Lecture 1

<u>Goal</u>: Formula-free "inevitable" def. of W. We will actually use some formulas, but only in a superficial way — they are easily removed.

1. <u>p-derivations</u> + <u>δ-rings</u>

p = prime (fixed)
R = ring

<u>Def</u>: A ring endo $\varphi : R \to R$ is a <u>Frobenius</u> <u>lift</u> if
$\forall x \in R$, $\varphi(x) \equiv x^p \mod pR$

<u>E.g.</u>: $\mathbb{Z}[x]$, $\varphi : x \mapsto x^p + p \cdot \text{anything}$
global class field theory
crystalline cohomology
Adams operations ψ^p

Goal: leads inevitably to Witt vectors

* {Rings with Frob. lift} naturally forms a category
.... but <u>not</u> a good one!

Problem: "lift" has a hidden $\exists$:
$$\forall x \, \exists x' \text{ s.t. } \varphi(x) = x^p + px'.$$
x' is unique up to p-torsion — no control over it if R is not p-tor. free

<u>Ex</u>: ① Category doesn't have pullbacks, intersections of sub-objects?

Solution: Provide x' itself as part of the structure, rather than the property of its mere existence.

i.e. want an operator $\delta : R \to R$ modelled on $\delta(x) = \dfrac{\varphi(x) - x^p}{p}$

Axioms? Write the ring-endo axioms for φ in terms of δ.

* $\varphi(x+y) = \varphi(x) + \varphi(y)$

$(x+y)^p + p\delta(x+y) \qquad x^p + p\delta(x) + y^p + p\delta(y)$

$\therefore\ p\delta(x+y) = p\delta(x) + p\delta(y) + x^p + y^p - (x+y)^p$ ⟿ equiv. to the additivity of φ

(i) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$ ⟿ equiv. only mod p-tor

* $\varphi(xy) = \varphi(x)\varphi(y)$

$(xy)^p + p\delta(xy) = (x^p + p\delta(x))(y^p + p\delta(y))$

(ii) $\delta(xy) = x^p\delta(y) + \delta(x)y^p + p\delta(x)\delta(y)$

(iii) $\delta(0) = 0$

(iv) $\delta(1) = 0$

"Leibniz rules" for δ under $+, \times, 0, 1$. Buium: $\delta = $ "$\partial/\partial p$"

<u>Def</u>: A function $\delta: R \to R$ is a <u>p-derivation</u> if it satisfies (i)-(iv).
A <u>δ-ring</u> is a ring equipped with a p-derivation.

* $\{\text{p-der on } R\} \longrightarrow \{\text{Frob. lifts on } R\}$

$\delta \longmapsto \varphi$, where $\varphi(x) = x^p + p\delta(x)$

This is a bijection if R is p-tor free (but <u>not</u> in general!)

<u>E.g.</u>: • Any p-tor-free ring with Frob. lift: $\mathbb{Z}[x]$, $\delta(x) = $ anything
• $R = K_0(\mathcal{C})$. $\delta = \lambda$-operation assoc. to $\frac{(x_1^p + x_2^p + \dots) - (x_1 + x_2 + \dots)^p}{p}$

<u>Rmk</u>: There is a formula-free def. of a δ-structure.

<u>Ex</u>: ② $R = \mathbb{F}_p$-alg: R admits a p-der. $\Leftrightarrow R = \{0\}$, whereas all such R have Frob. lifts!
③ Same for $\mathbb{Z}/p^n\mathbb{Z}$-algebras.

2. Witt vectors

Goal:
$$\begin{tikzcd} \delta\text{-rings} \arrow[d, shift left] \\ \text{Rings} \arrow[u, bend left] \arrow[u, bend right, "W"'] \end{tikzcd}$$

Compare:
$$\begin{tikzcd} \{\text{differential rings}\} \arrow[d] \\ \text{Rings} \arrow[u, bend left] \arrow[u, bend right, "\text{``}W^{diff}\text{''}"'] \end{tikzcd} \qquad \begin{tikzcd} C\text{-mod} \arrow[d] \\ Ab \arrow[u, bend left, "C\otimes -"] \arrow[u, bend right, "Hom(C,-)"'] \end{tikzcd} \qquad \begin{tikzcd} G\text{-sets} \arrow[d] \\ Sets \arrow[u, bend left, "G\times -"] \arrow[u, bend right, "(-)^G"'] \end{tikzcd}$$

* Warm up with differential rings

$$W^{diff}(A) = \text{“divided power series”} = \Big\{ \sum_{n\geq 0} a_n \overbrace{\frac{t^n}{n!}}^{\text{formal place holder}} \;\Big|\; a_n \in A \Big\}$$

obvious ring str: $\frac{t^m}{m!}\cdot\frac{t^n}{n!} = \binom{m+n}{m}\frac{t^{m+n}}{(m+n)!}$

derivation: $d\left(\frac{t^n}{n!}\right) = \frac{t^{n-1}}{(n-1)!}$

Universal property:
$$\begin{tikzcd} d \circlearrowright R \arrow[rr, dashed, "\tilde{g}", "\exists!\ \text{diff}"'] \arrow[dr, "g"'] & & W^{diff}(A) \arrow[dl] \\ & A & \end{tikzcd}$$

$$\tilde{g}(r) = \text{“}g(\text{Taylor series of } r)\text{”} = \sum_{n\geq 0} g(d^{\circ n}(r))\frac{t^n}{n!}$$

Check: $\tilde{g}$ is diff. ring map lifting g, and is unique.

Alternative point of view:

$W^{diff}(A) = A\times A\times \cdots$

$\sum a_n \frac{t^n}{n!} = (a_0, a_1, a_2, \dots)$

$\frac{d}{dt}$ = shift left

Then $\tilde{g}: r \longmapsto (g(r), g(d(r)), g(d^{\circ 2}(r)), \dots)$

We can see that $\tilde{g}$ is actually the unique <u>set map</u> lifting g + equivariant.
So from this point of view, the ring str. on

$$W^{diff}(A) = A \times A \times \cdots$$

is forced to make $\tilde{g}$ a ring map.

In fact, the ring str. is a "purely syntactic" re-expression of the Leibniz rules for $d^{\circ n}$:

$$\sum a_i \frac{t^i}{i!} \sum b_j \frac{t^j}{j!} = \sum \binom{i+j}{i} a_i b_j \frac{t^{i+j}}{(i+j)!}$$

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = \Big(\dots, \underbrace{\sum_{i+j=n} \binom{n}{i} a_i b_j}, \dots\Big)$$

$$d^{\circ n}(xy) = \underbrace{\sum_{i+j=n} \binom{n}{i} d^{\circ i}(x)\, d^{\circ j}(y)}$$

same

* back to W

$W(A) = A \times A \times \cdots$ $\rightsquigarrow$ ring str. at the n^{th} component given by the Leibniz rules for $\delta^{\circ n}$ w.r.t. <u>both</u> $\underline{+}$ <u>and</u> $\underline{\times}$

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = \Big(a_0 + b_0,\ a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_0^i b_0^{p-i}, \dots\Big)$$

$$(a_0, a_1, \dots) \times (b_0, b_1, \dots) = (a_0 b_0,\ a_0^p b_1 + a_1 b_0^p + p a_1 b_1, \dots)$$

$$0 = (0, 0, 0, \dots)$$

$$1 = (1, 0, 0, \dots)$$

$$\delta : (a_0, a_1, a_2, \dots) \longmapsto (a_1, a_2, \dots)$$

$$\begin{tikzcd} \delta \circlearrowright R \arrow[r, dashed, "\tilde{g}"] \arrow[dr, "g"'] & W(A) \arrow[d] \\ & A \end{tikzcd}$$

$$\tilde{g}(r) = \big(g(r), g(\delta(r)), g(\delta^{\circ 2}(r)), \dots\big)$$

$\tilde{g}$ is the unique set map lifting g (and compat. with δ)
~~It~~ It is also a ring map by construction.

$\therefore$ W is the right adjoint!

Next time: Our W is canonically isom. to the usual Witt vector construction.
But not by the identity map!

Ex: ④ Prove the poly $P_n^+(x_0, y_0, \dots, x_n, y_n)$ s.t. $\delta^{\circ n}(x+y) = P_n^+(x, y, \dots, \delta^{\circ n}(x), \delta^{\circ n}(y))$
is unique. Similarly for multiplication.

Lecture 2

References

- Copenhagen class 2016
- §1 of recent paper with Gurney, arxiv: 1905.10495
- Joyal's original paper
- Notes from these lectures

Last time: alternative def. of $W(A)$

- change of emphasis : universal way of inflating A s.t. it has a Frob lift
  - equally interesting for A p-tor-free — even traditionally, W is born in char 0
  - no formulas, no lemmas needed (Bhatt : derived Frobenius lift)
  - it has a few drawbacks (later)
  - A perfect $/\mathbb{F}_p$: other defs using deformation theory, more relevant to Fontaine theory
    <u>maps out</u> universal property
  - We're more interested in the case where $A \neq$ perfect $\mathbb{F}_p$-alg.
  - Why? de Rham-Witt, foundations, prismatic cohomology, moduli...

3. Moduli interpretation

$R =$ δ-ring
$X = \mathrm{Spec}(\bar{R})$

$$\begin{tikzcd} X(A) \arrow[d, dashed] \arrow[r, phantom, "="] & \mathrm{Hom}(R,A) \arrow[d, "\sim"] & g \arrow[d, mapsto] \\ X(W(A)) & \mathrm{Hom}_\delta(R, W(A)) \arrow[l, hook'] & \tilde{g} \end{tikzcd}$$

- If a moduli space has a δ-structure, then the objects it classifies have a theory of canonical lifts. It works over an <u>arbitrary</u> base.

E.g. Canonical lifts for arbitrary families of ordinary elliptic curves (Gurney)

4. p-differential operators

$$\begin{array}{ccc}
C\text{-mod} & \text{Diff. rings} & \delta\text{-rings} \\
C\otimes_{\mathbb{Z}} - \uparrow \downarrow \uparrow \mathrm{Hom}_{\mathbb{Z}}(C,-) & D\odot - \uparrow \downarrow \uparrow W^{diff} = \text{divided power series} & \Delta\odot - \uparrow \downarrow \uparrow W \\
\mathbb{Z}\text{-mod} & \text{Rings} & \text{Rings}
\end{array}$$

- C = free C-module on one generator "1"
  = {natural 1-ary operations on C-modules}
  = representing obj. for $\mathrm{Hom}_{\mathbb{Z}}(C,-)$

$C\otimes_{\mathbb{Z}} M$ = generators $c\otimes m$, relations ....

- D = free diff. ring on one gen e $= \mathbb{Z}[e, d, d^{\circ 2}, \ldots]$, $d^{\circ n} = d^{\circ n}(e)$
  = "alg. diff. operators", e = identity

  $\xi \in D$ defines $\xi : R \to R$

  and $D \xrightarrow{\sim} \{\text{natural 1-ary operations on diff. rings}\}$ (exercise)

  D also represents W^{diff}:

  $$\mathrm{Hom}(\mathbb{Z}[e,d,\ldots], A) = \prod_{\mathbb{N}} A = W^{diff}(A)$$

  $$D\odot A = \mathbb{Z}[d^{\circ n}(a) : a\in A] \big/ \left(d^{\circ n}(ab) = \cdots,\ d^{\circ n}(1) = \delta_{n,0},\ d^{\circ n}(a+b) = \cdots,\ d^{\circ n}(0) = 0\right) \qquad = \begin{cases}1\\0\end{cases}$$

- Δ = free δ-ring on one gen e
  $= \mathbb{Z}[e, \delta, \delta^{\circ 2}, \ldots]$, $\delta^{\circ n} = \delta^{\circ n}(e)$, $\varphi : \delta^{\circ n} \longmapsto (\delta^{\circ n})^p + p\,\delta^{\circ(n+1)}$
  = "p-differential operators"
  = {natural 1-ary operations on δ}

  Δ represents W:

  $$\begin{array}{ccc}
  \mathrm{Hom}(\mathbb{Z}[\ldots, \delta^{\circ n}, \ldots], A) & \longrightarrow & W(A) \\
  \alpha & \longmapsto & (\alpha(e), \alpha(\delta), \alpha(\delta^{\circ 2}), \ldots)
  \end{array}$$

  $$\Delta\odot A = \mathbb{Z}[\delta^{\circ n}(a) : a\in A] \big/ \left(\delta^{\circ n}(ab) = \cdots,\ \delta^{\circ n}(1) = \delta_{n,0},\ \delta^{\circ n}(a+b) = \cdots,\ \delta^{\circ n}(0) = 0\right)$$

Point: Suppose $\xi_0, \xi_1, \xi_2, \ldots \in \Delta$ is another free gen. set.
Then

$$W(A) = \mathrm{Hom}(\Delta, A) \xrightarrow{\sim} \prod_{\mathbb{N}} A$$
$$\alpha \longmapsto (\alpha(\xi_0), \alpha(\xi_1), \ldots)$$

But the ring str. on $W(A)$ when expressed in terms of the RHS will involve the Leibniz rules for the operators ξ_n, which in general will have nothing to do with those for the obvious generators $\delta^{\circ n}$.

- Traditional description of W is the one above for a certain list of free generating operators $\theta_0, \theta_1, \ldots \in \Delta$.

θ_n = the Witt operators
$\delta^{\circ n}$ = the Buium-Joyal operators

$$\theta_0 = \delta^{\circ 0} = e \quad , \quad \theta_1 = \delta^{\circ 1} \quad , \quad \theta_2 = \delta^{\circ 2} + \underbrace{\sum_{i=1}^{p} p^{i-2}\binom{p}{i} e^{p(p-i)}(\delta^{\circ 1})^i}_{\text{order } 1}$$

- Alternatively $W = \mathrm{Spec}(\Delta)$.
Two different coordinate systems $W \xrightarrow{\sim} \mathbb{A}^{\infty}_{\mathbb{Z}}$:
With coordinates, Buium-Joyal coordinates.
Buium-Joyal are usually better for conceptual purposes, but Witt are sometimes useful for computations.

So what are the Witt operators?

5. Witt components

Def: $\theta_n \in \Delta[1/p]$ recursively by

$$\underbrace{\varphi^{\circ n}}_{\text{iteration}} = \theta_0^{p^n} + p\,\theta_1^{p^{n-1}} + \cdots + p^n \theta_n$$

usual exponentiation

"Witt polynomials"

$\varphi^{\circ 0} = \theta_0 = e, \qquad \varphi^{\circ 1} = \theta_0^p + p\theta_1 \qquad \therefore\ \theta_1 = \delta$

$n=2$:
$$e^{p^2} + p\delta^p + p^2\theta_2 = \varphi^{\circ 2} = (e^p + p\delta)^p + p\delta^p + p^2\delta^{\circ 2}$$
$$\therefore\ p^2\theta_2 = p^2\delta^{\circ 2} + \sum_{i=1}^{p} \binom{p}{i} e^{p(p-i)} (p\delta)^i$$
$$\theta_2 = \delta^{\circ 2} + \sum_{i=1}^{p} \underbrace{\binom{p}{i} p^{i-2} e^{p(p-i)} \delta^i}_{\text{order 1 operator}}$$

Easy: the θ_n freely gen. $\Delta[1/p]$ as a $\mathbb{Z}[1/p]$-alg.

Pf: $\varphi^{\circ n} = p^n \theta_n$ + lower-order terms

Thm: (Joyal | Cartier-Dieudonné-Dwork)

(i) $\theta_n \in \Delta$

(ii) The θ_n gen Δ freely as a $\mathbb{Z}$-alg.

The proof requires an argument using non-trivial congruences. First time we've needed a non-formal argument.

Lecture 3

Remarks

- prismatic cohomology – δ-rings (Bhatt–Scholze) or $W(A)$ for A general (Drinfeld) (Arnab Saha's lectures)
- $W(A) = \mathrm{Hom}(\Delta, A)$ ⇝ (nearly) coord. indep. def of W.
- Ex: δ-rings have all limits + colimits, and the forgetful functor to rings preserves them.
- $\{\Delta \odot A \to C\} = \{A \to W(C)\}$

  $X = \mathrm{Spec}\, A$, $J(X) = \mathrm{Spec}\, \Delta \odot A$ ⇝ $J(X)(C) = X(W(C))$

  "arithmetic jet space" Buium p-differential alg. geom.

5. Witt components

Def: $\theta_n \in \Delta[1/p]$ recursively by

iterated composition

$$\varphi^{\circ n} = \theta_0^{p^n} + p\,\theta_1^{p^{n-1}} + \cdots + p^n\theta_n \qquad \text{"Witt polynomials"}$$

usual exponentiation

Thm: (Joyal | Cartier-Dieudonné-Dwork)

(i) $\theta_n \in \Delta$

(ii) The θ_n gen Δ freely as a $\mathbb{Z}$-alg.

Key fact: $\mathbb{Z}[\theta_0,\dots,\theta_n] = \mathbb{Z}[e,\dots,\delta^{\circ n}] \implies \delta(\theta_n) = \theta_{n+1} + (\text{terms in } \mathbb{Z}[\theta_0,\dots,\theta_n])$

Then $\delta^{\circ(n+1)} \equiv \theta_{n+1} + (\text{lower order terms})$ and can conclude by induction

Pf of key fact:

Expand $\varphi^{\circ(n+1)}$ in two ways

① $\varphi^{\circ(n+1)} = \sum_{i=0}^{n+1} p^i\theta_i^{p^{n+1-i}}$

②
$$\begin{aligned}
\varphi(\varphi^{\circ n}) &= \varphi\Big(\sum_{i=0}^{n} p^i\theta_i^{p^{n-i}}\Big)\\
&= p^n\varphi(\theta_n) + \sum_{i=0}^{n-1} p^i\varphi(\theta_i)^{p^{n-i}}\\
&= \text{---}\,''\,\text{---} + \sum_{i=0}^{n-1} p^i\big(\theta_i^p + p\,\delta(\theta_i)\big)^{p^{n-i}}\\
\text{induction} \leadsto\ &= \text{---}\,''\,\text{---} + \sum_{i=0}^{n-1} p^i\big(\theta_i^p + p\,f_i(\theta_0,\dots,\theta_{i+1})\big)^{p^{n-i}}\\
&\equiv p^n\varphi(\theta_n) + \sum_{i=0}^{n-1} p^i\theta_i^{p^{n+1-i}} \mod p^{n+1}\mathbb{Z}[\theta_0,\dots,\theta_n]
\end{aligned}$$

$\therefore\ p^n\varphi(\theta_n) \equiv p^n\theta_n^p + p^{n+1}\theta_{n+1} \mod p^{n+1}\mathbb{Z}[\theta_0,\dots,\theta_n]$

$\varphi(\theta_n) = \theta_n^p + p\,\theta_{n+1} \mod p\,\mathbb{Z}[\theta_0,\dots,\theta_n]$

$\quad \| $

$\theta_n + p\,\delta(\theta_n)$

$\therefore\ \delta(\theta_n) \equiv \theta_{n+1} \mod \mathbb{Z}[\theta_0,\dots,\theta_n]$ ☒

6. Ghost components

Consider the operators $\varphi^{\circ n} \in \Delta$

$$\varphi^{\circ 0} = e$$
$$\varphi^{\circ 1} = e^p + p\delta$$
$$\varphi^{\circ 2} = (e^p + p\delta)^p + p\delta^p + p^2\delta^{\circ 2}$$
$$\vdots$$

They do not gen. Δ, even mod p: $\varphi^{\circ n} \equiv e^{p^n} \bmod p\Delta$

$\therefore$ they gen. $\mathbb{F}_p[e] \subsetneq \mathbb{F}_p[e, \delta, \delta^{\circ 2}, \dots]$

But we can ignore that and proceed as if they did:

$$\begin{array}{ccc} W(A) & \xrightarrow{w} & \prod_{\mathbb{N}} A \\ \alpha & \longmapsto & \langle \alpha(e), \alpha(\varphi), \alpha(\varphi^{\circ 2}), \dots \rangle \end{array}$$

RHS has product ring str. because $\varphi^{\circ n}$ is additive + mult. So Leibniz rules are the easy ones

"Ghost map", RHS = "ghost components"

N.B.: the ghost map is not usually an isomorphism!

In δ-coordinates: $w(x_0, x_1, \dots) = \langle Z_0, Z_1, \dots \rangle$,
where Z_n is the poly $Z_n(x_0, x_1, \dots)$ s.t. $\varphi^{\circ n} = Z_n(e, \delta, \dots, \delta^{\circ n})$.

So $w(x_0, x_1, x_2)_\delta = \langle x_0, x_0^p + px_1, (x_0^p + px_1)^p + px_1^p + p^2x_2, \dots \rangle$

Similarly in Witt coords, but we have nice closed forms for the corresponding polys, by def. of the θ_n!

$$w(x_0, x_1, \dots)_{\text{Witt}} = \langle \dots, \sum_{i=0}^{n} p^i x_i^{p^{n-i}}, \dots \rangle$$

But they do freely generate $\Delta[1/p]$ as a $\mathbb{Z}[1/p]$-alg.

<u>Pf</u>: $\delta^{\circ n} = \left(\frac{\varphi - e^p}{p}\right)\circ\left(\frac{\varphi - e^p}{p}\right)\circ \cdots \circ\left(\frac{\varphi - e^p}{p}\right) = \frac{1}{p^n}\varphi^n + (\text{lower order terms})$

So $1/p \in A \Rightarrow W(A) = \mathrm{Hom}(\Delta, A) = \mathrm{Hom}(\Delta[\frac{1}{p}], A)$, so w is a bijection

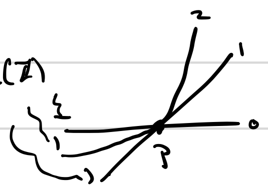
<u>Ex</u>: ① A p-tor free $\Rightarrow$ ghost map is injective.

$\therefore$ $W(A)$ is naturally a subring of the product ring $\prod_{\mathbb{N}} A$

② $W(\mathbb{Z}) = \{\langle a_0, a_1, \ldots\rangle \in \prod_{\mathbb{N}} A \mid a_{n+1} \equiv a_n \bmod p^{n+1}\}$

③ $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ and $W(\mathbb{Z}) \to W(\mathbb{F}_p)$ is $\langle a_0, a_1, \ldots\rangle \mapsto \lim_n a_n$

Spectrum: $\mathrm{Spec}(\mathbb{Z})$



* How to do a computation in $W(A)$, A general:
  ① Choose p-tor. free $\tilde{A} \twoheadrightarrow A$, and lift the problem to $W(\tilde{A})$
  ② Perform the computation in $W(\tilde{A})$ using ghost components (easy!)
  ③ Convert the answer back to the original components of $W(\tilde{A})$
  ④ Reduce back to $W(A)$.

<u>Ex</u>: If $x_0, y_0, x_1, y_1, \ldots$ are square-zero elements in an $\mathbb{F}_p$-algebra, compute $(x_0, x_1, \ldots) + (y_0, y_1, \ldots)$, and $\times$.

7. Teichmüller lifts

A ring, $\mathbb{Z}[A]$ = monoid alg. on mult. monoid of A

$$\begin{array}{rccc} \varphi : & \mathbb{Z}[A] & \longrightarrow & \mathbb{Z}[A] \\ & \Psi & & \Psi \\ & [a] & \longmapsto & [a^p] \end{array}$$

Frob. lift + $\mathbb{Z}[A]$ tor. free $\Rightarrow$ δ-str. $\delta([a]) = 0$.

$$\therefore \quad \begin{tikzcd} \mathbb{Z}[A] \arrow[rr, dashed, "\exists!"] \arrow[dr] & & W(A) \arrow[dl] \\ & A & \end{tikzcd}$$

In δ-coords: $[a] \longmapsto (a, 0, 0, \dots)$

Also true in Witt coordinates!

8. Conclusion

	Buium-Joyal	Witt	Ghost
Def	✓		✓
Category-th. props	✓		✓
φ-operator (F)			✓
δ-operator	✓		✓
Verschiebung		✓	✓
Ghost map closed form		✓	✓
Teich. elements	✓	✓	✓

Other topics

- plethystic formalism
- multiple primes, ramified, function field
- perfect Witt vectors: $W^{pf}(A) = \lim\limits_{\varphi} W(A)$, Fontaine's Θ, ...
- truncations
- de Rham-Witt interpretation