve out D.
- Intuitively, the idea is - say you wish to construct representations $\pi_m, m \in \mathbb{Z}$ of G with prescribed characteristic polynomials. If you know that for any character χ of $\mathbb{Z}$, the representation

$$\bigoplus_{m \in \mathbb{Z}} \pi_m \otimes \chi^m$$

of $G \times \mathbb{Z}$ exists. Then all the representations π_m exist.

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Main Theorems

- Main theorem can be proven for local systems arising from an algebraic representation of $Res_{F/\mathbb{Q}}GL_n$ as coefficients.
- We obtain the first corollary of this theorem for classical automorphic representations -

Corollary

Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ such that π_∞ is regular L-algebraic, and such that π_ν is unramified at all places $\nu \notin S$. Then there exists a unique continuous semisimple representation

$$\sigma_\pi : G_{F,S} \to GL_n(\bar{\mathbb{Q}}_p)$$

such that for all finite places $\nu \notin S$, the Satake parameters of π_ν agree with the eigenvalues of $\sigma_\pi(Frob_\nu)$.

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Galois representations
Torsion classes !

- For eigensystems valued in $\bar{\mathbb{F}}_p$, we get the existence of Galois representations -

Theorem

Let $\psi : \mathbb{T}_{F,S} \to \bar{\mathbb{F}}_p$ be a system of Hecke eigenvalues such that

$$H^i(X_K, \bar{\mathbb{F}}_p)[\psi] \neq 0.$$

Then, there exists a unique continuous semisimple representation

$$\sigma_\psi : G_{F,S} \to GL_n(\bar{\mathbb{F}}_p)$$

such that for all finite places $\nu \notin S$,

$$\det(1 - X.Frob_\nu|_{\sigma_\psi}) = 1 - \psi(T_{1,\nu})X + \ldots + (-1)^n \psi(T_{n,\nu})X^n.$$

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