Local Shtukas and Divisible Local Anderson Modules

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September 11, 2019

In the arithmetic of number fields elliptic curves and abelian varieties are important objects. Their theory has been vastly developed in the last two centuries and their moduli spaces have played a major role in Faltings's proof of the Mordell conjecture(1982), the proof of Fermat's Last Theorem by Wiles and Taylor(1993), and the proof of the Langlands correspondence for GL_n over non-archimedean local fields of characteristic zero by Harris and Taylor(2001).

A useful tool to study abelian varieties and their moduli spaces are p-divisible groups. More precisely, for an elliptic curve or an abelian variety E over a $\mathbb{Z}_p$-algebra R the *p-divisible group* $E[p^\infty] = \varinjlim E[p^n]$, also called *Barsotti-Tate group*, captures the local p-adic information of E. One reason why $E[p^\infty]$ is a useful tool to study E is that the complicated arithmetic data of a p-divisible group over a $\mathbb{Z}_p$-algebra R in which p is nilpotent can be faithfully encoded by an object of semi-linear algebra, its Dieudonné module.

Elliptic curves and abelian varieties have analogs in the arithmetic of function fields. Namely, Drinfeld invented the notions of *elliptic modules* (today called *Drinfeld modules*) and the dual notion of *F-sheaves* (today called *Drinfeld shtukas*). These structures are function field analogs of elliptic curves in the following sense. Their endomorphism rings are rings of integers in global function fields of positive characteristic or orders in central division algebras over the later. On the other hand, their moduli spaces are varieties over smooth curves over a finite field. Through these two aspects in which global function fields of positive characteristic come into play, Drinfeld shtukas and variants of them proved to be fruitful for establishing large parts of the Langlands program over local and global function fields of positive characteristic by Drinfeld, Laumon, Rapoport and Stuhler, L. Lafforgue and V. Lafforgue. Beyond this the analogy between Drinfeld modules and elliptic curves is abundant.

In this talk we will define Local Shtukas and Divisible Local Anderson Modules and will give some explicit examples. In theory of arithmetic of function fields p-divisible groups are replaced by divisible local Anderson modules, and Dieudonné modules are replaced by local shtukas.

Let $\mathbb{F}_q$ be a finite field with q elements and characteristic p. For a scheme S over $\operatorname{Spec}\mathbb{F}_q$ and a positive integer $n \in \mathbb{N}_{>0}$ we denote by $\sigma_{q^n} := \operatorname{Frob}_{q^n,S} \colon S \to S$ its absolute q^n-Frobenius endomorphism which acts as the identity on points and as the q^n-power map $b \mapsto b^{q^n}$ on the structure sheaf.

For an S-scheme X, respectively an $\mathcal{O}_S$-module M we write $\sigma_{q^n}^* X := X \times_{S,\sigma_{q^n}} S$, respectively $\sigma_{q^n}^* M := M \otimes_{\mathcal{O}_S,\sigma_{q^n}^*} \mathcal{O}_S$ for the pullback under σ_{q^n}. For $m \in M$ we also write $\sigma_{q^n}^*(m) := m \otimes 1 \in \sigma_{q^n}^* M$ and note that $\sigma_{q^n}^*(bm) = bm \otimes 1 = m \otimes b^{q^n} = b^{q^n} \cdot \sigma_q^* m$ for $b \in \mathcal{O}_S$ and $m \in M$.

Let z be an indeterminant over $\mathbb{F}_q$. Let $\mathcal{O}_S[\![z]\!]$ be the sheaf on S of formal power series in z. That is $\Gamma\big(U, \mathcal{O}_S[\![z]\!]\big) = \Gamma(U, \mathcal{O}_S)[\![z]\!]$ for open $U \subset S$ with the obvious restriction maps.

This is indeed a sheaf being the countable direct product of $\mathcal{O}_S$. Let ζ be an indeterminant over $\mathbb{F}_q$ and let $\mathbb{F}_q[\![\zeta]\!]$ be the ring of formal power series in ζ over $\mathbb{F}_q$. Let $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ be the category of $\mathbb{F}_q[\![\zeta]\!]$-schemes on which ζ is locally nilpotent.

For $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ let $\mathcal{O}_S(\!(z)\!)$ be the sheaf of $\mathcal{O}_S$-algebras on S associated with the presheaf $U \mapsto \Gamma(U, \mathcal{O}_U)[\![z]\!][\frac{1}{z}]$. If U is quasi-compact then $\mathcal{O}_S(\!(z)\!)(U) = \Gamma\big(U, \mathcal{O}_S[\![z]\!]\big)[\frac{1}{z}]$. Since ζ is locally nilpotent on S, the sheaf $\mathcal{O}_S(\!(z)\!)$ equals the sheaf associated with the presheaf $U \mapsto \Gamma\big(U, \mathcal{O}_S[\![z]\!]\big)[\frac{1}{z-\zeta}]$. We denote by σ_q^* the endomorphism of $\mathcal{O}_S[\![z]\!]$ and $\mathcal{O}_S(\!(z)\!)$ that acts as the identity on z and as $b \mapsto b^q$ on local sections $b \in \mathcal{O}_S$.

For a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules on S we let

$$\sigma_q^* M := M \otimes_{\mathcal{O}_S[\![z]\!], \sigma_q^*} \mathcal{O}_S[\![z]\!]$$

and

$$M[\frac{1}{z-\zeta}] := M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}] = M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$$

be the tensor product sheaves. Also for a section $m \in M$ we write $\sigma_q^* m := m \otimes 1 \in \sigma_q^* M$. Note that a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules which fpqc-locally on S is isomorphic to $\mathcal{O}_S[\![z]\!]^{\oplus r}$ is already Zariski-locally on S isomorphic to $\mathcal{O}_S[\![z]\!]^{\oplus r}$. We therefore call such a sheaf simply a *locally free sheaf of $\mathcal{O}_S[\![z]\!]$-modules of rank r.*

Let S be a scheme in $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$.

Definition

A *local shtuka of rank* (*or height*) r over S is a pair $\underline{M} = (M, F_M)$ consisting of a locally free sheaf M of $\mathcal{O}_S[\![z]\!]$-modules of rank r, and an isomorphism $F_M \colon \sigma_q^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$.

A *morphism* of local shtukas $f \colon (M, F_M) \to (M', F_{M'})$ over S is a morphism of the underlying sheaves $f \colon M \to M'$ which satisfies $F_{M'} \circ \sigma_q^* f = f \circ F_M$.

A *quasi-isogeny* between local shtukas $f \colon (M, F_M) \to (M', F_{M'})$ over S is an isomorphism of $\mathcal{O}_S(\!(z)\!)$-modules $f \colon M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \xrightarrow{\sim} M' \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$ with $F_{M'} \circ \sigma_q^*(f) = f \circ F_M$.

A morphism which is a quasi-isogeny is called an *isogeny*.

For any local shtuka (M, F_M) over $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ the homomorphism $M \to M[\frac{1}{z-\zeta}]$ is injective by the flatness of M and the following

Lemma

Let R be an $\mathbb{F}_q[\![\zeta]\!]$-algebra in which ζ is nilpotent. Then the sequence of $R[\![z]\!]$-modules

$$0 \longrightarrow R[\![z]\!] \longrightarrow R[\![z]\!] \longrightarrow R \longrightarrow 0$$

$$1 \longmapsto z-\zeta \ , \quad z \longmapsto \zeta$$

is exact. In particular $R[\![z]\!] \subset R[\![z]\!][\frac{1}{z-\zeta}]$.

Proof.

If $\sum_i b_i z^i$ lies in the kernel of the first map, that is, $0 = (z-\zeta)(\sum_i b_i z^i) = \sum_i (b_{i-1} - \zeta b_i) z^i$, then $b_i = \zeta b_{i+1} = \zeta^n b_{i+n}$ for all n. Since ζ is nilpotent, all b_i are zero. Also due to the nilpotency of ζ the second map is well defined and surjective. For exactness in the middle note that $\sum_i b_i \zeta^i = 0$ implies $\sum_i b_i z^i = \sum_i b_i (z^i - \zeta^i)$ which is a multiple of $z - \zeta$. □

Lemma

Let (M, F_M) be a local shtuka over S. Then locally on S there are $e, e', N \in \mathbb{Z}$ such that $(z-\zeta)^{e'} M \subset F_M(\sigma_q^* M) \subset (z-\zeta)^{-e} M$ and $z^N M \subset F_M(\sigma_q^* M)$. For any such e the map $F_M\colon \sigma_q^* M \to (z-\zeta)^{-e} M$ is injective, and the quotient $(z-\zeta)^{-e} M / F_M(\sigma_q^* M)$ is a locally free $\mathcal{O}_S$-module of finite rank.

Definition

A local shtuka $\underline{M} = (M, F_M)$ over S is called *effective* if F_M is actually a morphism $F_M\colon \sigma_q^* M \hookrightarrow M$. Let (M, F_M) be effective of rank $r = \operatorname{rk} \underline{M}$. We say that

1. (M, F_M) has *dimension* d if $\operatorname{coker} F_M$ is locally free of rank d as an $\mathcal{O}_S$-module.
2. (M, F_M) is *étale* if $F_M\colon \sigma_q^* M \xrightarrow{\sim} M$ is an isomorphism.
3. F_M is *topologically nilpotent* if locally on S there is an integer n such that $\operatorname{im}(F_M^n) \subset zM$, where $F_M^n := F_M \circ \sigma_q^* F_M \circ \ldots \circ \sigma_{q^{n-1}}^* F_M\colon \sigma_{q^n}^* M \to M$.

More generally, let now S be an arbitrary $\mathbb{F}_q$-scheme.

Definition

A *finite* $\mathbb{F}_q$*-shtuka* over S is a pair $\underline{M} = (M, F_M)$ consisting of a locally free $\mathcal{O}_S$-module M on S of finite rank denoted $\operatorname{rk} \underline{M}$, and an $\mathcal{O}_S$-module homomorphism $F_M \colon \sigma_q^* M \to M$.
satisfying $f \circ F_M = F_{M'} \circ \sigma_q^* f$.

We denote the category of finite $\mathbb{F}_q$-shtukas over S by $\mathbb{F}_q\text{-Sht}_S$.

A finite $\mathbb{F}_q$-shtuka over S is called *étale* if F_M is an isomorphism. We say that F_M is *nilpotent* if there is an integer n such that $F_M^n := F_M \circ \sigma_q^* F_M \circ \ldots \circ \sigma_{q^{n-1}}^* F_M = 0$.

Proposition (Laumon)

If S is the spectrum of a field L every finite $\mathbb{F}_q$-shtuka $\underline{M} = (M, F_M)$ is canonically an extension of finite $\mathbb{F}_q$-shtukas

$$0 \longrightarrow (M_{\text{ét}}, F_{\text{ét}}) \longrightarrow (M, F_M) \longrightarrow (M_{\text{nil}}, F_{\text{nil}}) \longrightarrow 0$$

where $F_{\text{ét}}$ is an isomorphism and F_{nil} is nilpotent. $\underline{M}_{\text{ét}} = (M_{\text{ét}}, F_{\text{ét}})$ is the largest étale finite $\mathbb{F}_q$-sub-shtuka of $\underline{M}$ and equals $\text{im}(F_M^{\text{rk}\,\underline{M}})$. *If L is perfect this extension splits canonically.*

Example. Every effective local shtuka (M, F_M) of rank r over S yields for every $n \in \mathbb{N}$ a finite $\mathbb{F}_q$-shtuka $\left(M/z^nM, F_M \bmod z^n\right)$ of rank rn, and (M, F_M) equals the projective limit of these finite $\mathbb{F}_q$-shtukas.

Proposition

If S is the spectrum of a field L in $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ every effective local shtuka (M, F_M) is canonically an extension of effective local shtukas

$$0 \longrightarrow (M_{\text{ét}}, F_{\text{ét}}) \longrightarrow (M, F_M) \longrightarrow (M_{\text{nil}}, F_{\text{nil}}) \longrightarrow 0$$

where $F_{\text{ét}}$ is an isomorphism and F_{nil} is topologically nilpotent. $(M_{\text{ét}}, F_{\text{ét}})$ is the largest étale effective local sub-shtuka of (M, F_M). If L is perfect this extension splits canonically. □

Let C be a smooth, projective, geometrically irreducible curve over $\mathbb{F}_q$. For an $\mathbb{F}_q$-scheme S we set $C_S := C \times_{\mathbb{F}_q} S$ and we consider the endomorphism $\sigma_q := \mathrm{id}_C \otimes \mathrm{Frob}_{q,S} \colon C_S \to C_S$.

Definition

1. Let n and r be positive integers. A *global shtuka of rank r with n legs* over an $\mathbb{F}_q$-scheme S is a tuple $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \ldots, c_n, \tau_{\mathcal{N}})$ consisting of
   - a locally free sheaf $\mathcal{N}$ of rank r on C_S,
   - $\mathbb{F}_q$-morphisms $c_i \colon S \to C$ called the *legs of $\underline{\mathcal{N}}$* and
   - an isomorphism $\tau_{\mathcal{N}} \colon \sigma_q^* \mathcal{N}|_{C_S \smallsetminus \bigcup_i \Gamma_{c_i}} \xrightarrow{\sim} \mathcal{N}|_{C_S \smallsetminus \bigcup_i \Gamma_{c_i}}$ outside the graphs Γ_{c_i} of the c_i.

   We will only consider the case $\Gamma_{c_i} \cap \Gamma_{c_j} = \emptyset$ for $i \neq j$.
2. A global shtuka over S is a *Drinfeld shtuka* if $n = 2$, $\Gamma_{c_1} \cap \Gamma_{c_2} = \emptyset$, and $\tau_{\mathcal{N}}$ satisfies $\tau_{\mathcal{N}}(\sigma_q^* \mathcal{N}) \subset \mathcal{N}$ on $C_S \smallsetminus \Gamma_{c_2}$ with cokernel locally free of rank 1 as $\mathcal{O}_S$-module, and $\tau_{\mathcal{N}}^{-1}(\mathcal{N}) \subset \sigma_q^* \mathcal{N}$ on $C_S \smallsetminus \Gamma_{c_1}$ with cokernel locally free of rank 1 as $\mathcal{O}_S$-module.

Drinfeld shtukas were introduced by Drinfeld under the name F-sheaves.

An important class of special examples is defined as follows. Let $\infty \in C$ be a closed point and put $A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$. Then $\operatorname{Spec} A = C \smallsetminus \{\infty\}$. We will consider affine A-schemes $c\colon S = \operatorname{Spec} R \to \operatorname{Spec} A$ and the ideal

$$J := (a \otimes 1 - 1 \otimes c^*(a)\colon a \in A) \subset A_R := A \otimes_{\mathbb{F}_q} R$$

whose vanishing locus $\mathrm{V}(J)$ is the graph Γ_c of the morphism c. The endomorphism

$$\sigma_q := \mathrm{id}_C \otimes \mathrm{Frob}_{q,S}\colon C_S \to C_S$$

induces the ring endomorphism

$$\sigma_q^* := \mathrm{id}_A \otimes \mathrm{Frob}_{q,R}\colon A_R \to A_R,\ a \otimes b \mapsto a \otimes b^q$$

of A_R for $a \in A$ and $b \in R$. The following definition generalizes Anderson's notion of *t-motives*, which is obtained as the special case, where $C = \mathbb{P}^1$, $A = \mathbb{F}_q[t]$ and R is a field.

Definition

Let d and r be positive integers and let $S = \operatorname{Spec} R$ be an affine A-scheme. An *effective A-motive of rank r and dimension d* over S is a pair $\underline{N} = (N, \tau_N)$ consisting of a locally free A_R-module N of rank r and a morphism

$$\tau_N \colon \sigma_q^* N \to N$$

of A_R-modules, such that $\operatorname{coker} \tau_N$ is a locally free R-module of rank d and $J^d \cdot \operatorname{coker} \tau_N = 0$.

More generally, an *A-motive of rank r* over S is a pair $\underline{N} = (N, \tau_N)$ consisting of a locally free A_R-module N of rank r and an isomorphism $\tau_N \colon \sigma_q^* N|_{\operatorname{Spec} A_R \smallsetminus \mathrm{V}(J)} \xrightarrow{\sim} N|_{\operatorname{Spec} A_R \smallsetminus \mathrm{V}(J)}$ outside the vanishing locus $\mathrm{V}(J) = \Gamma_c$ of J.

Now I will explain how Drinfeld shtukas can be realized as A-motives and A-motives can be realized as Drinfeld shtukas.

(a) If $\underline{\mathcal{N}} = (\mathcal{N}, c_1, c_2, \tau_{\mathcal{N}})$ is a global shtuka of rank r over $S = \operatorname{Spec} R$ with two legs such that $c_1 = c$ and $c_2 \colon S \to \{\infty\} \subset C$, then $\underline{N}(\underline{\mathcal{N}}) := (N, \tau_N) := \big(\Gamma(\operatorname{Spec} A_R, \mathcal{N}), \tau_{\mathcal{N}}\big)$ is an A-motive of rank r over S.

(b) Conversely, if $\infty \in C(\mathbb{F}_q)$, every A-motive $\underline{N} = (N, \tau_N)$ over an affine A-scheme $c \colon S = \operatorname{Spec} R \to \operatorname{Spec} A$ can be obtained from a global shtuka $\underline{\mathcal{N}} = (\mathcal{N}, c_1, c_2, \tau_{\mathcal{N}})$ by taking $c_1 = c$ and $c_2 \colon S \to \{\infty\} \subset C$, and taking $\mathcal{N}$ as an extension to C_S of the sheaf associated with N on $\operatorname{Spec} A_R$, and $\tau_{\mathcal{N}} = \tau_N$.

These global objects give rise to finite and local shtukas, and that motivates the names for the latter.
(a) Let $i\colon D \hookrightarrow C$ be a finite closed subscheme and let $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \ldots, c_n, \tau_{\mathcal{N}})$ be a global shtuka of rank r over S such that $\tau_{\mathcal{N}}(\sigma_q^*\mathcal{N}) \subset \mathcal{N}$ in a neighborhood of $D_S := D \times_{\mathbb{F}_q} S$. (For example this is satisfied if $\underline{\mathcal{N}}$ is a Drinfeld-shtuka and $D_S \cap \Gamma_{c_2} = \emptyset$ or if $\underline{\mathcal{N}}$ is as in last Example with $\underline{N}(\underline{\mathcal{N}})$ an effective A-motive and $D \subset \operatorname{Spec} A$.) Then

$$(M, F_M) \;:=\; (i^*\mathcal{N}\,,\, i^*\tau_{\mathcal{N}})$$

is a finite $\mathbb{F}_q$-shtuka over S, because M is locally free over S of rank $r \cdot \dim_{\mathbb{F}_q} \mathcal{O}_D$. The sense in which $\underline{\mathcal{N}}$ is *global* and (M, F_M) is *finite*, is with respect to the *coefficients*: $\underline{\mathcal{N}}$ lives over all of C and $\underline{M}$ lives over the finite scheme D. This example gave rise to the name “finite $\mathbb{F}_q$-shtuka”.

b) Let $v \in C$ be a closed point defined by a sheaf of ideals $\mathfrak{p} \subset \mathcal{O}_C$, let $\hat{q}$ be the cardinality of the residue field $\mathbb{F}_v$ of v, let $f := [\mathbb{F}_v : \mathbb{F}_q]$, and let $z \in \mathbb{F}_q(C)$ be a uniformizing parameter at v. Let $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \ldots, c_n, \tau_{\mathcal{N}})$ be a global shtuka of rank r over $S = \operatorname{Spec} R$ such that for some i the elements of $c_i^*(\mathfrak{p})$ are nilpotent in R. Set $\zeta := c_i^*(z) \in R$. Then the formal completion of C_S along the graph Γ_{c_i} of c_i is canonically isomorphic to $\operatorname{Spf} R[\![z]\!]$. The formal completion M of $(\mathcal{N}, \tau_{\mathcal{N}})$ along Γ_{c_i} together with $\tau_M := \tau_{\mathcal{N}}^f : \sigma_q^{f*} M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ is a local shtuka over S of rank r (as in Definition of Local Shtuka with q and $\mathbb{F}_q[\![z]\!]$ and σ_q^* replaced by $\hat{q}$ and $\mathbb{F}_v[\![z]\!]$ and σ_q^{f*}). Again $\underline{M}$ is *local* with respect to the *coefficients* as it lives over the complete local ring $\widehat{\mathcal{O}}_{C,v} = \mathbb{F}_v[\![z]\!]$ of C at v. This gave rise to the name "local shtuka".

Now we will define Drinfeld A-modules, or more generally abelian Anderson A-modules. To define them, let $c\colon S = \operatorname{Spec} R \to \operatorname{Spec} A$ be an affine A-scheme.

Recall that for a smooth commutative group scheme E over $\operatorname{Spec} R$ the co-Lie module $\omega_E := \varepsilon_E^* \Omega^1_{E/R}$ is a locally free R-module of rank equal to the relative dimension of E over R.

Moreover, on the additive group scheme $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ the elements $b \in R$, and in particular $c^*(a) \in R$ for $a \in \mathbb{F}_q \subset A$, act via endomorphisms $\psi_b\colon \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ given by $\psi_b^*\colon R[x] \to R[x]$, $x \mapsto bx$. This makes $\mathbb{G}_{a,R}$ into an $\mathbb{F}_q$-module scheme.

In addition, let $\tau := F_{q,\mathbb{G}_{a,R}}$ be the relative q-Frobenius endomorphism of $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ given by $x \mapsto x^q$. It satisfies $\tau \circ \psi_b = \psi_{b^q} \circ \tau$.

We let

$$R\{\tau\} := \{\sum_{i=0}^{n} b_i\tau^i \colon n \in \mathbb{N}_0, b_i \in R\}$$

with $\tau b = b^q\tau$ be the non-commutative polynomial ring in the variable τ over R. There is an isomorphism of rings

$$R\{\tau\} \xrightarrow{\sim} \mathrm{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R})$$

sending an element $f = \sum_i b_i\tau^i \in R\{\tau\}$ to the $\mathbb{F}_q$-equivariant endomorphism $f \colon \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ given by $f^*(x) := \sum_i b_i x^{q^i}$.

Definition

Let d and r be positive integers. An *abelian Anderson A-module of rank r and dimension d* over an affine A-scheme $c\colon \operatorname{Spec} R \to \operatorname{Spec} A$ is a pair $\underline{E} = (E, \varphi)$ consisting of a smooth affine group scheme E over $\operatorname{Spec} R$ of relative dimension d, and a ring homomorphism $\varphi\colon A \to \operatorname{End}_{R\text{-groups}}(E)$, $a \mapsto \varphi_a$ such that

1. there is a faithfully flat ring homomorphism $R \to R'$ for which $E \times_R \operatorname{Spec} R' \cong \mathbb{G}_{a,R'}^d$ as $\mathbb{F}_q$-module schemes, where $\mathbb{F}_q$ acts on E via φ and $\mathbb{F}_q \subset A$,
2. $(a \otimes 1 - 1 \otimes c^*a)^d \cdot \omega_E = 0$ for all $a \in A$ under the action of $a \otimes 1$ induced from φ_a and the natural action of $1 \otimes b$ for $b \in R$ on the R-module ω_E,
3. the set $N := M_q(\underline{E}) := \operatorname{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(E, \mathbb{G}_{a,R})$ of $\mathbb{F}_q$-equivariant homomorphisms of R-group schemes is a locally free A_R-module of rank r under the action given on $m \in N$ by
$$A \ni a\colon \quad N \longrightarrow N, \quad m \mapsto m \circ \varphi_a$$
$$R \ni b\colon \quad N \longrightarrow N, \quad m \mapsto \psi_b \circ m$$

If $d = 1$ this is called a *Drinfeld A-module* over S.

The case in which $C = \mathbb{P}^1$, $A = \mathbb{F}_q[t]$, and R is a field was considered by Anderson under the name *abelian t-module*. Hartl gave a proof of the following relative version of Anderson's theorem.

Theorem

If $\underline{E} = (E, \varphi)$ is an abelian Anderson A-module of rank r and dimension d, we consider in addition on $N := M_q(\underline{E})$ the map $\tau_N^{\text{semi}} : m \mapsto F_{q,\mathbb{G}_{a,R}} \circ m$. Since $\tau_N^{\text{semi}}(bm) = b^q \tau_N^{\text{semi}}(m)$ for $b \in R$, the map τ_N^{semi} is σ_q-semilinear and induces an A_R-linear map $\tau_N : \sigma_q^* N \to N$ with $\tau_N^{\text{semi}}(m) = \tau_N(\sigma_q^* m)$. Then $\underline{M}_q(\underline{E}) := (N, \tau_N)$ is an effective A-motive of rank r and dimension d. There is a canonical isomorphism of R-modules

$$\operatorname{coker} \tau_N \xrightarrow{\sim} \omega_E\,, \quad m \bmod \tau_N(\sigma_q^* N) \longmapsto m^*(1)\,, \tag{1}$$

where $m^*(1)$ means the image of $1 \in \omega_{\mathbb{G}_{a,R}} = R$ under the induced R-homomorphism $m^* : \omega_{\mathbb{G}_{a,R}} \to \omega_E$.
The contravariant functor $\underline{E} \mapsto \underline{M}_q(\underline{E})$ is fully faithful. Its essential image consists of all effective A-motives $\underline{N} = (N, \tau_N)$ over R for which there exists a faithfully flat ring homomorphism $R \to R'$ such that $N \otimes_R R'$ is a finite free left $R'\{\tau\}$-module under the map $\tau : N \to N$, $m \mapsto \tau_N(\sigma_q^* m)$.

The next example will motivate the definition of Divisible local Anderson modules. Let $\underline{E} = (E, \varphi)$ be an abelian Anderson A-module over an affine A-scheme $c \colon \operatorname{Spec} R \to \operatorname{Spec} A$, and let $\underline{N} := \underline{M}_q(\underline{E})$ be its associated effective A-motive.

a) Let $\mathfrak{a} \subset A$ be a non-zero ideal. The $\mathfrak{a}$*-torsion submodule of* E, defined as the scheme-theoretic intersection

$$\underline{E}[\mathfrak{a}] \;:=\; \bigcap_{a \in \mathfrak{a}} \ker(\varphi_a \colon E \to E)\,,$$

is a finite locally free $A/\mathfrak{a}$-module scheme and a strict $\mathbb{F}_q$-module scheme over S, which satisfies $\underline{M}_q(\underline{E}[\mathfrak{a}]) = \underline{N}/\mathfrak{a}\underline{N}$ and $\underline{E}[\mathfrak{a}] = \mathrm{Dr}_q(\underline{N}/\mathfrak{a}\underline{N})$.

(b) Let $\mathfrak{p} \subset A$ be a maximal ideal and assume that the elements of $c^*(\mathfrak{p}) \subset R$ are nilpotent. Let $\hat{q}$ be the cardinality of the residue field $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$ and let $f := [\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]$. We fix a uniformizing parameter $z \in \mathbb{F}_q(C) = \mathrm{Frac}(A)$ at $\mathfrak{p}$ and set $\zeta := c^*(z) \in R$. We obtain an isomorphism $\mathbb{F}_\mathfrak{p}[\![z]\!] \xrightarrow{\sim} A_\mathfrak{p} := \varprojlim A/\mathfrak{p}^n$. As before the J-adic completion M of $\underline{N}$ together with $\tau_M := \tau_N^f \colon \sigma_q^{f*} M \to M$ is an effective local shtuka over R of rank r with q and $\mathbb{F}_q[\![z]\!]$ and σ_q^* replaced by $\hat{q}$ and $\mathbb{F}_\mathfrak{p}[\![z]\!]$ and σ_q^{f*}). The torsion module $\underline{E}[\mathfrak{p}^n]$ is a finite locally free strict $\mathbb{F}_\mathfrak{p}$-module scheme which satisfies $\mathrm{Dr}_{\hat{q}}(\underline{M}/\mathfrak{p}^n\underline{M}) = \underline{E}[\mathfrak{p}^n]$ and $\underline{M}/\mathfrak{p}^n\underline{M} = \underline{M}_{\hat{q}}(\underline{E}[\mathfrak{p}^n])$. Moreover, in the sense of next definition below, the inductive limit $\underline{E}[\mathfrak{p}^\infty] := \varinjlim \underline{E}[\mathfrak{p}^n]$ is a $\mathfrak{p}$-divisible local Anderson module over R.

Divisible local Anderson modules are the function field analogs of p-divisible groups. We give the definition below analogously to Messing. We fix the following notation. For an *fppf*-sheaf of $\mathbb{F}_q[z]$-modules G over a scheme S we denote the kernel of $z^n \colon G \to G$ by $G[z^n]$. Clearly $(G[z^{n+m}])[z^n] = G[z^n]$ for all $n, m \in \mathbb{N}$.

Definition

A *z-divisible local Anderson module* over scheme $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ is a sheaf of $\mathbb{F}_q[\![z]\!]$-modules G on the big *fppf*-site of S such that

1. G is *z-torsion*, that is $G = \varinjlim_n G[z^n]$,
2. G is *z-divisible*, that is $z\colon G \to G$ is an epimorphism,
3. For every n the $\mathbb{F}_q$-module $G[z^n]$ is representable by a finite locally free strict $\mathbb{F}_q$-module scheme over S, and
4. locally on S there exists an integer $d \in \mathbb{Z}_{\geq 0}$, such that $(z-\zeta)^d = 0$ on ω_G where $\omega_G := \varprojlim_n \omega_{G[z^n]}$.

We define the *co-Lie module* of a z-divisible local Anderson module G over S as $\omega_G := \varprojlim \omega_{G[z^n]}$. It can be proved that ω_G is a finite locally free $\mathcal{O}_S$-module and we define the *dimension of* G as $\operatorname{rk} \omega_G$. It is locally constant on S.
A z-divisible local Anderson module is called *(ind-)étale* if $\omega_G = 0$. Since ω_G surjects onto each $\omega_{G[z^n]}$ because $\omega_{i_n} : \omega_{G[z^{n+1}]} \twoheadrightarrow \omega_{G[z^n]}$ is an epimorphism, $\omega_G = 0$ if and only if all $G[z^n]$ are étale.
A *morphism of z-divisible local Anderson modules over S* is a morphism of *fppf*-sheaves of $\mathbb{F}_q[\![z]\!]$-modules.