Moduli of p-divisible groups
(after Fargues, Fontaine, Scholze, Weinstein)

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Lecture I: Moduli of p-divisible groups

1. Background:
   1. Finite flat group schemes
   2. p-divisible groups
   3. Isogenies
2. More background:
   1. Adic rings
   2. Formal groups
   3. Liftings
3. The Lubin Tate tower $\mathscr{M}_\infty$
   1. The Lubin Tate moduli space
   2. Drinfeld level structure and the tower
   3. The case $h = 1$

Motivation

A an abelian variety $/k$ field of $char. \neq \ell$

$$Gal(\bar{k}/k) \curvearrowright T_\ell A = \varprojlim A(\bar{k})[\ell^n] \simeq \mathbb{Z}_\ell^{2g}$$

- Tate/Faltings: $T_\ell A$ determines isogeny class of A when k is a finite/number field.
- Ogg-Néron-Shafarevitch: k a number field, then $T_\ell A$ unramified at $v \nmid \ell \Leftrightarrow A$ has good reduction at v.

In $char. p$: $T_\ell A$ inadequate to study *deformations* or *variation in families* (ℓ-adic and p-adic topologies incompatible). On the other hand $0 \leq rk\, T_p A \leq g$ (not enough information).
Solution: Consider $A[p^\infty]$ as a *p-divisible group* !

Theorem (Serre-Tate)

Given A/k, the category of deformations of A is naturally equivalent to the category of deformations of $A[p^\infty]$.

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Finite flat group schemes (ffgs)

- $G_{/R}$ finite flat (commutative) group scheme, $G = Spec(A)$, locally $A \simeq R^n$ as a module, $n = rk(G)$.
- Group structure $\leftrightsquigarrow$ (cocommutative) Hopf algebra structure on A

$$A \xrightarrow{m^*} A \otimes_R A, \quad A \xrightarrow{i^*} A, \quad A \xrightarrow{e^*} R.$$

Example

(1) Γ finite abelian group, $A = R^\Gamma = \prod_{\gamma \in \Gamma} R$, constant $\underline{\Gamma} = Spec(A)$.
(2) $\mu_n = Spec(R[X]/(X^n - 1))$, $m^*(X) = X \otimes X$, $i^*(X) = X^{-1}$, $e^*(X) = 1$.
(3) R: $\mathbb{F}_p$-algebra, $\alpha_p = Spec(R[X]/(X^p))$, $m^*(X) = X \otimes 1 + 1 \otimes X$, $i^*(X) = -X$, $e^*(X) = 0$.
(4) $\mathscr{A}_{/R}$ abelian scheme, $G = \mathscr{A}[m]$, $rk(G) = m^{2g}$.

- **Functor of points:** $G(-) : \mathrm{Alg}_R \to \mathrm{Ab}$, $S \mapsto G(S)$, *fppf* sheaf. *Caution: $G(S)$ need not be finite, but killed by $n = rk(G)$.*

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- Represents the functor

$$G^\vee(S) = Hom_{Sgps}(G_S, \mathbb{G}_{m,S}).$$

E.g. $\mu_n^\vee \simeq \underline{\mathbb{Z}/n\mathbb{Z}}$, $\alpha_p^\vee \simeq \alpha_p$, $\mathscr{A}[m]^\vee \simeq \mathscr{A}^t[m]$ (Weil pairing).

- **Lie algebra:**

$$\boxed{Lie(G) = \ker(G(R[\varepsilon]) \to G(R))}$$

an R-module ($[r] : a + b\varepsilon \mapsto a + rb\varepsilon$). Check: $Lie(G) \simeq$

$$\{\text{derivations of } G_{/R} \text{ centered at } 0\} \simeq Hom_R(\omega_{G/R}, R)$$

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- R Henselian local ring (e.g. complete), *lifting idempotents* $\rightsquigarrow$ **connected-étale exact sequence**

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- If R is a *perfect field*, the sequence *splits canonically*:

$$G^{red} \hookrightarrow G, \;\; G^{red} \simeq G^{et}.$$

- **The category** Ffgs_R is additive, but in general not abelian (unless R is a field). A sequence

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Definition

A p-divisible group *of height h* over R is a system (G_n, i_n, p_n) where G_n is a ffgs of rank p^{nh}, $i_n : G_n \hookrightarrow G_{n+1}$ is a closed immersion identifying G_n with $G_{n+1}[p^n]$, and $p_n : G_n \twoheadrightarrow G_{n-1}$ is faithfully flat and satisfies $p_{n+1} \circ i_n = i_{n-1} \circ p_n = [p]_{G_n}$.

- $G = \lim_{\rightarrow} G_n$, $G_n = G[p^n]$ as an **fppf sheaf**
- **Examples**: $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, μ_{p^∞}, $\mathscr{A}[p^\infty]$ hts $1, 1, 2g$
- Notion of **SES**
- **Cartier-Serre** duality: $(G_n, i_n, p_n)^\vee = (G_n^\vee, p_{n+1}^\vee, i_{n-1}^\vee)$, e.g.

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if $n \geq N$. Call this common module $Lie(G)$.

- *Facts*: (1) $Lie(G)$ is *locally free* of rank $d \leq h = ht(G)$. Call $d = \dim(G)$ the **dimension**.

$$(2)\ Lie(G_S) = Lie(G)_S, \quad (3)\ \dim(G) + \dim(G^\vee) = ht(G).$$

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Isogenies

- $Hom(G, G')$ is a flat $\mathbb{Z}_p$-module. Let

$$qHom(G, G') = Hom(G, G')[1/p].$$

  The category of *p-divisible groups up to isogeny* has the same objects but *Hom* is replaced by *qHom*. A *quasi-isogeny* is an isomorphism in this category. An *isogeny* is a quasi-isogeny which is a homomorphism.

- If G and G' are isogenous then they have the same height and dimension, and the kernel of any isogeny is a ffgs. Any quasi-isogeny has $ht \in \mathbb{Z}$ and $ht(f' \circ f) = ht(f) + ht(f')$.

- Important isogenies if R is an $\mathbb{F}_p$-algebra: **Frobenius** and **Verschiebung**

$$F_G : G \to G^{(p)}, \quad V_G : G^{(p)} \to G.$$

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Adic rings

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An *adic ring* is a complete and separated topological ring R for which $\exists$ ideal I s.t.

$$R \simeq \lim_{\leftarrow} R/I^n$$

(topologically). Any such I is an *ideal of definition*, and R is "I-adic". Adic = category of adic rings and continuous hom's.

- **Examples**: R discrete ($I = 0$); completion of any ring w.r.t. a f.g. ideal; $(\mathbb{Z}_p[[u]], I = (p, u))$; $(\mathbb{Z}_p\langle u\rangle, I = (p))$.
- J is also an ideal of definition if $I^m \subset J$, $J^n \subset I$.
- If R is I-adic, $J \subset I$ and J^n is closed for all n (e.g. R noetherian) then R is complete and separated in the J-adic topology as well.
- If $R \in \mathrm{Adic}$, $Nil(R)$ = ideal of topologically nilpotent elements.

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Let $R \in \mathrm{Adic}$, $A = R[[X_1, \ldots, X_d]]$. A *d-dimensional (commutative) formal group law* over R is $\Phi(X,Y) \in (A\hat{\otimes}_R A)^d$ such that

1. $\Phi(X,0) = X$
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- $\mathscr{G}_\Phi : \mathrm{Adic}_R \to \mathrm{Ab}$, $\mathscr{G}_\Phi(S) = (Nil(S)^d, [+]_\Phi)$. By Yoneda, determines Φ up to isomorphism.
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p-divisible formal groups

- Let $R \in \mathrm{Adic}_{\mathbb{Z}_p}$. $\mathscr{G}_\Phi$ is p-**divisible** if $[p]^* : A \to A$ is finite flat. (Examples: $\hat{\mathbb{G}}_m$ because $[p]^*(X) = pX + \cdots + X^p$, but not $\hat{\mathbb{G}}_a$ where $[p]^*(X) = pX$).
- *Fact:* $\mathscr{G}$ p-divisible $\Rightarrow \deg[p]^* = p^h$, $h = ht(\mathscr{G})$, otherwise $ht = \infty$.

Theorem (Tate, Messing)

Let $\mathscr{G}$ be a p-divisible formal group. Define

$$G = \mathscr{G}(p) = (\mathscr{G}[p^n], i_n, p_n)$$

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p-divisible formal groups

- Let $R \in \mathrm{Adic}_{\mathbb{Z}_p}$. $\mathscr{G}_\Phi$ is p-**divisible** if $[p]^* : A \to A$ is finite flat. (Examples: $\hat{\mathbb{G}}_m$ because $[p]^*(X) = pX + \cdots + X^p$, but not $\hat{\mathbb{G}}_a$ where $[p]^*(X) = pX$).
- *Fact:* $\mathscr{G}$ p-divisible $\Rightarrow \deg[p]^* = p^h$, $h = ht(\mathscr{G})$, otherwise $ht = \infty$.

Theorem (Tate, Messing)

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- Let $R \in \mathrm{Adic}_{\mathbb{Z}_p}$, $S \in \mathrm{Adic}_R$, I_S an ideal of def'n. If G is a p-divisible group over R we **re-define**

$$\boxed{G(S) = \lim_{\leftarrow} G(S/I_S^n).}$$

- Assume R local complete $\mathfrak{m}_R$-adic. If S is discrete or G is étale, have not changed the def'n. If G is connected, and $G = \mathscr{G}(p)$ then $G(S) = \mathscr{G}(S)$.
- **Example**: $\mu_{p^\infty}(\mathscr{O}_C) = \lim_{\leftarrow} \mu_{p^\infty}(\mathscr{O}_C/p^n) = \lim_{\leftarrow}(1+\mathfrak{m}_C \mod p^n) = 1+\mathfrak{m}_C = \hat{\mathbb{G}}_m(\mathscr{O}_C)$. Here $C = \mathbb{C}_p$.
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Liftings of formal groups

Theorem (Lazard, 1955)

Let $R \in \mathrm{Adic}$, $J \subset R$ a closed ideal. Then any formal group over R/J lifts to a formal group over R.

Reason: there exists a universal d-dimensional formal group and it is defined over a *free polynomial ring* over $\mathbb{Z}$.

Theorem (Rigidity of quasi-isogenies)

Let $\mathscr{F}, \mathscr{G}$ be p-divisible formal groups over $R \in \mathrm{Adic}_{\mathbb{Z}_p}$, J a closed topologically nilpotent ideal in R. Then (i)

$$Hom_R(\mathscr{F}, \mathscr{G}) \hookrightarrow Hom_{R/J}(\mathscr{F}, \mathscr{G})$$

is injective. (ii) Assume $J^2 = 0$ and $p^N J = 0$. Then for any $\alpha \in Hom_{R/J}(\mathscr{F}, \mathscr{G})$, $p^N \alpha$ lifts to $Hom_R(\mathscr{F}, \mathscr{G})$. (iii) If J and p are nilpotent

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Caution:

- $End_R(\hat{\mathbb{G}}_a) \to End_{R/J}(\hat{\mathbb{G}}_a)$ not injective ($\hat{\mathbb{G}}_a$ not p-div).
- $End_{\mathscr{O}_C}(\mathscr{G}) \to End_{\mathscr{O}_C/\mathfrak{m}_C}(\mathscr{G})$ not injective ($\mathfrak{m}_C$ not top. nilp.).
- $qEnd_{\mathbb{Z}_p}(\mathscr{G}) \to qEnd_{\mathbb{F}_p}(\mathscr{G})$ not surjective ($p\mathbb{Z}_p$ not nilpotent).

Example (One dimensional formal groups)

k algebraically closed field, char. p. For any $h \geq 1 \exists!$ 1-dimensional formal p-divisible group H_0 of height h.

- *Construction*: $H_{/\mathbb{Z}_{p^h}}$ Lubin-Tate formal group law with $[p]_H = pX + X^{p^h} \rightsquigarrow H_0 = H \times_{\mathbb{Z}_{p^h}} k$.
- *Endomorphisms*: Let $\mathscr{O}_D = \mathbb{Z}_{p^h}[\Pi]$, $\Pi^h = p$, $\Pi a = \sigma(a)\Pi$, the maximal order in the division algebra D of invariant $1/h$ with center $\mathbb{Q}_p$. Then
$$End_k(H_0) = \mathscr{O}_D.$$
- Exercise: H_0 defined over $\mathbb{F}_p$. Find *End* over $\mathbb{F}_p$ ($\mathbb{F}_{p^h}$), and which of them lift to endomorphisms of H over $\mathbb{Z}_p$ ($\mathbb{Z}_{p^h}$).

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The Lubin-Tate moduli space

- Let k, H_0 as above, $W = W(k)$. Let $\mathscr{C}_k \subset \mathrm{Adic}_W$ be the full subcategory of local complete noetherian rings with residue field k. Consider the *deformation functor* $\mathscr{M}^0 : \mathscr{C}_k \to \mathrm{Sets}$

$$\boxed{\mathscr{M}^0(R) = \{(G,\iota) |\, G_{/R}\ 1\ \text{dim}\ p\ \text{div gp},\ \iota : G \times_R k \simeq H_0\}/\simeq .}$$

- Rigid: $Aut(G,\iota) = \{1\}$
- $\mathscr{O}_D^\times \curvearrowright \mathscr{M}^0$ via $\delta(G,\iota) = (G, \delta \circ \iota)$
- Variant: ι quasi-isogeny, $\mathscr{M} = \bigsqcup_{ht(\iota)=i} \mathscr{M}^i$, $D^\times \curvearrowright \mathscr{M}$, $\Pi(\mathscr{M}^i) = \mathscr{M}^{i+1}$.

Theorem (Lubin-Tate)

$\mathscr{M}^0$ is representable by $Spf(A_0)$ where

$$A_0 = W[[u_1, \ldots, u_{h-1}]].$$

Remark

(i) $h = 1$: a unique deformation $= \hat{\mathbb{G}}_m$. (ii) Action of $\mathscr{O}_D^\times$ non-trivial!

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Sketch: Let $v_1, v_2, \ldots$ be variables. Define $b_i \in p^{-i}\mathbb{Z}_p[v_1, v_2, \ldots]$

$$b_0 = 1,\ \ pb_i = v_i + b_1 v_{i-1}^{p} + b_2 v_{i-2}^{p^2} + \cdots + b_{i-1} v_1^{p^{i-1}}$$

$$f = \sum_{i=0}^{\infty} b_i X^{p^i},\ \ F(X,Y) = f^{-1}(f(X) + f(Y)).$$

Lemma (Lazard, Hazewinkel)

(i) $F \in \mathbb{Z}_p[\underline{v}][[X,Y]]$ is a universal 1-dimensional formal group law over $\mathbb{Z}_p$-algebras. (ii)

$$\log_F = f \equiv X + p^{-1} v_h X^{p^h} \mod (v_1, \ldots, v_{h-1}, X^{p^h+1})$$

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If R is an $\mathbb{F}_p$ algebra and G is obtained from F by $v_i \mapsto 0$ $(1 \le i < h)$, $v_h \mapsto 1$ then G has height h.

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(Ext group to be discussed later), so of dimension $h-1$ (in general $(h-d)d$). Identify it with the tangent space to $A_0 \otimes_W k$. This essentially shows that $\mathscr{M}^0$ is representable by a quotient $A_0/\mathfrak{a}$.

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- Lubin-Tate paper (1966) is only 10 pages long. Recommended!
- May replace formal groups by "***formal A-modules***" (A a CDVR), p (uniformizer) by π, p (degree) by q etc. See Gross-Hopkins, Drinfeld. Works in the function field case too, theory of Drinfeld modules.
- When all $u_i = 0$ one gets the "***canonical lifting***". If $R = W$ get the Lubin-Tate formal group of height h over W. More generally, for every $[L : \mathbb{Q}_p] < \infty$ and uniformizer π of L get a unique-up-to-isomorphism "**Lubin-Tate formal group**" over $\mathscr{O}_L$ associated with π. It plays an important role in Class Field Theory. Over $\widehat{L^{nr}}$ the dependence on π disappears.
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Drinfeld Level Structure

Definition

A *Drinfeld level-n structure* on $(G, \iota) \in \mathscr{M}(R)$ is a homomorphism of ffgs/R

$$\alpha_n : (\underline{\mathbb{Z}/p^n\mathbb{Z}})^h \to G[p^n]$$

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- $\mathcal{M}_n(R) = \{(G,\iota,\alpha_n) | (G,\iota) \in \mathcal{M}(R), \alpha_n \text{ level } n \text{ structure}\}$.
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Theorem (Drinfeld)

(i) $\mathcal{M}_n^0 = Spf(A_n)$ is representable.
(ii) A_n is a regular complete local ring, finite flat over A_0.
(iii) if $M^0 = Spa(A_0, A_0)$ and $M_n^0 = Spa(A_n, A_n)$ (the adic spaces associated to these formal schemes) and

$$M_\eta^0 = M^0 \times_{Spa(W,W)} Spa(W[1/p], W),\ M_{n,\eta}^0 = \cdots$$

are their generic fibers, then $M_{n,\eta}^0 \xrightarrow{\pi_n} M_\eta^0$ is Galois étale of Galois group $GL_h(\mathbb{Z}/p^n\mathbb{Z})$.

Caution: M_η^0 ($M_{n,\eta}^0$) is not an affinoid, but an "open polydisk".

The Lubin-Tate tower

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- $A_\infty = (\lim_{\rightarrow} A_n)\hat{}$ (I-adic completion, $I = (p, u_1, \ldots, u_{h-1})$) non-noetherian but I f.g. so A_∞ is complete and separated. Then
$$\mathscr{M}^0_\infty = Spf(A_\infty) = \lim_{\leftarrow} \mathscr{M}^0_n.$$
- $M^0_\infty = Spa(A_\infty, A_\infty)$ is an adic space. (*A point to check*: its structure presheaf is sheafy, follows from the fact that A_∞ is a perfectoid ring).
- $M^0_{\infty,\eta}$ the generic fiber of M^0_∞ (open set of valuations where $|p| \neq 0$), is **the analytic Lubin-Tate space at the infinite level**.
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The Lubin-Tate tower, $h = 1$

- $A_0 = W$, unique deformation is $\hat{\mathbb{G}}_m$.
- α_n level-n structure iff $\Phi_{p^n}(\alpha_n(1)) = 0$ so

$$A_n = W[X]/(\Phi_{p^n}) = W[\zeta_{p^n}], \quad A_\infty = \mathscr{O}_L, \, L = \widehat{\mathbb{Q}_p^{ab}}.$$

Proposition

L is a perfectoid field, i.e. $\phi : \mathscr{O}_L/p \to \mathscr{O}_L/p$ is surjective.

Proof. ϕ is surjective on W/p and $\phi(\zeta_{p^{n+1}}) = \zeta_{p^n}$.

- Tilt: $L^\flat = \bar{\mathbb{F}}_p((t^{1/p^\infty}))$
- First instance of: "the Lubin-Tate tower at the infinite level" $M^0_{\infty,\eta}$ is a perfectoid space.
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The universal covering of G

- $R \in \mathrm{Adic}_{\mathbb{Z}_p}$, $G_{/R}$ a p-div gp. Recall $G(S) = \lim_{\leftarrow} G(S/I_S^n)$.

Definition

$\widetilde{G}(S) = \lim_{\leftarrow \times p} G(S) = \{(x_0, x_1 \dots) | \, x_i \in G(S), \, [p]_G(x_{i+1}) = x_i\}$.
Presheaf on Adic_R, values in $\mathbb{Q}_p$-vector spaces.

- Examples: $G = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$, $\widetilde{G} = \underline{\mathbb{Q}_p}$; $G = \mu_{p^\infty}$, $\widetilde{G}(\mathscr{O}_C) = 1 + \mathfrak{m}_{C^\flat}$.
- If $G \sim G'$ then $\widetilde{G} \simeq \widetilde{G'}$.

Lemma (Crystalline nature of $\widetilde{G}$)

$I \subset S$ closed topologically nilpotent $\Rightarrow$ $\widetilde{G}(S) = \widetilde{G}(S/I)$.

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Corollary. Let $T \twoheadrightarrow S \in \mathrm{Adic}_R$ be a (pro-)nilpotent thickening. For any lift G' of G to T, $\widetilde{G'}(T) = \widetilde{G}(S)$. Write $\widetilde{G'}(T) = \widetilde{G}(T)$.

Proposition

Assume R perfect $\mathbb{F}_p$-algebra (ϕ bijective), G formal. Then $\widetilde{G}$ is (locally on R) representable by a formal scheme

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Key idea: May replace $\lim_{\leftarrow \times p}$ by $\lim_{\leftarrow \times F}$ and get isomorphic groups. For this need to consider

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There exists a quasi-isogeny

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Much deeper than "rigidity of quasi-isogenies" because $\mathfrak{m}_C/(p)$ not nilpotent - relies on theorems of Fargues and "full-faithfullness" result of Scholze-Weinstein. Crucial ingredient: C is perfectoid.

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$\widetilde{G} \simeq Spf(\mathscr{O}_C[[X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}]])$, *hence its associated analytic space* $(Spa(\ldots,\ldots)_\eta)$ *is a perfectoid (a "perfectoid open unit polydisk").*

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- **Tate module:** $T_pG = \lim_{\leftarrow \times p} G[p^n]$, $V_pG = T_pG[1/p] \hookrightarrow \widetilde{G}$.
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- *Goal*: an exact sequence (S flat over $\mathbb{Z}_p$)

$$\text{(LOG)} \quad \boxed{0 \to V_pG(S) \to \widetilde{G}(S) \xrightarrow{\theta} Lie(G_S)[1/p].}$$

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Logarithms

- $G_{/R}$ formal p-div $\leftrightsquigarrow \mathscr{G} = Spf(A)$, $A = R[[X_1, \ldots, X_d]]$.
- $\omega_{G/R} = \oplus R dX_i|_0 \simeq \{\omega \in \Omega_{A/R} | \, m^*(\omega) = \omega \otimes 1 + 1 \otimes \omega\} =$ translation invariant differentials (*all closed*: $d\omega = 0$).
- R flat over $\mathbb{Z}_p$: Then $\forall \omega \; \exists! \lambda_\omega \in A[1/p]$ without constant term, $d\lambda_\omega = \omega$, $\lambda_\omega \in Hom_{R[1/p]}(G, \hat{\mathbb{G}}_a)$ (*formal Poincaré lemma*).
- $Hom(\underline{\omega}_{G/R}, \underline{Hom}(G, \hat{\mathbb{G}}_a)) = Hom(G, \underline{Hom}(\underline{\omega}_{G/R}, \hat{\mathbb{G}}_a))$.
- $\underline{Lie}(G) = \underline{Hom}(\underline{\omega}_{G/R}, \hat{\mathbb{G}}_a) \rightsquigarrow \log_G \in Hom_{R[1/p]}(G, \underline{Lie}(G))$.
- Let $\theta = \log_G \circ pr_0$ where $pr_0 : \widetilde{G} \to G$ is $x \mapsto x_0$. Then (LOG) is exact.
- If $pr_0 : \widetilde{G}(S) \to G(S)$ is surjective θ is surjective too.
- θ will be related to the "θ" of Fontaine's rings.

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- Let $\theta = \log_G \circ pr_0$ where $pr_0 : \widetilde{G} \to G$ is $x \mapsto x_0$. Then (LOG) is exact.
- If $pr_0 : \widetilde{G}(S) \to G(S)$ is surjective θ is surjective too.
- θ will be related to the "θ" of Fontaine's rings.

Logarithms

- $G_{/R}$ formal p-div $\leftrightsquigarrow \mathscr{G} = Spf(A)$, $A = R[[X_1 \dots, X_d]]$.
- $\omega_{G/R} = \oplus R dX_i|_0 \simeq \{\omega \in \Omega_{A/R} | \, m^*(\omega) = \omega \otimes 1 + 1 \otimes \omega\} =$ translation invariant differentials (*all closed*: $d\omega = 0$).

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A simple description of $\mathscr{M}_\infty$

- $R \in \mathscr{C}_k$, $(G,\iota) \in \mathscr{M}(R)$, $\iota : G \times_R k \dashrightarrow H_0$, $ht(H_0) = h$. Let $H_{/W}$ be the Lubin-Tate group ("canonical lifting") of H_0.

Theorem (Hedayatzadeh, 2015)

There exists a canonical alternating multilinear $\lambda_n : G[p^n]^h \to \mu_{p^n}$ satisfying a universal property. $\rightsquigarrow \lambda_G : \widetilde{G}^h \to \widetilde{\mathbb{G}}_m$.

- Via ι: $\widetilde{G}(S) = \widetilde{G}(S/\mathfrak{m}_S) \overset{\iota}{\simeq} \widetilde{H}(S/\mathfrak{m}_S) = \widetilde{H}(S)$. Get

$$\exists! \lambda(\iota) \in \mathbb{Q}_p^\times,\ ord_p(\lambda(\iota)) = ht(\iota),\ \lambda_H \circ \iota = \lambda(\iota) \cdot \lambda_G.$$

- $\widetilde{H} = Spf(W[[X^{1/p^\infty}]])$, $\widetilde{\mathbb{G}}_m = Spf(W[[T^{1/p^\infty}]])$, $\lambda_H \leftrightsquigarrow$

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A commutative diagram of formal schemes over $Spf(W)$ $(L = \widehat{\mathbb{Q}_p^{ab}})$

$$(\mathrm{LT}_\infty) \quad \boxed{\begin{array}{ccc} \mathscr{M}_\infty & \longrightarrow & \coprod_{ht(\iota)=i} Spf(\mathscr{O}_L) \\ \downarrow & & \downarrow \underline{t} \\ \widetilde{H}^h & \xrightarrow{\lambda_H} & \widetilde{\mathbb{G}}_{m,W} \end{array}}$$

- Right $\underline{t} \leftrightsquigarrow T^{1/p^n} \mapsto \left(\lim_m (\zeta_{p^m} - 1)^{p^{m-n-i}}\right)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \mathscr{O}_L$
- Top $(G, \iota, \alpha_\infty) \in \mathscr{M}_\infty^0 \mapsto (\zeta_{p^n} \mapsto \lambda(\iota)\cdot\lambda_{n,G}(\alpha_{n,1},\dots,\alpha_{n,h}))$.
- Left $(G, \iota, \alpha_\infty) \mapsto (\alpha_{n,1},\dots,\alpha_{n,h})_{n=1}^\infty \in \widetilde{G}^h \overset{\iota}{\simeq} \widetilde{H}^h$.

Theorem (Weinstein, 2016)

(i) The diagram is cartesian. (ii) Action of $g \in GL_h(\mathbb{Q}_p)$: on $\widetilde{H}^h$ via right action of g^{-1} on row vectors, on $\widetilde{\mathbb{G}}_{m,W}$ via $\det g^{-1}$, similarly on $\coprod_{ht(\iota)=i} Spf(\mathscr{O}_L)$ (p shifts between copies, $\mathbb{Z}_p^\times$ acts like inverse cyclotomic character) (iii) Action of $D^\times$: via $D = qEnd(H_0)$ on $\widetilde{H} = \widetilde{H}_0$, via Nrd on right column.

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Remark. $(i=0)$ $Spf(\mathscr{O}_L) \simeq T_p \hat{\mathbb{G}}^{prim}_{m,W} \subset V_p \hat{\mathbb{G}}_{m,W} \hookrightarrow \widetilde{\mathbb{G}}_{m,W}$. When $h=1$ the bottom row is the identity. In general $\mathscr{M}^0_\infty$ is the fiber of λ_H at the $\mathscr{O}_L$-point $\underline{t}$ of $\widetilde{\mathbb{G}}_{m,W}$. Unlike the horizontal maps, the vertical maps *do not make sense at finite levels*.

Corollary

(i) The group $(GL_h(\mathbb{Q}_p) \times D^\times)^{\det = Nrd}$ acts on the cartesian diagram (trivially on $\widetilde{\mathbb{G}}_{m,W}$).
(ii) Explicitly, let $t^{1/p^n} = \underline{t}^*(T^{1/p^n})$, $\mathscr{M}_\infty = Spf(A_\infty)$,

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Scholie: The Lubin-Tate tower at the *infinite* level is *infinitely* simpler than at finite levels, and in addition is a perfectoid!

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The universal vectorial extension

$G_{/R}$ p-div gp, $p^N = 0$ in R. The sequence of fppf sheaves on Alg_R

$$0 \to G[p^n] \to G \xrightarrow{p^n} G \to 0$$

is exact. Applying $RHom(-, \mathbb{G}_a)$ get a SES

$$0 \to Hom(G, \mathbb{G}_a)/p^n \to Hom(G[p^n], \mathbb{G}_a) \to Ext(G, \mathbb{G}_a)[p^n] \to 0.$$

- $Hom(G, \mathbb{G}_a) = 0$ since G is p-divisible but $p^N \mathbb{G}_a = 0$.
- $n \geq N \Rightarrow Ext(G, \mathbb{G}_a) = Hom(G[p^n], \mathbb{G}_a) = \{a \in A_n | \, m_G^*(a) = a \otimes 1 + 1 \otimes a\} = Lie(G^\vee[p^n]) = Lie(G^\vee) = Hom(\omega_{G^\vee}, R)$.
- Similarly for any R-module $Ext(G, \underline{M}) \simeq Hom(\omega_{G^\vee/R}, M)$.
- Taking $M = \omega_{G^\vee/R}$ and the identity $\rightsquigarrow$ **"universal" extension**

$$0 \to \underline{\omega}_{G^\vee/R} \to EG \to G \to 0$$

from which *any* (fppf sheaf) extension of G by a vector-group $\underline{M}$ is gotten by a unique push-out.

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The Grothendieck-Messing crystal MG

Take $\underline{Lie}(-)$. Get a SES of vector groups ($MG = \underline{Lie}(EG)$)

$$\boxed{0 \to \underline{\omega}_{G^\vee/R} \to MG \to \underline{Lie}(G) \to 0.}$$

- $\forall S \in \mathrm{Alg}_R$, $MG(S)$ a locally free module, $rk(MG) = ht(G)$.
- **Goal**: enhance MG to a *crystal* of modules on the crystalline site. *Need*: MG classifies *rigidified extensions* of $G^\vee$ by $\mathbb{G}_a$.
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Any two rigidifications differ by a homomorphism from $Lie(G)$ to $\mathbb{G}_a$, i.e. by an element of $\omega_{G/R}$. The group of rigidified extensions $Ext^\natural(G^\vee, \mathbb{G}_a)$ sits in an exact sequence

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$$\begin{array}{ccc} T & \overset{pd}{\twoheadrightarrow} & S \\ & & \uparrow \\ & & R \end{array}$$

  $S \in \mathrm{Alg}_R$, T a *nilpotent divided powers* thickening of S. If S is $\mathbb{Z}_p$-flat: $x \in I = \ker(T \twoheadrightarrow S) \Rightarrow x^n/n! \in I$, and $\exists N$ s.t. $(x_1^{n_1}/n_1!)\cdots(x_r^{n_r}/n_r!) = 0$ if $x_i \in I$, $\sum n_i \geq N$. *Morphisms* "preserve the pd structure".

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- A locally free coherent sheaf, $MG(S) = M(G_S)$ is $MG(S \twoheadrightarrow S)$.

Explanation (Katz): $R \in \mathrm{Adic}_{\mathbb{Z}_p}$, $\mathbb{Z}_p$-flat. Let $\mathscr{F}_{/R}$ a p-divisible formal group, $\mathscr{F} = Spf(R[[X_1, \ldots, X_d]])$.

- $H^1_{dR}(\mathscr{F}/R) = \{[\eta] | \, \eta \text{ closed}, m^*_{\mathscr{F}}(\eta) - \eta \otimes 1 - 1 \otimes \eta \text{ exact}\}$ translation invariant cohomology classes.

- $\omega_{\mathscr{F}/R} = \{\eta | \, m^*_{\mathscr{F}}(\eta) = \eta \otimes 1 + 1 \otimes \eta\} \hookrightarrow H^1_{dR}(\mathscr{F}/R)$, because R is p-adic, and $\mathscr{F}$ is p-divisible, so η exact and translation invariant $\Rightarrow \eta = 0$. (Logarithms need $R[1/p]$.)

$$\begin{array}{ccccccccc} 0 & \to & \omega_{\mathscr{F}/R} & \to & H^1_{dR}(\mathscr{F}/R) & \xrightarrow{\partial} & H^2(\mathscr{F}; \mathbb{G}_a)_s & \to & 0 \\ & & || & & || & & || & & \\ 0 & \to & \omega_{\mathscr{F}/R} & \to & Ext^\natural(\mathscr{F}; \mathbb{G}_a) & \to & Ext(\mathscr{F}; \mathbb{G}_a) & \to & 0 \end{array}$$

- Bottom row identified, when $G^\vee \leftrightsquigarrow \mathscr{F}$, with

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$$\begin{array}{ccccccccc} 0 & \to & \omega_{\mathscr{F}/R} & \to & H^1_{dR}(\mathscr{F}/R) & \xrightarrow{\partial} & H^2(\mathscr{F};\mathbb{G}_a)_s & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \omega_{\mathscr{F}/R} & \to & Ext^\natural(\mathscr{F};\mathbb{G}_a) & \to & Ext(\mathscr{F};\mathbb{G}_a) & \to & 0 \end{array}$$

- Bottom row identified, when $G^\vee \leftrightsquigarrow \mathscr{F}$, with

$$0 \to \omega_{G^\vee/R} \to MG(R) \to Lie(G) \to 0.$$

- We explain the map ∂ :

- A locally free coherent sheaf, $MG(S) = M(G_S)$ is $MG(S \twoheadrightarrow S)$.

Explanation (Katz): $R \in \mathrm{Adic}_{\mathbb{Z}_p}$, $\mathbb{Z}_p$-flat. Let $\mathscr{F}_{/R}$ a p-divisible formal group, $\mathscr{F} = Spf(R[[X_1, \dots, X_d]])$.

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$$H^2(\mathscr{F};\mathbb{G}_a)_s = \frac{\{\Delta(X,Y)\in R[[X;Y]]|\,\text{symm.}, \delta(\Delta)=0\}}{\{\delta(f)=f(X[+]Y)-f(X)-f(Y)\}}$$

$$\delta(\Delta) = \Delta(Y,Z)-\Delta(X[+]Y,Z)+\Delta(X,Y[+]Z)-\Delta(X,Y).$$

The map from $H^1_{dR}(\mathscr{F}/R)$ is: find a primitive $f(X)\in R[1/p][[X]]$ for η, let $\Delta=\delta(f)$. $[\eta]$ is translation invariant $\Rightarrow \Delta$ is integral: $\delta(\eta)=d\Delta$. Set $\partial([\eta])=[\Delta]$.

The identification $H^2(\mathscr{F};\mathbb{G}_a)_s \simeq Ext(\mathscr{F};\mathbb{G}_a)$ is standard, that of $H^1_{dR}(\mathscr{F}/R)\simeq Ext^\natural(\mathscr{F};\mathbb{G}_a)$ requires only a little more work.

Lemma

Let $\mathscr{F}',\mathscr{F}''$ be liftings of $\mathscr{F}$ to $T\overset{pd}{\twoheadrightarrow}R$. Let $\varphi:\mathscr{F}'\to\mathscr{F}''$ be a morphism of pointed Lie varieties reducing to the identity on R. Then (i) $\varphi^*:H^1_{dR}(\mathscr{F}''/T)\simeq H^1_{dR}(\mathscr{F}'/T)$ (preserving the invariance under the group law). (ii) φ^* is independent of φ. (iii) Similarly, if φ reduces to an endomorphism φ_0 of $\mathscr{F}$, φ^* is a homomorphism that depends only on φ_0.

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Proof.

$(d=1)$ Let $\eta = df$, $f \in T[1/p][[X]]$, represent $[\eta] \in H^1_{dR}(\mathscr{F}''/T)$. Let $I = \ker(T \overset{pd}{\twoheadrightarrow} R)$, $\varphi_1, \varphi_2 \in T[[Y]]$, $\varphi_i(0) = 0$, $\varphi_1 \equiv \varphi_2 \mod I$. Then by Taylor

$$\varphi_2^*(\eta) - \varphi_1^*(\eta) = d\left(\sum_{n=1}^{\infty} f^{(n)}(\varphi_1) \cdot \frac{(\varphi_2 - \varphi_1)^n}{n!}\right)$$

and $(\cdots) \in T[[Y]]$ since I has divided powers and $f^{(1)}$ is already integral. This shows (ii) $\varphi_2^*([\eta]) = \varphi_1^*([\eta])$. A similar argument proves (i) and (iii). □

- *Explains phrase*: "$MG'_T(T)$ depends functorially only on G".
- It is *blatantly false* that φ^* maps $\omega_{\mathscr{F}''/T}$ to $\omega_{\mathscr{F}'/T}$.
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Dieudonné modules

k perfect field, $char.p$, $W = W(k)$, σ the Frobenius automorphism. $G_{/k}$ p-div gp. Its Dieudonné module is

$$D(G) = M(G^\vee)(W \twoheadrightarrow k).$$

- *Contra*variant, free W-module rk $h = ht(G)$.
- $F_G : G \to G^{(p)}$, dual isogeny $V_{G^\vee} : G^{\vee(p)} \to G^\vee$. By functoriality of $M(-)$ get $F : D(G)^{(p)} \to D(G)$, i.e. σ-linear $D(G) \to D(G)$. Similarly, $V_G : G^{(p)} \to G \rightsquigarrow \sigma^{-1}$-linear V.

$$F \circ V = V \circ F = p.$$

- $(D(G), F, V)$ - an F-crystal over k. Form an additive category.

Theorem (Dieudonné-Manin)

$D(-)$ is an anti-equivalence between pdivgp_k *and* Fcryst_k.

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F-isocrystals

- $M(G^\vee)(k) = D(G)/pD(G)$ ($\simeq H^1_{dR}(\mathscr{A}/k)$ if $G = \mathscr{A}[p^\infty]$).
- $\omega_{G/k} \simeq VD(G)^{(p^{-1})}/pD(G)$.
- Original equivalent def'n: $D(G) = Hom_k(G, CW)$ where CW is the group of co-Witt vectors $\curvearrowleft F, V$.

F-**isocrystals** (N, F, V) - N a f.dim. $W[1/p]$-vector space, F, V as above. An equivalence of categories between "p-div gps up to isogeny" and "F-isocrystals containing an invariant F-crystal".

- Standard example: $(r, s) = 1$, $s > 0$, $\lambda = r/s$. Let $N_\lambda = \sum_{i=1}^{s} W[1/p]e_i$, $Fe_i = e_{i+1}$ $(i < s)$, $Fe_s = p^r e_1$. Call λ the (Frobenius) *slope*.

Theorem

Let k be alg. closed. The category of F-isocrystals over k is semisimple. Its simple objects are the N_λ. An F-isocrystal contains an F-crystal iff all its slopes are contained in $[0, 1]$.

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- $End(N_\lambda) = D_{-\lambda}$, division ring over $\mathbb{Q}_p$ with invariant $-\lambda$ mod 1. (If $N = D(G)[1/p]$ this means $qEnd(G) \simeq D_\lambda$.)
- Lubin-Tate case $\lambda = 1/h$.
- *Exercise*: If $0 \leq r \leq s$ extend e_i' by $e_{i+ms}' = p^m e_i'$, define the F-crystal

$$M_\lambda = \sum_{i=1}^{s} We_i', Fe_i' = e_{i+r}', Ve_i' = e_{i+s-r}'.$$

  Then N_λ has a lattice isomorphic to M_λ (but there are others).
- Let k be perfect. Call N *isoclinic of slope* λ if $N \otimes_k \bar{k} \simeq N_\lambda^n$.

Proposition (Slope decomposition)

Let k be perfect and N an F-isocrystal over k. Then $N = \oplus_{\lambda \in \mathbb{Q}} N(\lambda)$ where $N(\lambda)$ is isoclinic of slope λ.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be the slopes of N. Then the **Newton polygon** $NP(N)$ is convex, starts at $(0,0)$, and has slopes λ_i with horizontal length $rk(N(\lambda_i))$. Break points are in $\mathbb{Z}^2$.

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Lecture III: Period maps

1. The Grothendieck-Messing (GM) period map
    1. The quasi-logarithm and a big diagram
    2. Rapoport-Zink deformation spaces
    3. The Grothendieck-Messing period map
    4. Period domains and admissibility
2. The Hodge-Tate (HT) period map
    1. The Hodge-Tate decomposition
    2. Hodge-Tate period map
3. Example: Drinfeld's p-adic symmetric domain
    1. Rapoport-Zink spaces with PEL structure
    2. The Drinfeld moduli problem

Quasi logarithms and a big diagram

- $R, S \in \mathrm{Adic}_{\mathbb{Z}_p}$, $\pi : S \overset{pd}{\twoheadrightarrow} R$, $S \simeq \lim_{\leftarrow} S/(\ker \pi)^n$. Assume S flat over $\mathbb{Z}_p$, e.g. $\mathscr{O}_C \twoheadrightarrow \mathscr{O}_C/p$.
- Let $G_{/S}$ lift $G_{0/R}$. Both $\widetilde{G}$ and MG have a "crystalline nature". We relate them. Top maps to bottom via log's (not shown).

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\begin{array}{ccccccccc}
 & \omega_{G^\vee/S} & \hookrightarrow & EG_0(S) & \cdots & \to & G(S) \\
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| & & & & & & \downarrow \\
\vdots & \omega_{G^\vee/S,\mathbb{Q}} & \hookrightarrow & MG_0(S)_{\mathbb{Q}} & \cdots & \to & Lie(G)_{\mathbb{Q}} \\
\downarrow & \nearrow & & \nearrow \mathrm{qlog} & & \nearrow & \\
\boldsymbol{V_pG(S)} & \hookrightarrow & \boldsymbol{\widetilde{G}_0(S)} & \underset{\theta}{\to} & \boldsymbol{Lie(G)_{\mathbb{Q}}} & &
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\end{array}
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- $s_G(x_0, x_1, \dots) = \lim [p^n]_{EG}(\xi_n)$, if $EG(S) \ni \xi_n \mapsto x_n \in G(S)$.
- $\alpha_G = s_G|_{T_pG(S)}$ has the following interpretation:

$$x \in T_pG(S) \rightsquigarrow x_n \in G(S)[p^n] = Hom_S(G^\vee[p^n], \hat{\mathbb{G}}_m)$$

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- $\mathrm{qlog}_G = \log_{EG} \circ s_G$. If $G \leftrightsquigarrow \mathscr{G}$, fix coordinates on $E\mathscr{G}$, let $x = (x_0, x_1, \dots)$ and ξ_n as above, then

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Generalize Lubin-Tate space: k alg. closed char. p, $W = W(k)$, $H_{0/k}$ p-div gp, $ht = h$, $\dim = d$. Set

$$\boxed{M_0 = MH_0(W \twoheadrightarrow k) = D(H_0^\vee) \simeq W^h.}$$

- $S \in \mathrm{Nilp}_W = W$-algebras on which p is loc. nilp., e.g. $\mathcal{O}_C/p^N$.

$$\boxed{\mathcal{D}(S) = \{(G,\iota) | G_{/S}\; p\,\mathrm{div\,gp}, \iota : G \times_S S/p \overset{q.i.}{\dashrightarrow} H_0 \times_k S/p\}.}$$

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Rapoport-Zink deformation spaces

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Example

1) $d=1$, $h=2$, $H_0 = \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}$. Since $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ and μ_{p^∞} do not deform,

$$\mathscr{D}^0(S) = Ext_S(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \simeq \widehat{\mathbb{G}_m}(S)$$

("Serre-Tate canonical coordinate"). Note $\mathscr{D}^0(k)$ is a point.
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The Grothendieck-Messing period map

- Take $S = \mathscr{O}_C$. For $(G, \iota) \in \mathscr{D}(\mathscr{O}_C)$ we have a quotient map

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- This defines a "period map" from the moduli space to a Grassmanian

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The assignment $G\mapsto Lie(G)$ is a bijection between the liftings G of H_0 to W (up to strict isomorphism) and the liftings of $MH_0(k)\twoheadrightarrow Lie(H_0)$ to a free quotient $M_0\twoheadrightarrow L$ over W.

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The period map (Lubin-Tate case)

- $\pi_{GM} : \mathscr{M}(\mathscr{O}_C) \to \mathbb{P}^{h-1}(C)$ studied by Gross-Hopkins.

Theorem (Gross-Hopkins)

(i) π_{GM} *is* $D^\times$*-equivariant.*

(ii) $\pi_{GM}(G,\iota) = \pi_{GM}(G',\iota') \Leftrightarrow \exists f : G \overset{q.i.}{\to} G',\ \iota' \circ \bar{f} = \iota.$

(iii) $\pi^0_{GM} : \mathscr{M}^0(\mathscr{O}_C) \to \mathbb{P}^{h-1}(C)$ *is surjective.*

(iv) $M_{\infty,\eta} \to M_\eta \to \mathbb{P}^{h-1}_C$ *gives* $\mathbb{P}^{h-1}_C = M_{\infty,\eta}/GL_h(\mathbb{Q}_p)$.

- Part (i) follows from the definitions. Action of $D^\times$ on $\mathbb{P}^{h-1}(C)$ is via the (projective) regular representation. The element Π acts (in appropriate coordinates) like

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Period domains and admissibility

- The "if" part in (ii) follows from the definitions. The "only if" follows from Grothendieck-Messing.
- Part (iii) is proved in [G-H] by a detailed analysis of π_{GM} "in coordinates".
- Part (iv) follows from (ii) and (iii).

In general, the image of π_{GM} is restricted by the notion of "weak admissibility". Given an exact sequence

$$0 \to Fil \to M_{0,C} \to W \to 0$$

with associated filtration $Fil^0 = M_{0,C} \supset Fil^1 = Fil \supset Fil^2 = 0$, $\underline{N} = (N, Fil) = (M_{0,C}, Fil)$ becomes a *filtered F-isocrystal.*

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- If N' is a sub-F-isocrystal let $Fil' = Fil \cap (N' \otimes_{W[1/p]} C)$, $\underline{N}' = (N', Fil')$.
- For any filtered F-isocrystal $\underline{N}$ define

$$t_{Newton}(\underline{N}) = v_p(\det(F))$$

(independent of the matrix representing F, since this matrix is unique up to σ-conjugation),

$$t_{Hodge}(\underline{N}) = \sum i \dim gr^i_{Fil^\bullet} = \dim Fil.$$

- Call $\underline{N} = (N, Fil)$ *weakly admissible* if for any sub F-isocrystal $N' \subset N$

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- Given $H_{0/k}$, the *weakly admissible period domain* is an open subspace $\mathfrak{F}^{wa} \subset Gr(d, M_0)^{ad}_{\eta}$ such that $\mathfrak{F}^{wa}(C)$ consists of all d-dimensional quotients

$$M_{0,C} \to W$$

for which $\underline{N}$ is weakly admissible.

Theorem

(i) The image of $\pi_{GM} : \mathscr{D}^{ad}_{\eta} \to Gr(d, M_0)^{ad}_{\eta}$ factors through $\mathfrak{F}^{wa}$.
(ii) The image contains all the classical points of $\mathfrak{F}^{wa}$ (points whose residue field is a finite extension of $K_0 = W[1/p]$).

Remarks: (i) is easy. (ii) (Colmez-Fontaine) "weakly admissible filtered isocrystals are admissible". We shall later relate it to the geometry of the Fargues-Fontaine curve. Hartl describes the *non-classical* points in $\mathfrak{F}^{a} = \mathrm{Im}(\pi_{GM})$. In general $\mathfrak{F}^{a} \neq \mathfrak{F}^{wa}$.

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The Hodge-Tate decomposition

Recall the map $\alpha_G : T_pG(R) \to \omega_{G^\vee/R}$. Let $R = \mathscr{O}_C$ and let $-(1)$ denote Tate twist. The following theorem was the begining of p-adic Hodge theory, 50 years ago.

Theorem (Tate)

(i) There is an exact sequence

$$0 \to Lie(G_C)(1) \overset{\alpha^\vee_{G^\vee}(1)}{\to} T_pG(\mathscr{O}_C) \otimes_{\mathbb{Z}_p} C \overset{\alpha_G}{\to} \omega_{G^\vee/C} \to 0.$$

(ii) (Hodge-Tate decomposition) If G is defined over $\mathscr{O}_K$ where $K \subset C$ is a complete discrete valuation field, then the sequence splits canonically (respecting $\Gamma_K = Gal(\bar{K}/K)$ action)

$$T_pG(\mathscr{O}_C) \otimes_{\mathbb{Z}_p} C \simeq \omega_{G^\vee/C} \oplus Lie(G_C)(1).$$

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- The map $\alpha^\vee_{G^\vee}(1)$ sends $Lie(G_C)(1) = Hom(\omega_{G/C}, T_p\mu_{p^\infty} \otimes C)$ to
$$Hom(T_pG^\vee \otimes C, T_p\mu_{p^\infty} \otimes C) \simeq T_pG \otimes C.$$

- To get (ii) from (i) invoke Tate's theorems that $H^0(\Gamma_K, C(i)) = H^1(\Gamma_K, C(i)) = 0$ if $i \neq 0$ and both cohomology groups are 1-dimensional if $i = 0$. In the absence of Galois action, there is no canonical splitting of (i).

- Let $G = \mathscr{A}[p^\infty]$. Dualizing, (i) is equivalent to the existence of a spectral sequence (Faltings: the Hodge-Tate spectral sequence)
$$E^2_{i,j} = H^i(\mathscr{A}, \Omega^j_{\mathscr{A}/C})(-j) \Rightarrow H^{i+j}_{et}(\mathscr{A}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C.$$
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- The fact that the Hodge-Tate decomposition is not valid in families, only the HT *filtration*, leads to the HT *period map*, just as over $\mathbb{C}$ the fact that only the Hodge *filtration* varies holomorphically in families lies behind the classical period map to classifying spaces of Hodge structures.

The **Hodge-Tate period map**. Consider (Lubin-Tate case)

$$(G,\iota,\alpha_\infty)\in\mathscr{M}_\infty(\mathscr{O}_C).$$

Use $\alpha_\infty:\mathbb{Z}_p^h\simeq T_pG(\mathscr{O}_C)\rightsquigarrow(\alpha_G\otimes 1)\circ(\alpha_\infty\otimes 1):C^h\to\omega_{G^\vee/C}$, whose kernel is a *line*. Mapping (G,ι,α_∞) to this line is

$$\pi_{HT}:\mathscr{M}_\infty(\mathscr{O}_C)\to\mathbb{P}^{h-1}(C).$$

Unlike π_{GM}, π_{HT} is defined only on $\mathscr{M}_\infty$. It goes *canonically* to $\mathbb{P}^{h-1}(C)$ while π_{GM} landed in $\mathbb{P}(M_0)(C)\simeq\mathbb{P}^{h-1}(C)$.

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- *Fact*: π_{HT} comes from an *analytic* map $M_{\infty,\eta} \to (\mathbb{P}^{h-1})^{ad}_{\eta}$. In *our "basic" case* (but not always), it is also *étale.*
- For $\delta \in D^{\times}$, $\pi_{HT} \circ \delta = \pi_{HT}$ (obvious).
- π_{HT} intertwines the actions of $GL_h(\mathbb{Q}_p)$ on $M_{\infty,\eta}$ and $\mathbb{P}^{h-1}$ (obvious).

A global detour ($h = 2$): *modular curves at the infinite level.* Let Y_n be the (open) modular curve of level p^n over $\mathbb{Q}_p$ and Y_∞ the scheme $\lim_{\leftarrow} Y_n$. A point of $Y_\infty(C)$ *is* an elliptic curve $E_{/C}$ equipped with an isomorphism $\alpha_\infty : \mathbb{Z}_p^2 \simeq T_p E$. As above, we get $\pi_{HT} : Y_\infty(C) \to \mathbb{P}^1(C)$. Let $\mathfrak{X} = \mathbb{P}^1(C) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ (the Drinfeld p-adic upper half plane).

Theorem

The map $\pi_{HT} : Y_\infty(C) \to \mathbb{P}^1(C)$ is surjective. We have $\pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p)) = Y_\infty(C)^{ord}$ (the pairs (E, α_∞) where E has bad, or good ordinary reduction) and $\pi_{HT}^{-1}(\mathfrak{X}) = Y_\infty(C)^{ss}$ (the pairs where E has good supersingular reduction).

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- Note the anomaly: at infinite level the "fat" set $Y_\infty(C)^{ord}$ gets mapped to the "meager" $\mathbb{P}^1(\mathbb{Q}_p)$ and the meager $Y_\infty(C)^{ss}$ fills up its complement $\mathfrak{X}$.
- If E is ordinary, $G = E[p^\infty] \rightsquigarrow T_pG^0$, the Tate module of the "kernel of reduction", a line in T_pG, spans $\ker(\alpha_G \otimes 1)$. This proves $\pi_{HT}(E, \alpha_\infty) \in \mathbb{P}^1(\mathbb{Q}_p)$. Conversely, if E is defined over a CDVF K and $\pi_{HT}(E, \alpha_\infty) \in \mathbb{P}^1(\mathbb{Q}_p)$ then $\Gamma_K \curvearrowright T_pG$ is potentially reducible, so E is ordinary. This proves the theorem, except for the surjectivity. In general:

Theorem

(i) The image of π_{HT} is the Drinfeld p-adic symmetric domain

$$\mathfrak{X}(C) = \mathbb{P}^{h-1}(C) \setminus \bigcup_{a \in (\mathbb{P}^{h-1})^*(\mathbb{Q}_p)} H_a.$$

(ii) π_{HT} induces $M_{\infty,\eta}/D^\times \simeq \mathfrak{X}(C)$ (on the level of C-points, so far).

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Rapoport-Zink spaces with PEL structure

- Recall k alg. closed field, char. p, $W = W(k)$, $H_{0/k}$ p-div gp, dim d, ht h, $M_0 = MH_0(W \twoheadrightarrow k)$ the covariant Dieudonné module.
- Ignore **Level** (L) - treated in [R-Z] by "multi-chains of lattices".
- **Endomorphisms** (E) -
  - Semi-simple algebra B over $\mathbb{Q}_p$ with a maximal order $\mathcal{O}_B \hookrightarrow End_k(H_0)$. Then $B \curvearrowright V = M_{0,\mathbb{Q}}$ (linear action) commuting with Frobenius, and $\mathcal{O}_B$ stabilizes the lattice $\Lambda = M_0$.
  - Fix B-stable decomposition $V = V_0 \oplus V_1$, dim $V_0 = d$, dim $V_1 = h - d$, $\Lambda \cap V_1$ reducing modulo p to $\omega_{H_0^\vee/k} \subset MH_0(k)$ and $\Lambda \cap V_0$ mapping onto $Lie(H_0)$.
- **Polarization** (P) - a quasi-isogeny $\lambda_0 : H_0 \overset{\sim}{\dashrightarrow} H_0^\vee$ inducing
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PEL deformation functor: Let $S \in \mathrm{Nilp}_W$.

- $\mathscr{D}(S) = \{(G,\iota,\lambda)\}/\simeq$ where:
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The functor $\mathscr{D}$ is "representable" by a formal scheme over W whose ideal of definition is locally finitely generated. Every irreducible component of its (reduced) special fiber is proper over k.

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The Drinfeld moduli problem

Definition. A *special formal $\mathcal{O}_D$-module* over $S \in \mathrm{Nilp}_W$ is a formal group $\mathscr{G}$ over S, of height h^2 and dimension h, equipped with $\mathcal{O}_D \hookrightarrow End_S(\mathscr{G})$, such that the induced representation of $\mathbb{Z}_{p^h} \subset \mathcal{O}_D$ on $Lie(\mathscr{G})$ is the regular representation (note $\mathbb{Z}_{p^h} \subset W \to S$).

- Fix $H_{0/k}$ a special formal $\mathcal{O}_D$-module (all isogenous). Explicitly:

$$H_0 = \mathrm{H}_0 \times \mathrm{H}_0^{\sigma} \times \cdots \times \mathrm{H}_0^{\sigma^{h-1}}$$

  where $\mathrm{H}^{\sigma^i}_{0/\mathbb{F}_p}$ is "the H_0" of the Lubin-Tate moduli problem of dim 1 and ht h with the $\mathcal{O}_D$-action defined over $\mathbb{F}_{p^h}$ (see Exercise) and twisted by σ^i.

- Besides the action of $\mathcal{O}_D$, note a (commuting) action of $GL_h(\mathbb{Q}_p)$ on H_0 by quasi-isogenies.
- $V \simeq D \otimes_{\mathbb{Z}_p} W$, $F = \Pi_r \otimes \sigma, \ldots$ EL moduli problem $\mathscr{D}^{Drin}$.
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  where $\mathrm{H}_{0/\mathbb{F}_p}^{\sigma^i}$ is "the H_0" of the Lubin-Tate moduli problem of dim 1 and ht h with the $\mathcal{O}_D$-action defined over $\mathbb{F}_{p^h}$ (see Exercise) and twisted by σ^i.

- Besides the action of $\mathcal{O}_D$, note a (commuting) action of $GL_h(\mathbb{Q}_p)$ on H_0 by quasi-isogenies.
- $V \simeq D \otimes_{\mathbb{Z}_p} W$, $F = \Pi_r \otimes \sigma, \ldots$ EL moduli problem $\mathscr{D}^{Drin}$.
- $GL_h(\mathbb{Q}_p) \curvearrowright \mathscr{D}^{Drin}$.

The Drinfeld moduli problem

Definition. A *special formal $\mathscr{O}_D$-module* over $S \in \mathrm{Nilp}_W$ is a formal group $\mathscr{G}$ over S, of height h^2 and dimension h, equipped with $\mathscr{O}_D \hookrightarrow End_S(\mathscr{G})$, such that the induced representation of $\mathbb{Z}_{p^h} \subset \mathscr{O}_D$ on $Lie(\mathscr{G})$ is the regular representation (note $\mathbb{Z}_{p^h} \subset W \to S$).

- Fix $H_{0/k}$ a special formal $\mathscr{O}_D$-module (all isogenous). Explicitly:

$$H_0 = \mathrm{H}_0 \times \mathrm{H}_0^{\sigma} \times \cdots \times \mathrm{H}_0^{\sigma^{h-1}}$$

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The moduli problem $\mathscr{D}^{Drin}$ had been considered by Drinfeld. It is the moduli problem of deformations of special formal $\mathscr{O}_D$-modules.

Theorem (Drinfeld)

The formal scheme $\mathscr{X}$ representing $\mathscr{D}^{Drin}$ is such that $\mathscr{X}^{an} \simeq \mathfrak{X}$.

In fact, the formal scheme structure on $\mathfrak{X}$ can be "read" from a reduction map

$$r : \mathfrak{X}(C) \to |\mathscr{B}\mathscr{T}|$$

to the Bruhat-Tits building of $PGL_h(\mathbb{Q}_p)$.

- When $h = 2$ the special fiber of $\mathscr{X}$ is a tree of $\mathbb{P}^1$'s, each intersecting transversally $p+1$ others at the $\mathbb{F}_p$-rational points. $|\mathscr{B}\mathscr{T}|$ is the $p+1$-regular tree; $r_v^{-1}(v)$, for a vertex v, is an affinoid isomorphic to the affinoid obtained from $\mathbb{P}^1$ upon removal of the $p+1$ $\mathbb{Q}_p$-rational residue disks, and $r^{-1}(\varepsilon)$, for an edge ε, is an open annulus.

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Lecture IV: Geometrization via the Fargues-Fontaine curve

1. The Fargues-Fontaine curve X^{FF}
   1. A review of some of Fontaine's rings
   2. The Fargues-Fontaine curve
   3. Line bundles and divisors
2. Vector bundles on the Fargues-Fontaine curve
   1. Vector bundles on X
   2. (B, v)-pairs and vector bundles
   3. p-divisible groups over $\mathcal{O}_C/p$ up to isogeny
   4. Cohomology of vector bundles
3. Vector bundles associated to p-divisible groups over $\mathcal{O}_C$
   1. Filtered F-isocrystals
   2. Modification of vector bundles

A review of some of Fontaine's rings

- $F = C^\flat$, $\mathcal{O}_F = \lim_{\leftarrow \times p} \mathcal{O}_C/p$, complete alg. closed (in particular *perfect*) non-arch. field, $char.F = p$.
  - $G_{\mathbb{Q}_p} = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on F.

  - Fix $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$, $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_F$.
  - If $x = (x^{(0)}, x^{(1)}, x^{(2)}, \dots) \in \mathcal{O}_F$ then $x^\sharp = \lim(\tilde{x}^{(m)})^{p^m} \in \mathcal{O}_C$ exists. But note that the definition of $x^\sharp$ *is not intrinsic to F: it presumes* the knowledge of F as the tilt of C !

- $A_{inf} = W(\mathcal{O}_F) \overset{\Theta}{\to} \mathcal{O}_C$, $\Theta(\sum_{n=0}^\infty p^n[x_n]) = \sum_{n=0}^\infty p^n x_n^\sharp$, a homomorphism!
  - $\ker(\Theta) = (\xi)$, $\xi = p - [p^\flat]$ "primitive element of degree 1".
  - $G_{\mathbb{Q}_p}$ acts by functoriality, compatible with Θ.
  - Frobenius φ acts on A_{inf}, does not preserve $\ker(\Theta)$.

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- $A_{cris} = p$ -adic completion of $A_{inf}[\xi^n/n!] \subset A_{inf}[1/p]$ (initial object in the category of p-adic pd-thickenings of $\mathscr{O}_C$).
  - $G_{\mathbb{Q}_p}$ and φ actions extend to A_{cris}.
  - $t = \log[\varepsilon] \in A_{cris}$, $\sigma(t) = \chi_{cyc}(\sigma)t$, $\varphi(t) = pt$, t unique up to $\mathbb{Z}_p^\times$, $\Theta(t) = 0$.
  - $B_{cris}^+ = A_{cris}[1/p] \xrightarrow{\Theta} C$, $B_{cris} = B_{cris}^+[1/t]$.

Lemma

The ring $B_e = B_{cris}^{\varphi=1} = \bigcup_{n=0}^{\infty} t^{-n} B_{cris}^{+,\varphi=p^n}$ (increasing union) is a PID and $B_e \cap B_{cris}^+ = \mathbb{Q}_p$. Moreover, $B_e^\times = \mathbb{Q}_p^\times$.

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- $B_{dR}^{+} = \lim_{\leftarrow} A_{inf}[1/p]/(\xi^n) \supset B_{cris}^{+}$, but much cruder.
  - a CDVR, t a uniformizer, $B_{dR} = B_{dR}^{+}[1/t]$ a CDVF, v_{dR} the corresponding normalized valuation.
  - $\Theta : B_{dR}^{+}/tB_{dR}^{+} \simeq C$ (residue field).
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Theorem (Fundamental exact sequence of p-adic Hodge theory)

The following sequence is exact:

$$\text{(FES)} \quad \boxed{0 \to \mathbb{Q}_p \to B_{cris}^{\varphi=1} \to B_{dR}/B_{dR}^{+} \to 0.}$$

Remarks. (i) It is instructive to view (FES) as the analogue of

$$0 \to \mathbb{C} \to \mathbb{C}[z] \to \mathbb{C}((1/z))/\mathbb{C}[[1/z]] \to 0,$$

$B_{cris}^{+} \leftrightsquigarrow \mathbb{C}[x,y]$, $B_{cris}^{+,\varphi=p^n} \leftrightsquigarrow \mathbb{C}[x,y]^{\text{hom.deg.}n}$, $z = x/y$, $t \leftrightsquigarrow y$.
(ii) Works for any *perfectoid pair* (R, R^{+}) replacing $(C, \mathcal{O}_C)$.

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The Fargues-Fontaine curve

Definition

$X = X^{FF} = Proj(P)$, where $P = \oplus_{n=0}^{\infty} B_{cris}^{+,\varphi=p^n}$.

Theorem

(a) X is an integral, noetherian, regular, 1-dim scheme (a "curve").
(b) $\Theta \rightsquigarrow$ a closed point $\infty \in X$ with residue field C.
(c) $H^0(X - \{\infty\}, \mathcal{O}_X) \simeq B_e$.
(d) $\widehat{\mathcal{O}}_{X,\infty} \simeq B_{dR}^+$.
(e) (FES) $\Rightarrow H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$.

Although X is not of finite type over $P_0 = \mathbb{Q}_p$ *(e)* is an indication that X is "complete". We shall see that it has a *theory of divisors* and behaves as if it had genus 0 (taken with a grain of salt...).

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Remark. The closed points $|X|$ *are* in bijection with "Frobenius orbits of **untilts** of F". More precisely:

- An untilt of F is a pair (C', ι') where C' is a complete algebraically closed non-archimedean field containing $\mathbb{Q}_p$ and $\iota' : (C')^\flat \simeq F$. Let $|Y|$ be the set of untilts, up to equivalence. (Kedlaya and Temkin have shown that untilts need not be isomorphic to C even as abstract topological fields, ignoring ι).
- Frobenius acts: $\varphi(C', \iota') = (C', \varphi \circ \iota')$, and $|X| \simeq |Y|/\varphi^{\mathbb{Z}}$.
- $|Y|$ is the set of closed points of an analytic (adic) space $\mathfrak{Y}$ and $\mathfrak{Y} \to \mathfrak{X} = X^{an}$ is étale.
- If $(C', \iota') = y \in |Y|$, $\exists \Theta_y : \mathscr{O}_{\mathfrak{Y},y} \twoheadrightarrow C'$ just like $\Theta = \Theta_\infty$.
- If $y \mapsto x \in |X|$ then (*) $\widehat{\mathscr{O}}_{X,x} \simeq \widehat{\mathscr{O}}_{\mathfrak{X},x} \simeq \widehat{\mathscr{O}}_{\mathfrak{Y},y}$.
- The choice of C as a distinguished untilt dictates $\infty \in \mathfrak{Y}$, above $\infty \in \mathfrak{X}$. The element t is a uniformizer at $\infty \in \mathfrak{Y}$, but by (*) becomes a uniformizer at $\infty \in X$.

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If $f \in Frac(B_e)$, define the divisor $div(f) = \sum_{x \in |X|} ord_x(f) \cdot [x]$ as usual. Such a divisor is called principal.

Theorem

A divisor is principal if and only if it is of degree 0.

For $d \in \mathbb{Z}$ let $\mathscr{O}(d)$ be the line bundle on X associated with the graded module $P(d) = \oplus_{n \in \mathbb{Z}} B_{cris}^{+,\varphi = p^{n+d}}$. An equivalent formulation is:

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Line bundles and divisors on X

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Vector bundles on X

Vector bundles: On $\mathbb{P}^1_{\mathbb{C}}$ every vector bundle is a direct sum of line bundles (Grothendieck). Here the analogy between X and $\mathbb{P}^1_{\mathbb{C}}$ *breaks down* for the first time. Let $\lambda = d/h \in \mathbb{Q}$ (reduced, $h > 0$) and let $\mathscr{O}(\lambda)$ be the vector bundle associated with the graded module

$$P(\lambda) = \oplus_{n\in\mathbb{Z}}(N_{-\lambda} \otimes_{W[1/p]} B^+_{cris})^{\varphi=p^n}.$$

($N_{-\lambda}$ is the standard isocrystal of slope $-\lambda$). Recall that the degree of a vector bundle $\mathscr{V}$ is the degree of the line bundle $\det(\mathscr{V})$.

Theorem

(i) $\mathscr{O}(\lambda)$ is a vector bundle of rank h, degree d and Harder - Narasimhan slope λ.
(ii) Every vector bundle on X is $\oplus_\lambda \mathscr{O}(\lambda)^{m_\lambda}$ for unique $m_\lambda \in \mathbb{N}$.
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Theorem says that the functor $\mathscr{E} : \mathrm{FIsocrys}_k \rightsquigarrow \mathrm{VecBun}_X$,

$$\boxed{\mathscr{E}(N,\varphi) = (\oplus_{n\in\mathbb{Z}}(N \otimes_{W[1/p]} B^+_{cris})^{\varphi=p^n})^{\sim},}$$

is essentially surjective. But it is far from being an equivalence!

Corollary

$\pi_1^{et}(X) \simeq Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Key step: Theorem $\Rightarrow$ if $f : X' \to X$ is finite étale, $f_*\mathscr{O}_{X'} \simeq \mathscr{O}_X^{\deg(f)}$.

- *Alternative description of vector bundles over* $\mathbb{P}^1_{\mathbb{C}}$ (**Beauville - Laszlo gluing**): a rk r vector bundle $\mathscr{V} \leftrightsquigarrow$ a finite free $\mathbb{C}[z]$-module V, a finite free $\mathbb{C}[[1/z]]$-module V_∞, and

$$\rho : V \otimes_{\mathbb{C}[z]} \mathbb{C}((1/z)) \simeq V_\infty \otimes_{\mathbb{C}[[1/z]]} \mathbb{C}((1/z)).$$

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Theorem says that the functor $\mathscr{E} : \mathrm{FIsocrys}_k \rightsquigarrow \mathrm{VecBun}_X$,

$$\boxed{\mathscr{E}(N,\varphi) = (\oplus_{n\in\mathbb{Z}}(N\otimes_{W[1/p]} B^+_{cris})^{\varphi=p^n})^\sim,}$$

is essentially surjective. But it is far from being an equivalence!

Corollary

$\pi_1^{et}(X) \simeq Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Key step: Theorem $\Rightarrow$ if $f : X' \to X$ is finite étale, $f_*\mathscr{O}_{X'} \simeq \mathscr{O}_X^{\deg(f)}$.

- *Alternative description of vector bundles over* $\mathbb{P}^1_{\mathbb{C}}$ (**Beauville - Laszlo gluing**): a rk r vector bundle $\mathscr{V} \leftrightsquigarrow$ a finite free $\mathbb{C}[z]$-module V, a finite free $\mathbb{C}[[1/z]]$-module V_∞, and

$$\rho : V\otimes_{\mathbb{C}[z]}\mathbb{C}((1/z)) \simeq V_\infty\otimes_{\mathbb{C}[[1/z]]}\mathbb{C}((1/z)).$$

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(B,ν)-pairs and vector bundles

Definition (Berger)

A (B,ν)-pair is $\underline{M} = (M_e, M_{dR}^+, \rho)$ where M_e is a finite free B_e-module, M_{dR}^+ a finite free B_{dR}^+-module and

$$\rho : M_e \otimes_{B_e} B_{dR} \simeq M_{dR}^+ \otimes_{B_{dR}^+} B_{dR} =: M_{dR}.$$

The degree $\deg(\underline{M})$ is defined by the same procedure as above, replacing ν_∞ by ν_{dR}.

Proposition

The category VecBun_X *of vector bundles over* X *is equivalent to the category of* (B,ν)*-pairs. The map is*

$$\mathscr{V} \mapsto (H^0(X - \{\infty\}, \mathscr{V}), \mathscr{V} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_{X,\infty}),$$

and in the opposite direction by "Beauville-Laszlo gluing". The correspondence respects ranks and degrees, hence slopes.

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Remark. Let $N \in \mathrm{FIsocryst}_k$. The vector bundle $\mathscr{E}(N,\varphi)$ corresponds to the pair

$$((N \otimes B_{cris})^{\varphi=1}, N \otimes B_{dR}^+).$$

However, the Harder-Narasimhan slope of $\mathscr{E}(N,\varphi)$ is the *negative* of the Frobenius slope of N.

Corollary

Canonically, $H^0(X, \mathscr{E}(N,\varphi)) \simeq (N \otimes B_{cris}^+)^{\varphi=1}$.

Proof.

$H^0(X, \mathscr{E}(N,\varphi)) = (N \otimes B_{cris})^{\varphi=1} \cap N \otimes B_{dR}^+ = (N \otimes B_{cris}^+)^{\varphi=1}$ (the last equality needs justification, even if N is trivial). □

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p-div gps over $\mathscr{O}_C/p$ up to isogeny

Recall: The category "p-div gps over k up to isogeny" is equivalent to the full subcategory of F-isocrystals whose slopes lie in $[0,1]$ (Dieudonné-Manin). We examine the same category, but over $\mathscr{O}_C/p$. Its "objects up to isomorphism" are in bijection with those of the same category over k (a consequence of the *isotriviality theorem*) but the category is much richer, and far from semi-simple!

- If G is a p-div gp over $\mathscr{O}_C/p$ let

$$M_{cris}(G) = MG(A_{cris} \twoheadrightarrow \mathscr{O}_C/p)$$

("**crystalline Dieudonné module**"). Then $M_{cris}(G)[1/p]$ is a finite projective B^+_{cris}-module. Let $\mathscr{E}(G)$ be the vector-bundle associated to the graded P-module

$$\oplus_{n=0}^{\infty}(M_{cris}(G)[1/p])^{\varphi=p^{n+1}}.$$

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- Let H_0 be a p-div gp over k such that $G \overset{q.i.}{\dashrightarrow} H_0 \times_k \mathscr{O}_C/p$. The given quasi-isogeny determines an isomorphism $M_{cris}(G) \simeq A_{cris} \otimes_W M_0$, hence

$$\mathscr{E}(G) \simeq \mathscr{E}(M_0(1), \varphi).$$

However, the *functorial dependence of $\mathscr{E}(G)$ on G can not be read from H_0 alone!*

- *Example*: Lubin-Tate case: H_0 unique formal p-div gp of ht h, dim 1: Then $M_0 \simeq N_{(h-1)/h}$, $M_0(1) \simeq N_{-1/h}$ so $\mathscr{E}(G) \simeq \mathscr{O}(1/h)$ (the isom. *depending* on the q.i. above).

Theorem (Full-faithfullness, Scholze-Weinstein)

(i) The functor $M_{cris}(-)$ is fully faithful, i.e.

$$Hom_{\mathscr{O}_C/p}(G, G') \simeq Hom_{A_{cris},\varphi}(M_{cris}(G), M_{cris}(G')).$$

(ii) The functor $\mathscr{E}(-)$ is an equivalence between the category of "p-div gps over $\mathscr{O}_C/p$ up to isogeny" and the full subcategory of vector bundles over X all of whose slopes lie in the interval $[0,1]$.

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Corollary (Universal covering as global sections)

Canonically,

$$\widetilde{G}(\mathcal{O}_C/p) \simeq H^0(X, \mathcal{E}(G)), \; MG(\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/p)[1/p] \simeq i_\infty^* \mathcal{E}(G)(-1).$$

Proof.

We have $\widetilde{G}(\mathcal{O}_C/p) \simeq Hom_{\mathcal{O}_C/p}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, G)[1/p]$. By "full-faithfulness" (in the isogeny category is enough!) this is

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But $M(\underline{\mathbb{Q}_p/\mathbb{Z}_p})[1/p] = D(\mu_{p^\infty})[1/p] \simeq N_1$ so we get

$$\widetilde{G}(\mathcal{O}_C/p) \simeq Hom_{B^+_{cris},\varphi}(B^+_{cris} \otimes N_1, M_{cris}(G)[1/p])$$

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Lemma

Let $\mathscr{V}$ be a vector bundle on X. Then $H^i(X,\mathscr{V})=0$ for $i\geq 2$, and if (M_e, M_{dR}^+, ρ) is the associated (B,ν)-pair there is a "Mayer-Vietoris" exact sequence

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This enables one to calculate the cohomology. We have already seen (i) of the following theorem.

Theorem

(i) Let $\lambda = d/h$ (reduced, $h>0$). Then $H^0(X,\mathscr{O}(\lambda))=0$ if $\lambda<0$ and is equal to $(B_{cris}^+)^{\varphi^h=p^d}$ otherwise.
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Remarks. (i) Once again, $H^1(X, \mathscr{O}_X) = 0$ may be counted as an indication of "genus 0", but note the *second* time the analogy with $\mathbb{P}^1_{\mathbb{C}}$ breaks down: $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}}(-1)) = 0$.
(ii) The spaces $H^0(X, \mathscr{O}(\lambda))$ are $\mathbb{Q}_p$-vector spaces, but for $\lambda > 0$ they are *never* finite dimensional. In fact they belong to a very interesting category of "Banach-Colmez vector spaces". More to come soon, when we relate them to the (LOG) exact sequence.

- **Extensions**. The vector bundles $\mathscr{O}(\lambda)$ are indecomposable, but not irreducible. In general, if

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

is a short exact sequence of vector bundles, both *rk* and deg are "Euler-Poincaré characteristics" so the slope $\mu = \deg / rk$ satisfies the usual Harder-Narasimhan formalism

$$\mu(\mathscr{E}) = \frac{rk(\mathscr{E}')}{rk(\mathscr{E})}\mu(\mathscr{E}') + \frac{rk(\mathscr{E}'')}{rk(\mathscr{E})}\mu(\mathscr{E}'').$$

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Proposition

A vector bundle on the Fargues-Fontaine curve is semistable if and only if it is isoclinic (has only one slope).

Example. For $n \leq 0$, $Ext^1(\mathscr{O}(1-n), \mathscr{O}(n)) \simeq H^1(X, \mathscr{O}(2n-1)) \neq 0$, so there is a non-split extension

$$(*) \quad 0 \to \mathscr{O}(n) \to \mathscr{O}(1/2) \to \mathscr{O}(1-n) \to 0.$$

Take $n = 0$, fix $\mathscr{E}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq \mathscr{O}$, $\mathscr{E}(G) \simeq \mathscr{O}(1/2)$, $\mathscr{E}(\mu_{p^\infty}) \simeq \mathscr{O}(1)$. By the equivalence of categories, $(*) \leftrightsquigarrow$ a unique

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Note that α and β are only quasi-homomorphisms and that modulo $\mathfrak{m}_C$ we have $\alpha \equiv \beta \equiv 0$.

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Filtered F-isocrystals

Goal: Upgrade "p-div gps over $\mathcal{O}_C/p$ up to isogeny" to the same category over $\mathcal{O}_C$. *Insight* (Grothendieck): should add a "Hodge filtration". We have already seen the following definition when we discussed the weakly admissible period domain $\mathfrak{F}^{wa}$. Let $K_0 = W(k)[1/p] \subset K$ be a finite ext'n.

Definition

A K-filtered F-isocrystal (over k) is $\underline{D} = (D, \varphi, Fil^\bullet)$ where (D, φ) is an F-isocrystal and $Fil^\bullet$ is a separated exhaustive descending filtration on D_K. Define the *slope* μ *by*

$$t_{Newton}(\underline{D}) = v_p(\det(\varphi)),\ \ t_{Hodge}(\underline{D}) = \sum i \dim gr^i_{Fil^\bullet} D_K,$$

$$\deg(\underline{D}) = t_{Hodge}(\underline{D}) - t_{Newton}(\underline{D}),\ \ \mu(\underline{D}) = \deg(\underline{D})/rk(\underline{D}).$$

Call $\underline{D}$ *semistable* if for any strict sub-object $\underline{D}'$ (*strict* means that the filtration on D'_K is induced by that of $\underline{D}$) $\mu(\underline{D}') \leq \mu(\underline{D})$, and *weakly admissible* $\Leftrightarrow$ s.st. of slope 0.

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Modifications of vector bundles

$\underline{D}$ a K-filtered F-isocrystal, $\mathscr{E}(\underline{D}) = \mathscr{E}(D, \varphi, Fil^\bullet)$ v.b. associated with the (B, ν)-pair $((D \otimes_{K_0} B_{cris})^{\varphi=1}, Fil^0(D_K \otimes_K B_{dR}))$.

- The degree (slope) of $\mathscr{E}(\underline{D})$ are the *same as* those of $\underline{D}$.
- $Fil^1 D = 0 \Rightarrow Fil^0(D_K \otimes_K B_{dR}) \subset D \otimes_{K_0} B_{dR}^+ \rightsquigarrow$ exact sequence

$$0 \to \mathscr{E}(D, \varphi, Fil^\bullet) \to \mathscr{E}(D, \varphi) \to i_{\infty,*}(D \otimes_{K_0} B_{dR}^+ / Fil^0) \to 0,$$

  last term a finite length “skyscraper sheaf” supported at ∞.
- Gives a *modification* of vector bundles at ∞. Similarly define modifications with “legs” at several points. Relax “$Fil^1 D = 0$” by allowing the first arrow to go backwards. Notion (similar to Drinfeld's “shtukas”) is key to the geometrization of LLC (Scholze and Fargues).

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Lecture V: Applications: Classification over $\mathcal{O}_C$, Galois representations and duality

1. p-divisible groups over $\mathcal{O}_C$ up to isogeny
   1. The big diagram revisited
   2. A classification
2. Applications to Galois representations of G_K
   1. Crystalline Galois representations
   2. Weakly admissible is admissible
3. Duality between the Lubin-Tate and Drinfeld towers
   1. The Drinfeld tower
   2. A simple proof of a theorem of Faltings

- G - p-div gp over $\mathscr{O}_C$, $G_0 = G \times_{\mathscr{O}_C} \mathscr{O}_C/p$, and as in RZ $\iota : G_0 \overset{q.i.}{\dashrightarrow} H_0 \times_k \mathscr{O}_C/p$, $h = ht$, $d = \dim$.
- $M_0 = D(H_0^\vee)$ covariant Dieudonné module of H_0. Then

$$\mathscr{E}(G_0) \overset{\iota_*}{\simeq} \mathscr{E}(M_0(1)_{\mathbb{Q}}, \varphi) := \mathscr{E}.$$

Define the trivial vector bundle

$$\mathscr{F} = V_p G(\mathscr{O}_C) \otimes_{\mathbb{Q}_p} \mathscr{O}_X.$$

Theorem (Scholze-Weinstein)

(i) There is a natural modification of vector bundles associated with (G, ι)

$$\boxed{0 \to \mathscr{F} \to \mathscr{E} \to i_{\infty,*}(Lie\, G_C) \to 0.}$$

Furthermore, if G is defined over $\mathscr{O}_K$ for a finite K/K_0, $\mathscr{F} \simeq \mathscr{E}(D, \varphi, Fil^\bullet)$ where $D = M_0(1)_{\mathbb{Q}}$, $Fil^{-1} = D_K$, Fil^0 is of rank $h - d$, $Fil^1 = 0$, and $(D, \varphi, Fil^\bullet)$ is weakly admissible.

The big diagram revisited

Theorem (continued)

(ii) The global sections of the exact sequence in (i) are identified canonically with the exact seqence

$$\text{(LOG)} \quad \boxed{0 \to V_p G(\mathcal{O}_C) \to \widetilde{G}(\mathcal{O}_C) \xrightarrow{\theta} \mathit{Lie}\, G_C \to 0.}$$

(iii) The fiber at ∞ of the exact sequence in (i) (i.e. taking $- \otimes_{\mathcal{O}_{X,\infty}} C$) is

$$\boxed{V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} C \to MG(\mathcal{O}_C)[1/p] \to \mathit{Lie}\, G_C \to 0.}$$

The first arrow factors through the Hodge-Tate map

$$\boxed{V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} C \overset{\alpha_G}{\twoheadrightarrow} \omega_{G^\vee/C} \hookrightarrow MG(\mathcal{O}_C)[1/p].}$$

(iv) The map "global sections to fiber at ∞" is "q log".

Example. Back to Lubin-Tate, $(G,\iota)\in\mathscr{M}(\mathscr{O}_C)$. As already seen,

$$\widetilde{G}(\mathscr{O}_C)=\widetilde{G}_0(\mathscr{O}_C/p)\simeq H^0(X,\mathscr{E})=B_{cris}^{+,\varphi^h=p},$$

and 1-dim'l $Lie\, G_C$ may be identified with C (choice of a parameter). The map θ is then identified with

$$\Theta: B_{cris}^{+,\varphi^h=p}\to C.$$

The relation between "p-divisible groups" and "modifications" allows to give a ***complete classification of*** p***-div gps over*** $\mathscr{O}_C$ ***up to isogeny***.

- Let $\mathscr{C}$ be the category of modifications

$$(*)\ 0\to\mathscr{F}\to\mathscr{E}\to i_{\infty,*}(W)\to 0$$

where (i) $\mathscr{F}$ and $\mathscr{E}$ are vector bundles over X (ii) $\mathscr{F}\simeq\mathscr{O}_X^h$ for some $h\in\mathbb{N}$ (iii) W is a f. dim'l C-vector space (B_{dR}^+-module killed by t, modification is *minuscule*).

- Let $\mathscr{C}'$ be the category of pairs (V, W) where V is a $\mathbb{Q}_p$-v.sp. and $W \subset V_C$ a sub C-v.sp. (*no extra structure!*).

Theorem (Scholze-Weinstein, Fargues-Fontaine)

The categories of p-divisible groups over $\mathscr{O}_C$ up to isogeny, $\mathscr{C}$ and $\mathscr{C}'$ are all naturally equivalent.

sketch.

To pass from $\mathscr{C}$ to $\mathscr{C}'$ let $V = H^0(X, \mathscr{F})$. To go backwards let $\mathscr{F} = V \otimes \mathscr{O}_X$. In both directions, we relate the extension $(*)$ to $W \hookrightarrow V_C$ as follows. A basic computation shows that $Ext(i_{\infty,*}C, \mathscr{O}_X) \simeq C$, hence $Ext(i_{\infty,*}(W), \mathscr{F}) \simeq Hom_C(W, V_C)$. Here, the extension $(*)$ associated to a homomorphism $u : W \to V_C$ is the pull-back of

$$0 \to V \otimes \mathscr{O}_X \to V \otimes \mathscr{O}_X(1) \to V \otimes i_{\infty,*}(C) \to 0$$

under $i_{\infty,*}u$. Note $\mathscr{E}$ is locally free $\Leftrightarrow$ u is injective. □

cont'd.

The above construction also shows that in any extension like $(*)$ we have

$$\mathscr{O}_X^h \subset \mathscr{E} \subset \mathscr{O}_X(1)^h$$

so all the slopes of $\mathscr{E}$ lie in $[0,1]$, by semistability of isoclinic vector bundles. Thus $\mathscr{E} = \mathscr{E}(G_0)$ for a p-div gp G_0 over $\mathscr{O}_C/p$, unique up to isogeny. We wish to upgrade the equivalence $G_0 \rightsquigarrow \mathscr{E}(G_0)$ to an equivalence between p-div gps (up to isogeny) over $\mathscr{O}_C$ and the category of modifications $\mathscr{C}$.

We have already seen how to associate with a p-div gp G over $\mathscr{O}_C$ a modification in $\mathscr{C}$ with $V = V_pG(\mathscr{O}_C)$, $W = Lie(G_C)$, $\mathscr{F} = V \otimes \mathscr{O}_X$, $\mathscr{E} = \mathscr{E}(G_0)$. This is functorial, and the key steps are to prove (i) that it is fully faithful (ii) that it is essentially surjective. For the details, see [S-W], §5.2. □

cont'd.

We only remark that one works first at the generic fiber of the adic spaces, building G_η^{ad} out of the multiplicative group $G' = T_pG(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ mimicking the construction of the modification $(*)$ out of $W \hookrightarrow V_C$. Namely, one defines G_η^{ad} as the fiber product

$$\begin{array}{ccc} G_\eta^{ad} & \to & W \otimes \mathbb{G}_a \\ \downarrow & & \downarrow \\ (G')_\eta^{ad} & \to & V_C \otimes \mathbb{G}_a \end{array}.$$

The special features of C are involved in the reconstruction of the formal group G from its generic fiber G_η^{ad}, which is pretty delicate. □

Remark. A remarkable feature of the classification over $\mathcal{O}_C$ is that it is in terms of linear algebra alone, and not semi-linear algebra as Dieudonné theory over k.

- Given G, the pair $W \hookrightarrow V_C$ is identified with the Hodge-Tate map $\alpha_{G^\vee}^\vee : Lie(G_C) \hookrightarrow V_p G(\mathscr{O}_C) \otimes_{\mathbb{Q}_p} C$ (we've ignored Tate twists).
- **Example: Assume** $h = 2$, $d = 1$. The only possibilities for $\mathscr{O}_X^2 \subset \mathscr{E} \subset \mathscr{O}_X(1)^2$ are $\mathscr{E} \simeq \mathscr{O} \oplus \mathscr{O}(1)$ or $\mathscr{E} \simeq \mathscr{O}(1/2)$. As we have seen before, the first occurs if $W = L_C$ for a $\mathbb{Q}_p$-rat'l line $L \subset V$ and the second otherwise.
- **Exercise**: Write all the possibilities when $h = 5$, $d = 2$ (there are 7 such) and the corresponding Newton polygons.

Crystalline Galois representations

- $k = \bar{\mathbb{F}}_p$, $K_0 = W(k)[1/p]$, $K_0 \subset K \subset C$ a finite extension.
- V - h-dim'l continuous $\mathbb{Q}_p$-rep'n of $G_K = Gal(\bar{K}/K)$.
- Define

$$D_{cris}(V) = (V \otimes B_{cris})^{G_K}, \; D_{dR}(V) = (V \otimes B_{dR})^{G_K}.$$

  The first is a K_0-v.sp. and inherits an action of φ, the second is a K-v.sp. and inherits a filtration $Fil^\bullet$.
- $\dim_{K_0} D_{cris}(V) \leq h$, $\dim_K D_{dR}(V) \leq h$ and V is called **crystalline** (resp. de-Rham) if equalities hold. We have $D_{cris}(V)_K \subset D_{dR}(V)$ (with equality if V is crystalline), so with the induced filtration $\underline{D}(V) = (D_{cris}(V), \varphi, Fil^\bullet)$ becomes a K-filtered φ-module.
- If V is crystalline, $D = D_{cris}(V)$, one recovers (Fontaine)

$$V = (D \otimes_{K_0} B_{cris})^{\varphi=1} \cap Fil^0(D_K \otimes B_{dR}).$$

- **Interpretation via the Fargues-Fontaine curve**: Assume V is crystalline.

$$V \rightsquigarrow \mathscr{E}(V) := \mathscr{E}(D_{cris}(V), \varphi, Fil^\bullet), \;\; V = H^0(X, \mathscr{E}(V)).$$

Lemma

V crystalline $\Rightarrow$ $\underline{D}(V)$ weakly admissible.

Proof.

Write $\underline{D} = \underline{D}(V)$ and $\mathscr{E}(V) \simeq \oplus \mathscr{O}(\lambda_i)$. $\dim_{\mathbb{Q}_p} H^0(X, \mathscr{E}(V)) < \infty$ implies all $\lambda_i \leq 0$. Since this dimension is exactly $h = rk\mathscr{E}(V)$, all $\lambda_i = 0$ and $\mathscr{E}(V)$ is trivial. This implies $\mu(\underline{D}) = 0$.
Suppose $\underline{D}' \subset \underline{D}$ is a *strict* sub-object. Then $\mathscr{E}(\underline{D}') \subset \mathscr{E}(\underline{D}) \simeq \mathscr{O}_X^h$ is a *sub-bundle*. But $\mathscr{E}(\underline{D})$ isoclinic $\Rightarrow$ semi-stable, so $\mu(\underline{D}') = \mu(\mathscr{E}(\underline{D}')) \leq 0$, showing that $\underline{D}$ is semi-stable. □

- The converse is the celebrated "weakly admissible = admissible" theorem.

Weakly admissible is admissible

Theorem (Colmez-Fontaine)

Every weakly admissible K-filtered φ-module is $\underline{D}(V)$ for a crystalline representation V.

Proof.

Suppose $\underline{D}$ is weakly admissible. Then $\mathscr{E}(\underline{D})$ is semistable of slope 0, so by the classification of vector bundles on X must be trivial. i.e. isomorphic to $\mathscr{O}_X^h$. This means (using the language of (B, ν_{dR})-pairs) that

$$V = H^0(X, \mathscr{E}(D)) \simeq (D \otimes_{K_0} B_{cris})^{\varphi=1} \cap Fil^0(D_K \otimes_K B_{dR})$$

is h-dimensional. But it is known that this equality of dimensions forces $D = D_{cris}(V)$, hence V is crystalline. □

- Change notation: $M_n^{LT} = M_{n,\eta} = \mathscr{M}_{n,\eta}^{ad}$ the generic fiber of the Lubin-Tate tower, similarly $M_\infty^{LT} = \mathscr{M}_{\infty,\eta}^{ad}$, a perfectoid space over $Spa(W[1/p], W)$.
- $M_0^{Drin} = (\mathscr{D}^{Drin})_\eta^{ad}$ the generic fiber of the formal scheme representing the Drinfeld moduli problem, over $Spa(W[1/p], W)$.
- As in the Lubin-Tate case, there is a *finite étale cover* $M_n^{Drin} \to M_0^{Drin}$ (in the category of adic spaces) representing triples (H, ι, α_n) where $(H, \iota) \in M_0^{Drin}$ and $\alpha_n : \mathscr{O}_D/p^n \to H[p^n]$ is an "analytic $\mathscr{O}_D$ level p^n structure" (meaning that at any geometric point it induces an isomorphism onto the p^n-torsion, compatible with the action of $\mathscr{O}_D$).

- The role of the groups $D^\times$ and GL_h is interchanged: $GL_h(\mathbb{Q}_p)$ acts on each M_n^{Drin}. The Galois group of $M_n^{Drin} \to M_0^{Drin}$ is $(\mathscr{O}_D/p^n)^\times$.
- *Fact:* $\exists M_\infty^{Drin}$, adic space over $Spa(W[1/p], W)$, representing the functor on complete affinoid $(W[1/p], W)$ algebras (R, R^+)

$$M_\infty^{Drin}(R,R^+) = \{(H,\iota,\alpha) | (H,\iota) \in M_0^{Drin}(R,R^+), \alpha ...\}/\simeq$$

where $\alpha : \mathscr{O}_D \to T_p H_\eta^{ad} \sim \lim_\leftarrow H[p^n]$ is $\mathscr{O}_D$-compatible and induces an isomorphism on any geometric point of $Spa(R,R^+)$.

- *Fact:* M_∞^{Drin} is (pre)perfectoid, $M_\infty^{Drin} \sim \lim_\leftarrow M_n^{Drin}$.
- There are analytic maps defined as in LT case

$$\pi_{GM}^{Drin} : M_\infty^{Drin} \to M_0^{Drin} = \bigsqcup_{i \in \mathbb{Z}} \mathfrak{X} \to \mathfrak{X} \simeq M_\infty^{Drin}/D^\times$$

$$\pi_{HT}^{Drin} : M_\infty^{Drin} \to (\mathbb{P}_W^{h-1})_\eta^{ad} \simeq M_\infty^{Drin}/GL_{h-1}(\mathbb{Q}_p).$$

The duality theorem

Theorem (Faltings, Fargues, Scholze-Weinstein)

There is a canonical isomorphism of adic spaces $M_\infty^{LT} \simeq M_\infty^{Drin}$, compatible with the action of $GL_h(\mathbb{Q}_p) \times D^\times$, under which the period maps are interchanged:

$$\pi_{GM}^{Drin} = \pi_{HT}^{LT}, \ \pi_{GM}^{LT} = \pi_{HT}^{Drin}.$$

The original proof was difficult, partly because of missing language. [S-W], Theorem 7.2.3 and [F-F] 8.3.5 gave a conceptual proof using the equivalence of the moduli problems represented by the towers with categories of modifications of vector bundles on the FF curve. We shall outline the main construction at the level of $(C, \mathscr{O}_C)$-points, as usual.

- Let $(G, \imath, \alpha_\infty) \in M_\infty^{LT}(C, \mathcal{O}_C)$. This gives a *trivialized* modification
$$0 \to \mathcal{O}_X^h \to \mathcal{E} \to i_{\infty,*}W \to 0$$
where $W = Lie(G_C)$, $\mathcal{E} = \mathcal{E}(H_0) = \mathcal{O}(1/h)$ and $\mathcal{E}(G_0)$ is identified with $\mathcal{E}$ using $\imath$. The *trivialization* is the identification of the kernel of the map to the Lie algebra, canonically given as $V_pG(\mathcal{O}_C) \otimes \mathcal{O}_X$, with $\mathcal{O}_X^h$. It uses α_∞.
- The group $GL_h(\mathbb{Q}_p) \simeq Aut(\mathcal{O}_X^h)$ acts on such a trivialized modification by push-out of the first factor. It does not change the modification class, but only its trivialization. The group $D^\times$ acts by changing the identification of $\mathcal{E}$ with $\mathcal{E}(G_0)$. This action yields a new modification.
- Apply the sheaf-hom functor $\mathcal{RH}om(-, \mathcal{O}(1/h))$. Get an exact sequence
$$0 \to \mathcal{E}nd(\mathcal{O}(1/h)) \to \mathcal{O}(1/h)^h \to \mathcal{E}xt^1(i_{\infty,*}W, \mathcal{O}(1/h)) \to 0.$$

- We used $\mathscr{E}xt^1(\mathscr{O}(1/h), \mathscr{O}(1/h)) = 0$ (easy).
- The first factor is canonically $D \otimes \mathscr{O}_X$, D acting naturally.
- D acts on the second factor via $D \simeq End(\mathscr{O}(1/h))$.
- The last factor is a skyscraper sheaf at ∞. Since W is 1-dimensional it is $i_{\infty,*}W'$ where

$$W' \simeq Ext^1_{B^+_{dR}}(C, (B^+_{dR})^h) \simeq C^h$$

  and $\mathbb{Q}_{p^h} \subset D$ acts with h distinct characters, each with multiplicity 1.
- $\rightsquigarrow$ a "special trivialized modification of vector bundles with D-action", which (by an analogue of the main theorem with PEL structure) corresponds to a triple $(G', \iota', \alpha'_\infty) \in M^{Drin}_\infty(C, \mathscr{O}_C)$.

- The D action described above was obtained from the action of D on $\mathscr{O}(1/h)$ when we took $\mathscr{RH}om(-, \mathscr{O}(1/h))$ and therefore did not change the new trivialized modification, but *enhanced* it to correspond to a "moduli problem with endomorphisms in D". It was an *algebra action.*
- We still have the $D^\times \times GL_h(\mathbb{Q}_p)$ *group action* on the *set* of all "special trivialized modifications with D-action" and one checks that the functor $\mathscr{RH}om(-, \mathscr{O}(1/h))$ between the two categories of modifications preserves these actions.
- Finally, one constructs in a similar way a quasi-inverse, establishing the duality between $M_\infty^{Drin}(C, \mathscr{O}_C)$ and $M_\infty^{LT}(C, \mathscr{O}_C)$.